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A MESH INDEPENDENCE PRINCIPLE FOR INEXACT NEWTON–TYPE METHODS AND THEIR DISCRETIZATIONS

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Abstract. In this manuscript we study inexact Newton-like methods for the solution of nonlinear operator equations in a Banach space and their discretized versions in connection with the mesh independence principle. This principle asserts that the behavior of the discretized process is asymptotically the same as that for the original iteration and consequently, the number of steps required by the two processes to converge to within a given tolerance is essentially the same. So far this result has been proved by others using Newton's method for certain classes of boundary value problems and even more generally by considering a Lipschitz uniform discretization. In some of our earlier papers we extended these results to include Newton-like methods under more general conditions. However, all previous results assume that the iterates can be computed exactly. This is not true in general. That is why we use inexact Newton-like methods and even more general conditions. Our results, on the one hand, extend, and on the other hand, make more practical and applicable all previous results.

I. Introduction

The goal of this study is to extend the validity of the mesh independence principle to include perturbed Newton-like methods. Let us consider the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on some convex subset D of a Banach space E_1 with values in a Banach space E_2.

Let $x_0 \in D$ be fixed and define the inexact Newton-like method for all $n \geq 0$ by

$$y_n = x_n - A(x_n)^{-1}F(x_n) \tag{2}$$

and

$$x_{n+1} = y_n - z_n. \tag{3}$$

Here, $A(x_n)$ denotes a linear operator which is a conscious approximation to the Fréchet derivative $F'(x_n)$ of F evaluated at $x = x_n$ for all $n \geq 0$. The points $z_n \in D$ for all $n \geq 0$, and are determined in such a way that the iteration $\{x_n\}(n \geq 0)$ converges to a solution x^* of equation (1). By setting $z_n = 0$ for all $n \geq 0$, we obtain the Newton-like method. Moreover, setting $A(x_n) = F'(x_n)$ for all $n \geq 0$, we obtain Newton's method. The convergence of both methods have been examined extensively by others and us [1], [2], [3], [7-12], [16], in connection with the mesh independence principle. Since the iterates of the inexact Newton-like method (2)-(3) (for $z_n = 0,\ n\ \geq 0$ or not) can rarely be computed in infinite dimensional spaces, (2)-(3) can be replaced in practice by a family of discretized equations

$$P(a) = 0 \tag{4}$$

indexed by some real number $h > 0$, where P is a nonlinear operator between finite dimensional spaces E^1 and E^2. Let the discretization on E_1 be defined by the bounded linear operators $L : E_1 \to E^1$. Consider also the iteration $\{a_n\}(n \geq 0)$ given for all $n \geq 0$ by

$$b_n = a_n - Q(a_n)^{-1}P(a_n), \quad a_0 = L(x_0) \tag{5}$$

and

$$a_{n+1} = b_n - d_n. \tag{6}$$

Here $Q(a_n)$ denotes a linear operator which is a conscious approximation to the Fréchet derivative $P'(a_n)$ of P evaluated at $a = a_n$ for all $n \geq 0$. The points $d_n \in E^1$ for all $n \geq 0$, and are determined in such a way that the iteration $\{a_n\}(n \geq 0)$ converges to a solution a^* of equation (4). Note that all symbols introduced from (4)-(6) really depend on h. That is $P = P_h,\ L = L_h,\ Q = Q_h$, etc. But we do not use the latter to simplify the notation.

In practice, the iterates y_n or even b_n can rarely be computed exactly. That is why we need to "correct" at every step by introducing z_n or d_n, respectively, for the iterations under consideration. This is the factor that the results in the studies mentioned above have not taken into account when proving the mesh independence principle. The mesh independence principle (proved in the studies above) asserts that the number of steps required by the two processes to converge to within a given tolerance is essentially the same.

Here we show that this is true for our inexact Newton-like method (2)-(3). Our results can be reduced to the ones obtained earlier for appropriate choices of the factors involved. We make use of the sufficient conditions for the convergence of the perturbed Newton-like method (2)-(3) that we found in [4], [5].

The importance of the formulation of an efficient mesh size strategy based upon the mesh-independence has been extensively discussed in [1], [2], [3], [7], [11], [12], [16] and the references there. We finally apply our results to a two point boundary value problem.

II. Preliminaries

The norms in all spaces will be denoted by the same symbol $\| \ \|$. For any bounded linear operator from E_1 to E_2 or from E^1 to E^2, the induced norm will be used.

We find it convenient to introduce the following:

(G_1) Let $R > 0$ be fixed and using the notation $U(x_0, R) = \{x \in E_1 \mid \|x - x_0\| \leq R\}$, assume there exists $x_0 \in D$ and a function $\overline{\alpha} : U^3(x_0, R) \to [0, +\infty)$ such that

$$\|A(x_0)^{-1}[F(y) - F(x) - A(x)(y - x) + F'(y)(z - y)]\| \leq \overline{\alpha}(x, y, z) \tag{7}$$

for all $x, z \in U(y, r(x, z)) \subseteq U(x_0, R) \subseteq D$, where

$$r(x, z) = \max\{\|y - x\|, \ \|y - z\|\} \leq R - \|y - x_0\|.$$

(G_2) There exist continuous, nondecreasing functions w, w_1 and w_2 such that

$$w : U(x_0, R) \to [0, +\infty), \quad w_1, w_2 : [0, R] \to [0, +\infty)$$

with $w_1(0) = w_2(0) = 0$ and a sequence $\{z_n\} (n \geq 0)$ of points with

$$\|z_i\| \leq w(z_i) \leq w_1(r) \quad \text{for all } i \geq 0 \tag{8}$$

and for all $k \in N$

$$\sum_{i=0}^{k+1} \|z_i\| \leq \sum_{i=0}^{k+1} w(z_i) \leq w_2(r) \tag{9}$$

for all $z_i \in U(x_0, r) \subseteq U(x_0, R)$ for $r \in (0, R]$.

(G_3) There exist continuous, nondecreasing functions $w_3, w_4 : [0, R] \to \to [0, +\infty)$ with $w_3(0) = w_4(0) = 0$ such that

$$\overline{\alpha_i} = \overline{\alpha}(x_i, y_i, x_{i+1}) \leq w_3(r) \quad \text{for all } i \geq 0 \tag{10}$$

and for all $k \in N$

$$\sum_{i=0}^{k+1} \overline{\alpha_i} \leq w_4(r) \tag{11}$$

for all $x_i, y_i, x_{i+1} \in U(x_0, r) \subseteq U(x_0, R)$.

We will need the definition of a divided difference of order one [6], [13], [15].

Definition. Let F be a nonlinear operator defined on a subset D of a linear space E_1 with values in a linear space E_2, and let x, y be two points of D. A linear operator from E_1 into E_2, denoted by $[x, y]$ which satisfies the condition

$$[x, y](x, y) = F(x) - F(y) \tag{12}$$

is called a divided difference of order one of F at the points x and y.

Condition (12) does not uniquely determine the divided difference, with the exception of the case when E_1 is one-dimensional. An operator $[.,.] : D \times D \to \to L(E_1, E_2)$ satisfying condition (12) is called a divided difference of order one of F on D. If we fix the first variable, we get an operator $[x_0, .] : D \to \to L(E_1, E_2)$.

From now on we assume that D is convex, and E_1 and E_2 are Banach spaces.

(G_4) Let $F : D \subseteq E_1 \to E_2$ be a nonlinear Fréchet differentiable operator whose divided difference $[x, y]$ satisfies

$$\|A(x_0)^{-1}([x + l_1, x + l_2] - A(x))\| \leq C_1(r, r + \|l_1\|) + C_2(r, r + \|l_2\|) \tag{13}$$

and

$$\|A(x_0)^{-1}(A(x) - A(x_0))\| \leq C_0(\|l\|) \tag{14}$$

for all $x \in U(x_0, r) \subseteq U(x_0, R)$ and $0 \leq \|l\|,\ \|l_1\|,\ \|l_x\| \leq R_r$. The functions C_1 and C_2 are assumed to be continuous in both variables on $[0, R] \times [0, R]$ and such that if one variable is fixed, then they are nondecreasing functions of the other on $[0, R]$ with $C_1(0, 0) = C_2(0, 0) = 0$. We also set $C = C_r + C_2$. The function C_0 is continuous and nondecreasing on $[0, R]$ with $C_0(0) = 0$.

We can now show the following useful lemma.

Lemma. *Let $F : D \subseteq E_1 \to E_2$ be a nonlinear operator and assume:*

(a) For any $x, y \in D_0 \subseteq D$ there is a divided difference $[x, y] \in L(E_1, E_2)$ such that condition (12) is satisfied;

(b) Condition (13) is satisfied for all $x \in U(x_0, r) \subseteq U(x_0, R) \subseteq D_0$. Then

$$F'(x) = [x, x] \quad for\ all\ x \in D_0. \tag{15}$$

Proof. Let us choose $x \in \text{Int}\ U(x_0, r)$, $r \leq R$ and $\delta > 0$ such that $U(x, \delta) \subseteq U^0(x, r)$. For $\|l\| \leq \delta$, using (13) we obtain

(16)

$$\begin{aligned}\|A(x_0)^{-1}(F(x+l) - F(x) - [x,x]l)\| &= \|A(x_0)^{-1}([x+l,x] - [x,x])l\| = \\ &= \|A(x_0)^{-1}[([x+l,x] - A(x)) + (A(x) - [x,x])]l\| \leq \\ &\leq \|A(x_0)^{-1}([x+l,x] - A(x))l\| + \|A(x_0)^{-1}(A(x) - [x,x])l\| \leq \\ &\leq [C_1(r, r + \|l\|) + 2C_2(r,r) + C_1(r,r)]\|l\|.\end{aligned}$$

The above inequality proves that $F'(x) = [x, x]$ for all $x \in \text{Int}\ U(x_0, r)$ when $C_1(r, r) + C_2(r, r) \neq 0$ and $\|l\| \to 0$. To cover the case when $C_1(r, r) = C_2(r, r) = 0$ (note that $r = 0$ then) we observe that by (13) $\exists L \in L(E_1, E_2)$ such that $[x, y] = L$ for all $x, y \in U^0(x, r)$. Therefore by (12) we can choose δ arbitrarily above and set $F'(x) = l$. That completes the proof of the lemma.

Let us define the functions $\varphi, \varphi_1, \varphi_2 : [0, R] \to [0, +\infty)$ by

$$\varphi(r) = r - w_2(r) = \frac{1}{1 - C_0(r)} \left[\int_0^r C(r,t)dt + C(r,r)r + w_4(r) \right], \tag{17}$$

$$\varphi_1(r) = r - \varphi(r) \tag{18}$$

and

$$\varphi_2(r) = \frac{1}{1 - C_0^*(r)} \int_0^1 C^*(0, rt)dt + w_5(r) \quad \text{for all } t \in [0,1], r \in [0, R]. \tag{19}$$

Here $C^* = C_1^* + C_2^*$ and the functions C_1^*, C_2^*, C_0^* and w_5 are as the corresponding functions above without the "stars".

By the hypotheses on the C and w functions above, there exist constants $p, h_1, \delta_0, \delta_1, \delta_2, \delta_3, c_4, c_3$ with $c_4 > \delta_3$ such that for

$$0 < \delta_0 \leq c_4 - c_5 h^p \leq r(h) \leq c_3 h^p \leq \delta_1 \leq R_0 \tag{20}$$

the following is true

$$0 < \delta_2 \leq \varphi_1(r(h)) \leq \delta_3 < 1. \tag{21}$$

We can now show that for all $h \in (0, h_2]$, where

$$h_2 = \min\left\{ h_1, \left(\frac{\delta_2}{c_3 - c_6}\right)^{1/p}, \left(\frac{c_4 - \delta_3}{c_2 + c_5}\right)^{1/p} \right\} \quad \text{with } c_3 > c_6, \tag{22}$$

the following is true

$$0 < \varphi(r(h)) \leq c_6 h^p. \tag{23}$$

Indeed from (20) $r(h) - \delta_3 \leq \varphi(r(h)) \leq r(h) - \delta_2$. It is enough to show that $r(h) - \delta_3 > 0$ and $r(h) - \delta_2 \leq c_9 h^p$, which will be true if $c_4 - c_5 h^5 - \delta_3 \geq c_2 h^p$ and $c_3 h^p - \delta_2 \leq c_6 h^p$, respectively. The last inequalities are true by the choice of h and (22).

Similar arguments can show that for sufficiently small h there exist $\delta_4, \delta_5, \delta_6, \delta_7, c_7, c_8, c_9$ such that for

$$0 < \delta_4 \leq c_7 - c_8 h^p \leq r^*(h) \leq c_9 h^p \leq \delta_5 \leq R_1 \leq r^* \tag{24}$$

the following is true

$$0 < \delta_6 \leq \varphi(r^*(h)) \leq \delta_7 < 1. \tag{25}$$

III. Convergence analysis

We will need the following result on local convergence.

Theorem 1. *Let* $F : D \subset E_1 \to E_2$ *be a nonlinear operator and assume:*

(a) There exists a solution $x^* \in D$ *of the equation* $F(x) = 0$ *such that the linear operator* $A(x^*)$ *has a bounded inverse;*

(b) F *is a Fréchet-differentiable operator whose divided difference* $[x, y]$ *satisfies*

$$\|A(x^*)^{-1}([x + l_1, x + l_2] - A(x)\| \le C_1^*(r, r + \|l_1\|) + C_2^*(r, r + \|l_2\|) \tag{26}$$

and

$$\|A(x^*)^{-1}(A(x) - A(x^*))\| \le C_0^*(r, r + \|l\|) \tag{27}$$

for all $x \in U(x_0^*, r) \le U(x_0^*, R)$ *and* $0 \le \|l\|, \|l_1\|, \|l_2\| \le R - r$ *for some* $R > 0$ *and* $r \in [0, R]$ *with* $U(x^*, R) \subseteq D$. *Moreover,* C_0^*, C_1^*, C_2^* *are real continuous functions of two variables with* $C_0^*(0, 0) = C_1^*(0, 0) = C_2^*(0, 0) = 0$ *and such that if one variable is fixed, then they are nondecreasing functions of the other on* $[0, R]$*, and*

(c) There exists a sequence $\{z_n\}(n \ge 0)$ *of points from* D *such that for all* $x_n, y_n, z_n \in U(x_0, r)$

$$\|z_n\| \le T_n = T(z_n) \le w_5(r), \tag{28}$$

where $T : U(x^*, R) \to [0, +\infty)$ *is continuous, and* $w_5 : [0, R] \to [0, +\infty)$ *is continuous and nondecreasing with* $w_5(0) = 0$.

Then the following are true:

(i) There exists a sufficiently large number $r^* \in (0, R]$ *such that*

$$C_0^*(0, r^*) < 1 \tag{29}$$

and

$$0 < \varphi_2(r^*) < 1 \quad \text{for all} \quad t \in [0, 1] \tag{30}$$

where $C^* = C_1^* + C_2^*$.

(ii) The sequence $\{y_n\}$, $\{x_n\}(n \ge 0)$ *are well defined, remain in* $U(x^*, r^*)$ *for all* $n \ge 0$ *and* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$, *provided that* $x_0 \in U(x^*, r^*)$.

Moreover, the following estimates are true for all $n \geq 0$:

$$\|x_{n+1} - x^*\| \leq \gamma_n \|x_n - x^*\| \leq \gamma \|x_n - x^*\| \tag{31}$$

and

$$\|y_n - x^*\| \leq \delta_n \|x_n - x^*\| \leq \delta \|x_n - x^*\|, \tag{32}$$

where

$$\delta_n = \delta_n(t) = \frac{\int_0^1 C^*(0, \|x_n - x^*\| t) dt}{1 - C_0^*(\|x_n - x^*\|)}, \tag{33}$$

$$\delta = \frac{\int_0^1 C^*(0, r^* t) dt}{1 - C_0^*(r^*)}, \tag{34}$$

$$\gamma_n = \gamma_n(t) = \delta_n(t) + T_n \tag{35}$$

and

$$\gamma = \delta = w_5(r^*). \tag{36}$$

Proof. (i) By hypotheses (b) and (c) we have $C_0^*(0,0) = C_1^*(0,0) = C_2^*(0,0) = C^*(0,0) = w_5(0) = 0$, and that all these functions are continuous and nondecreasing on $[0, R]$. Hence, we can find a sufficiently large $r^* \in (0, R]$ such that estimates (29) and (30) are satisfied.

(ii) Using the Banach lemma on invertible operators [13] and condition (27), we deduce

$$\|A(x^*)^{-1}(A(x) - A(x^*))\| \leq C_0^*(\|x - x^*\|) \leq C_0^*(r^*) < 1 \tag{37}$$

(by (29)) for all $x \in U(x^*, r^*)$. Hence we deduce that the linear operator $A(x)$ is invertible on $U(x^*, r^*)$ and

$$\|A(x)^{-1} A(x^*)\| \leq \frac{1}{1 - C_0^*(\|x - x^*\|)} \leq \frac{1}{1 - C_0^*(r^*)}. \tag{38}$$

In particular, (37), (38) are true for $x = x_0$, since $x_0 \in U(x^*, r^*)$. Let us assume that $x \in U(x^*, r^*)$ for $m = 0, 1, 2, \dots, k$. Using (2) we get

$$y_k - x^* = -A(x_k)^{-1}[F(x_k) - F(x^*) - A(x_k)(x_k - x^*)]. \tag{39}$$

We also introduce the approximation

$$F(x_k) - F(x^*) - A(x_k)(x_k - x^*) = \int_0^1 [F'(x^* + t(x_k - x^*)) - A(x_k)](x_k - x^*)dt =$$

$$= \int_0^1 [[x^* + t(x_k - x^*),\ x^* + t(x_k - x^*)] - A(x_k)](x_k - x^*)dt. \tag{40}$$

We now compose both sides of (40) by $A(x^*)$ and then by taking norms and using (26), we obtain that the left hand side of (40) is bounded above by

$$\begin{aligned} &\int_0^1 [C_1^*(0, 0 + t\|x_k - x^*\|) + C_2^*(0, 0 + t\|x_k - x^*\|)]\|x_k - x^*\|dt = \\ &\qquad = \int_0^1 C^*(0, \|x_k - x^*\|t)\|x_k - x^*\|dt. \end{aligned} \tag{41}$$

From (38), (39) and (41) we now have

$$\|y_k - x^*\| \le \|A(x_k)^{-1}A(x^*)\| \cdot$$

$$\cdot A(x^*)^{-1} \int_0^1 [[x^* + t(x_k - x^*),\ x^* + t(x_k - x^*)] - A(x_k)](x_k - x^*)dt \le$$

$$\le \delta_k \|x_k - x^*\| \le \delta \|x_k - x^*\|. \tag{42}$$

The above estimate shows that (32) is true and that $y_k \in U(x^*, r^*)$ since $\delta_k \le \delta < 1$ (by (30)).

Moreover, from (3), (42) and (28), we get

$$\|x_{k+1} - x^*\| \le \|y_k - x^*\| + \|z_k\| \le \gamma_k \|x_k - x^*\| \le \gamma \|x_k - x^*\| \tag{43}$$

which shows (31) and that $x_{k+1} \in U(x^*, r^*)$. Hence the sequences $\{x_n\}$, $\{y_n\}$ are well defined, remain in $U(x^*, r^*)$ and satisfy (31) and (32) for all $n \geq 0$.

Let $m \geq 0$. Then by (31) we get

$$\begin{aligned} \|x_{n+m} - x^*\| &\leq \gamma_{n+m-1}\|x_{n+m-1} - x^*\| \leq \\ &\leq \gamma_{n+m-1}\gamma_{n+m-2}\|x_{n+m-2} - x^*\| \leq \\ &\leq \ldots \leq \gamma^m\|x_n - x^*\|. \end{aligned} \tag{44}$$

Similarly by (32) and (31) we get

$$\|y_{n+m} - x^*\| \leq \delta \cdot \gamma^m \|x_n - x^*\|. \tag{45}$$

Finally, by letting $m \to \infty$ in (44) and (45), we obtain $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$ (since $0 < \gamma < 1$). That completes the proof of the theorem.

We now state the position of a theorem that we will use here, whose proof can be found in [4], [5].

Theorem 2. *Let $F : D \subseteq E_1 \to E_2$ be a nonlinear Fréchet differentiable operator whose divided difference is denoted by $[x, y]$ for $x, y \in D$. Assume:*

(a) conditions (G_1), (G_2), (G_3) and (G_4) are satisfied;

(b) there exists a minimum nonnegative number r_0 such that

$$\|y_0 - x_0\| \leq s_0 \leq \varphi(r_0); \tag{46}$$

(c) the following estimates are true:

$$r_0 \leq R, \quad \int_0^1 C(0, (1-t)R + tr_0)dt < 1 \tag{47}$$

and

$$C_0(0, r_0) < 1. \tag{48}$$

Then the sequences $\{x_n\}$, $\{y_n\}$ $(n \geq 0)$ generated by relations (2) and (3) are well defined, remain in $U(x_0, r_0)$ for all $n \geq 0$, and converge to a solution x^* of the equation $F(x) = 0$, which is unique in $U(x_0, R)$.

In many applications it turns out that the solution x^* of equation (1) as well as the iterates x_n, y_n have "better smoothness" properties than the elements of E_1. This is a motivation for considering a subset $E_3 \subseteq E_1$ such that

$$x^* \in E_3, \quad x_n, y_n \in E_3, \quad x_n - x^*, y_n - x^* \in E_3, \\ x_{n+1} - x_n, y_{n+1} - y_n \in E_3 \quad (n \geq 0). \tag{49}$$

We consider a family of triplets

$$\{P, L, L_0\}, \quad h > 0, \tag{50}$$

where

$$P \; : \; E_4 \subseteq E^1 \to E^2, \quad h > 0$$

are nonlinear operators and

$$L \; : \; E_1 \to E^1, \quad L_0 \; : \; E_2 \to E^2, \quad h > 0$$

are bounded linear discretization operators such that

$$L(E_3 \cap U(x^*, r^*)) \subseteq E_4, \quad h > 0. \tag{51}$$

The discretization (50) is called uniform if there exists a number R_0 such that

$$U(L(x^*), R_0) \subseteq E_4, \quad h > 0 \tag{52}$$

and the triplet $(P, Q, L(x^*))$ satisfies the "G" conditions that the triplet (F, A, x_0) satisfy for all $h > 0$ in the ball $U(L(x^*), r_0)$.

Moreover, the discretization family (50) is called: bounded if there is a constant $q > 0$ such that

$$\|L(x)\| \leq q\|x\|, \quad x \in E_3, \quad h > 0, \tag{53}$$

stable if there is a constant $\sigma > 0$ such that

$$\|Q(L(x))^{-1}\| \leq \sigma, \quad x \in E_3 \cap U(L(x^*), r^*), \quad h > 0, \tag{54}$$

consistent of order $p > 0$ if there are two constants $c_0, c_1 > 0$ such that

$$\|Q(L(x^*))(L_0 F(x) - P(L(x)))\| \leq c_0 h^p, \quad x \in E_3 \cap U(L(x^*), r^*), \quad h > 0 \tag{55}$$

and

$$\|Q(L(x^*))(L_0(F'(x)(y) - Q(L(x))L(y)))\| \leq c_1 h^p, \\ x \in E_3 \cap U(L(x^*), r^*), \quad y \in E_3, \quad h > 0. \tag{56}$$

To simplify the notation, and without loss of generality, we will identify the functions C_1^*, C_2^*, C_0^* and w_5 by the functions C_1, C_2, C_0 and w_2 respectively.

With the notation introduced above we can now formulate our main result.

Theorem 3. *Let $F : D \subseteq E_1 \to E_2$ be a nonlinear operator. Assume:*

(a) the hypotheses of Theorem 1 are satisfied;

(b) the discretization (50) is bounded, stable and consistent of order p.

Then

(i) equation (4) has a locally unique solution

$$a^*(h) = a^* = L(x^*) + O(h^p) \tag{57}$$

for all $h \in (0, \overline{h_0}]$ with $\overline{h_0}$ being a fixed constant.

(ii) There exist constants $\overline{h_1} \in (0, \overline{h_0}]$, $r_1 \in (0, r^*]$ such that the discrete iteration (5)-(6) converges to a^*; and

(iii) if there exist constants c_{10}, c_{11} with $4(c_0 + c_1) < c_{10} \leq c_{11}$, such that for all $n \geq 0$

$$\|d_n - L(z_n)\| \leq (c_{11} - c_{10})h^p, \quad h \in (0, \overline{h_0}],\ r \in (0, r^*], \tag{58}$$

then there exist constants $\overline{h_1} \in (0, \overline{h_0}]$, $r_3 \in (0, r_1]$ such that the following estimates are true for all $n \geq 0$

$$b_n = L(y_n) + O(h^p), \tag{59}$$

$$a_n = L(x_n) + O(h^p), \tag{60}$$

$$Q(b_n)^{-1}P(b_n) = Q(b_n)^{-1}L_0(F(y_n)) + O(h^p), \tag{61}$$

$$Q(a_n)^{-1}P(a_n) = Q(a_n)^{-1}L_0(F(x_n)) + O(h^p), \tag{62}$$

$$b_n - a^* = L(y_n - x^*) + O(h^p) \tag{63}$$

and

$$a_n - a^* = L(x_n - x^*) + O(h^p). \tag{64}$$

Proof. (i) The functions C_0 and C are continuous and $C_0(0,0) = C(0,0) = 0$. Hence we can find intervals $(0,h_0]$ and $(0,R_0]$ ($R_0 \leq R$) such that conditions (47) and (48) are satisfied for all $h \in (0,h_0]$ and $r(h) \in (0,R_0]$. Set $h_3 = \min\left\{h_0, h_2, \left(\frac{R_0}{C_3}\right)^{1/p}\right\}$. Then using (20), (54) and (55) we obtain in turn

$$s_0(h) = \|Q'(L(x^*))^{-1}P(L(x^*))\| \leq \sigma\|P(L(x^*)) - L_0(F(x^*))\| \leq$$

$$\leq \sigma c_0 h^p \leq c_2 h^p \leq \varphi(r(h)), \quad c_2 \geq \sigma c_0 \tag{65}$$

and

$$r(h) \leq c_3 h^p \leq R_0 \quad \text{for all } h \in (0,h_3], \tag{66}$$

which shows that (46), (47) and (48) hold for all $h \in (0,h_3]$. Since all hypotheses of Theorem 2 are satisfied equation (4) has a solution $a^*(h) = a^* \in U(L(x^*), r(h))$ which is a unique solution of (4) in $U(L(x^*), R_0)$. Thus (57) follows from

$$\|a^* - L(x^*)\| \leq r(h) \leq c_3 h^p \leq R_0 \tag{67}$$

by setting $\overline{h_0} = h_3$.

(ii) The functions C_0^*, C^* and w_5 are continuous and $C_0^*(0,0) = C^*(0,0) = w_5(0) = 0$. Hence we can find intervals $(0,h_3]$ and $(0,R_1]$ such that conditions (24) and (30) are satisfied for all $h \in (0,h_3]$ and $r^*(h) \in (0,R_1]$. By applying Theorem 1 to (4) we see that the sequence (5)-(6) converges to a^* if

$$\|L(x_0) - a^*\| < r^*(h) \tag{68}$$

and

$$U(a^*, \|L(x_0) - a^*\|) \subseteq U(L(x^*), R_0). \tag{69}$$

But (69) holds if

$$\|a^* - L(x^*)\| + \|L(x_0) - a^*\| \leq R_0. \tag{70}$$

By (53) and (66) we obtain

$$\|L(x_0) - a^*\| \leq \|L(x_0) - L(x^*)\| + \|L(x^*) - a^*\| \leq q\|x_0 - x^*\| + c_3 h^p. \tag{71}$$

Thus (68), (69) are satisfied if

$$q\|x_0 - x^*\| + 2c_3h^p \leq R_0 \tag{72}$$

and

$$q\|x_0 - x^*\| + c_3h^p \leq c_7 - c_8h^p \tag{73}$$

hold respectively. Conditions (72) and (73) will certainly hold if

$$q\|x_0 - x^*\| \leq \frac{R_0}{2}, \quad 2c_3h^p \leq \frac{R_0}{2}, \quad q\|x_0 - x^*\| \leq \frac{c_7}{2} \quad \text{and}$$

$$c_3h^p \leq \frac{c_7}{2} - c_8h^p.$$

We choose

$$\|x_0 - x^*\| \leq r_1 = \min\left\{\frac{R_0}{2q}, \frac{c_7}{2q}\right\}$$

and

$$h_4 = \min\left\{h_2, h_3, \left(\frac{R_0}{4c_3}\right)^{1/p}, \left[\frac{c_7}{2(c_3+c_8)}\right]^{1/p}\right\}.$$

It is now easily verified that (68) and (69) are satisfied for all $h \in (0, h_4]$ and $x_0 \in U(x^*, r_1)$. Therefore, for these h and x_0, the iteration (5)-(6) converges to a^*.

(iii) We shall now show that there exist $\overline{h_1} \in (0, \overline{h_0}], r_3 \in (0, r_1]$ such that

$$\|a_n - L(x_n)\| \leq c_{11}h^p \tag{74}$$

for $n = 0$, (74) is true since $a_0 = L(x_0)$. Suppose that (74) holds for $n = 0, 1, \ldots, i$. We note that if we show that

$$\|b_n - L(y_n)\| \leq c_{10}h^p, \tag{75}$$

then from (5), (6), (75), (58) and the estimate

(76)
$$\|a_{i+1} - L(x_{i+1})\| = \|b_i - d_i - L(y_i - z_i)\| \leq \|b_i - L(y_i)\| + \|d_i - L(z_i)\| \leq$$
$$\leq c_{10}h^p + (c_{11} - c_{10})h^p = c_{11}h^p,$$

we can complete the induction for (74). But (75) is true for $n = 0$ by (58). We now suppose that (75) is true for $n = 0, 1, \ldots, i$. Using (2), (3), (5) and (6) we can obtain the approximation

$$
\begin{aligned}
b_i - L(y_i) = Q(a_i)^{-1}\{&[Q(a_i)(a_i - L(x_i)) - P(a_i) + P(L(x_i))]+\\
&+ [(Q(a_i) - Q(L(x_i)))L(A(x_i)^{-1}F(x_i))]+\\
&+ [Q(L(x_i))L(A(x_i)^{-1}F(x_i)) - L_0(F(x_i))]+\\
&+ [L_0(F(x_i)) - P(L(x_i))]\}.
\end{aligned} \tag{77}
$$

From (53) and (74) we obtain

$$
\begin{aligned}
\|a_i - L(x^*)\| &\le \|a_i - L(x_i)\| + \|L(x_i) - L(x^*)\| \le\\
&\le c_{11}h^p + qr_1.
\end{aligned} \tag{78}
$$

As in (37), (38) and using (14) and (78) we obtain

$$
\|Q(a_i)^{-1}Q(L(x^*))\| \le \frac{1}{1 - C_0(\|a_i - L(x^*)\|)} \le \frac{1}{1 - (c_{11}h^p + qr_1)}. \tag{79}
$$

By composing the first bracket in (77) by $Q(L(x^x))^{-1}$ and by taking norms, using (13) and (78), we obtain that this term is bounded above by

(80)

$$
\begin{aligned}
&\int_0^1 \|Q(L(x^*))^{-1}[P'(a_i + t(L(x_i) - a_i)) - Q(a_i)](a_i - L(x_i))\|dt \le\\
&\quad\le \int_0^1 C(\|a_i - L(x^*)\|, \|a_i - L(x^*)\| + t\|L(x_i) - a_i\|)\|a_i - L(x_i)\|dt \le\\
&\quad\le \int_0^1 C(c_{11}h^p + qr_1, qr_3 + (t + c_{11}h^p))c_{11}h^p dt.
\end{aligned}
$$

Moreover, by adding and subtracting $Q(L(x^*))^{-1}$ inside the parenthesis of the second bracket, composing by $Q(L(x^*))^{-1}$ and using (14), (78) and (53), we obtain that this term is bounded above by

$$
\begin{aligned}
&[C_0(\|a_i - L(x^*)\|) + C_0(\|L(x^*) - L(x_i)\|)]q\|b_i - L(y_i)\| \le\\
&\quad\le [C_0(c_{11}h^p + qr_1) + C_0(r_1)]qc_{10}h^p.
\end{aligned} \tag{81}
$$

Furthermore, using (56) and (55), we obtain that the third and fourth brackets in (77), after being composed by $Q(L(x^*))^{-1}$, are bounded above by c_1h^p and c_0h^p respectively.

Finally, we collect all the above majorizations, denoted by $B(h)h^p$, to obtain that

$$\|b_i - L(y_i)\| \le \frac{1}{1-(c_{11}h^p + qr_1)} B(h)h^p. \tag{82}$$

Estimate (75) will now be true if

$$\frac{B(h)}{1-(c_{11}h^p + qr_1)} \le c_{10}, \tag{83}$$

which will certainly be true if

$$\frac{1}{c_{10}} \left[\int_0^1 C(c_{11}h^p + qr_1, qr_3 + (t + c_{11})h^p) c_{11} dt + q(C_0(c_{11}h^p + qr_1)) + \right.$$

$$\left. + C_0(r_1)c_{10} \right] + \frac{c_1 + c_0}{c_{10}} + c_{11}h^p + qr_1 \le 1. \tag{84}$$

Inequality (84) will certainly be true if each term at the left hand side of it is bounded above by $\frac{1}{4}$. Since the functions C and C_0 vanish at the origin, we can find $h_5, \ r_2 > 0$ such that this will happen for the first term. By the choice of c_{10} (see (58)), the second term is also bounded above by $\frac{1}{4}$. Finally, set $\overline{h_1} = = \min\left\{h_4, h_5, \left(\frac{1}{4c_{11}}\right)^{1/p}\right\}$ and $r_3 = \min\{r_1, r_2\}$. With the above choices of $\overline{h_1}$ and r_3, estimates (59) and (60) follow.

By the lemma and the continuity of P there exists $b > 0$ such that

$$\|P'(x)\| \le b, \quad x \in U(L(x^*), R_0). \tag{85}$$

Using (55), (75) and (85), we obtain

(86)

$$\begin{aligned} \|Q(b_n)^{-1}(P(b_n) - L_0(F(y_n)))\| &\le \|Q(b_n)^{-1}(P(b_n) - P(L(y_n)))\| + \\ &+ \|Q(b_n)^{-1}(P(L(y_n)) - L_0(F(y_n)))\| \le \\ &\le \frac{1}{1-(c_{10}h^p + qr_3)}(b\|b_n - L(y_n)\| + c_0 h^p) = \\ &= \frac{Mc_{10} + c_0}{1-(c_{10}h^p + qr_3)} h^p, \end{aligned}$$

which shows (61). Estimate (62) is obtained similarly by replacing b_n, y_n and c_{10} by a_n, x_n and c_{11}.

Moreover, from (75) and (66) we obtain

$$\begin{aligned}\|(b_n - a^*) - L(y_n - x^*)\| &\leq \|b_n - L(y_n)\| + \|a^* - L(x^*)\| \leq \\ &\leq c_{10}h^p + c_3 h^p = (c_{10} + c_3)h^p\end{aligned} \tag{87}$$

which shows (63).

Furthermore, from (74) and (66) we finally obtain

$$\begin{aligned}\|(a_n - a^*) - L(x_n - x^*)\| &\leq \|a_n - L(x_n)\| + \|a^* - L(x^*)\| \leq \\ &\leq c_{11}h^p + c_3 h^p = (c_{11} + c_3)h^p\end{aligned} \tag{88}$$

from which (64) follows. That completes the proof of the theorem.

We can now prove the mesh-independence principle for perturbed Newton-like methods.

Theorem 4. *Assume:*

(a) the hypotheses of Theorem 3 are true;

(b) there exists a constant $\delta > 0$ such that

$$\liminf_{h \to 0} \|L(u)\| \geq \delta \|u\| \quad \text{for each } u \in E_3. \tag{89}$$

Then for some $r_6 \in (0, r_3]$, and for any fixed $\varepsilon > 0$ and $x_0 \in U(x^*, r_6)$ there exists a constant $\overline{h} = \overline{h}(\varepsilon, x_0) \in (0, \overline{h_1}]$ such that

$$|\min\{n \geq 0, \quad \|x_n - x^*\| < \varepsilon\} - \min\{n \geq 0, \quad \|a_n - a^*\| < \varepsilon\}| \leq 1 \tag{90}$$

for all $h \in (0, \overline{h}]$.

Proof. By hypotheses there exists a unique integer $i > 0$ such that

$$\min\{n \geq 0, \quad \|x_n - x^*\| < \varepsilon \leq \|y_i - x^*\|\} \tag{91}$$

with $h_6 = h_6(x_0)$ such that

$$\|L(y_i - x^*)\| \geq \delta \|y_i - x^*\| \quad \text{for all } h \in (0, h_6]. \tag{92}$$

We will prove that the theorem holds for

$$r_6 = \min\left\{r_3, \frac{r_4}{q}\right\}, \ \beta = \min\left\{\delta, 2q, \frac{1 - C_0(r_5)}{2q\int\limits_0^1 C(0, r_5)}\right\}, \tag{93}$$

(94) $$\overline{h} = \min\left\{\overline{h_1}, h_6, \left(\frac{\beta\varepsilon}{2c_{12}}\right)^{1/p}, \left(\frac{\delta\varepsilon}{2c_{13}}\right)^{1/p}\right\},$$

where $c_{12} = c_3 + c_{11}, c_{13} = c_{10} + c_3$ and $r_5 = R_0 + r_4$.

From (88) and (94) it follows that

(95) $$\|a_{i+1} - a^*\| \leq \|L(x_{i+1} - x^*)\| + c_{12}h^p \leq q\varepsilon + \frac{\beta\varepsilon}{2} < 2q\varepsilon.$$

Using (93), (94) and Theorem 1 we obtain in turn that

(96) $$\begin{aligned} \|b_{i+1} - a^*\| &\leq \frac{\int_0^1 C(0, \|a_{i+1} - a^*\|t)dt}{1 - C_0(\|a_{i+1} - a^*\|)} \|a_{i+1} - a^*\| \leq \\ &\leq \frac{\int_0^1 C(0, r_5)dt}{1 - C_0(r_5)} 2\beta q\varepsilon < \varepsilon \end{aligned}$$

(since $\|a_{i+1} - a^*\| \leq \|a_0 - a^*\| = \|L(x_0) - a^*\| \leq \|L(x_0) - L(x^*)\| + \|L(x^*) - -a^*\| \leq qr_3 + c_3h^p \leq r_4 + R_0 = r_5$).

Moreover from (92) and (87) we obtain

$$\varepsilon \leq \|y_i - x^*\| \leq \frac{1}{\delta}\|L(y_i - x^*)\| \leq \frac{1}{\delta}(\|b_i - a^*\| + c_{13}h^p)$$

or

(97) $$\|b_i - a^*\| \geq \delta\varepsilon - c_{13}h^p \geq \delta\varepsilon - \frac{\delta\varepsilon}{2} = \frac{\delta\varepsilon}{2}.$$

Furthermore, if $\|a_i - a^*\| < \varepsilon$ then as in (96) we get

$$\|b_i - a^*\| < \frac{1}{2}\beta\varepsilon \leq \frac{\delta\varepsilon}{2}$$

which contradicts (97).

Hence we must have

(98) $$\|a_i - a^*\| \geq \varepsilon.$$

The result now follows from (91), (96) and (98). That completes the proof of the theorem.

As it was observed in [1], [2], [3], [7], the condition (89) follows from the condition

$$\lim_{h\to 0} \|L(u)\| = \|u\| \quad \text{for each } u \in E_3, \tag{99}$$

which is standard in most discretization studies. In fact, for some discretization studies, we have

$$\lim_{h\to 0} \|L(u)\| = \|u\| \quad \text{uniformly for } u \in E_3 \tag{100}$$

(see Remark 5 that follows). If this is the case, we can have a stronger version of the mesh independence principle.

Corollary. *Assume:*

(a) the hypotheses of Theorem 3 are true;

(b) condition (100) holds uniformly for $u \in E_3$*.*

Then there exists a constant $r_7 \in (0, r_3]$ *and for any fixed* $\varepsilon > 0$*, some* $\overline{h_2} = \overline{h_2}(\varepsilon) \in (0, \overline{h_1}]$ *such that (90) holds for all* $h \in (0, \overline{h_2}]$ *and all starting points* $x_0 \in (x^*, r_7)$*.*

IV. Applications - Remarks

Remarks.

(1) In [4], [5], [6] we showed how to choose the functions $\overline{\alpha}$, w, w_1, w_2, w_3, w_4, w_5 and the sequence $\{z_n\}$ $(n \geq 0)$. We also showed that special choices of the above can reduce our results to earlier ones involving single step methods (Newton's method, Secant method, the method of tangent parabolas, the method of tangent hyperbolas and other) as well as two step methods [4], [5], [6], and by making use of Theorem 2. Several examples involving the solution of nonlinear integral equations were also given there.

(2) As an application of Theorem 1, we note that this theorem can be realized for operators F which satisfy an autonomous differential equation of the form

$$F'(x) = B(F(x)), \quad \text{for some given operator } B \text{ [7].}$$

Assume for simplicity that $A(x) = F'(x)$ for all $x \in D$. As $F'(x^*) = B(0)$, the inverse $F'(x^*)^{-1}$ can be evaluated without knowing the actual solution x^*. Consider, for example, the scalar equation

$$F(x) = 0, \tag{101}$$

where F is given by $F(x) = e^x - s,\ s > 0$. Note that $F'(x) = F(x) + s$. That is $F'(x^*) = s$. Under the hypotheses of Theorem 1 and provided that $x_0 \in U(x^*, r^*)$, the iteration (2)-(3) converges to the solution $x^* = \ln(s)$ of equation (101).

(3) It can easily be seen that our results can be reduced to the ones in [1] for $A(x_n) = F'(x_n)$ and $z_n = 0$ for all $n \geq 0$. Moreover, they can be reduced to the ones in [2], [3] for $z_n = 0$ for all $n \geq 0$. Furthermore, our condition (90) and the corresponding ones in [1], [2], [3], [7] state that if

$$\min\{n \geq 0, \|x_n - x^*\| < \varepsilon\} = i + 1, \quad i > 0, \tag{102}$$

then

$$\min\{n \geq 0, \|a_n - a^*\| < \varepsilon\} = i + 1, \text{ or } i, \text{ or } i + 2. \tag{103}$$

However we can actually show that if (102) is true, then

$$\min\{n \geq 0, \|a_n - a^*\| < \varepsilon\} = i + 1, \text{ or } i, \tag{104}$$

which improves (103).

Let us assume that $q \in (0, \gamma)$ for some $\gamma \in (0,1)$, and under the hypotheses of Theorem 4, set

$$\overline{h_3} = \min\left\{\overline{h}, \quad \left(\frac{(\gamma - q)\varepsilon}{c_{12}}\right)^{1/p}\right\}. \tag{105}$$

The estimate (95) can also be written as

$$\|a_{i+1} - a^*\| \leq q\varepsilon + c_{12}h^p \leq \gamma\varepsilon < \varepsilon \quad \text{for } h \in (0, \overline{h_3}], \tag{106}$$

which shows (104).

(4) Concerning the choices of the "corrector" sequences $\{z_n\}$ and $\{d_n\}$ $(n \geq\ \geq 0)$ appearing in (3) and (6) respectively we state the following. Once the z_n's $(n \geq 0)$ are chosen (see Remark 1), then the d_n's will be chosen in such a way that conditions (58) are satisfied. Note that condition (58) will certainly

be satisfied if we simply set $d_n = L(z_n)$ for all $n \geq 0$, which is a logical choice but not the only one.

(5) The results obtained here in Theorems 1-4 and the Corollary can be extended to include operator equations involving nondifferentiable operators of the form

$$(1)' \qquad F_1(x) = F(x) + F_2(x)$$

and

$$(4)' \qquad P_1(a) = P(a) + P_2(a),$$

where F (and P) are as before and the operator F_2 (and P_2) satisfy an estimate of the form

$$(107) \qquad \|A(x_0)^{-1}(F(x) - F(y))\| \leq w_6(r)\|x - y\|$$

for all $x, y \in U(x_0, r) \subseteq U(x_0, R)$ and w_6 is a nondecreasing continuous function on $[0, R]$ with $w_6(0) = 0$. Our results will now be true for equations $(1)'$ and $(4)'$ if we make the following modifications: inside the bracket of (17) add the term $w_6(r)$; at the numerator of (19) add the term $w_6(r)$; at the numerators of (33) and (34) add the terms $\|x_n - x^*\|$ and $w_6(r^*)$ respectively; F_1, P_1 will replace F, P everywhere and the condition (107) will be added to the hypotheses of Theorems 1-4 and the Corollary. For the computational details on operator equations of the form $(1)'$ see also [6], [11], [12], [17].

(6) As we showed in [2], [3] (see also [1], [7], [16]) the discretization method $\{P, L, L_0\}$ can be used to solve boundary value problems involving operators F of the form

$$F(y) = \{y'' - f(x, y, y');\ 0 \leq x \leq 1,\ y(0) - v,\ y(1) - w\}$$

or

$$F(y) = \{y' - f(x, y),\ 0 \leq x \leq 1,\ sy(0) + ty(1) - v\},$$

or integral operators of the form

$$(F(y))(x) = y(x) - \int_0^1 f(x, b, y(t))dt + g(x), \quad 0 \leq x \leq 1$$

or operators of the form

$$F(y) = \{-y_{x_1 x_1} - y_{x_2 x_2} + f(x_1, x_2, y, y_{x_1}, y_{x_2}) \text{ in } \Omega, y = 0 \text{ on } \partial\Omega\}$$

involving partial differential equation boundary value problems. We leave the details to the motivated reader.

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