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ON COMPLETELY ADDITIVE FUNCTIONS SATISFYING A CONGRUENCE

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Abstract. A fascinating connection was conjectured on sums of completely additive functions satisfying a congruence by Kátai. The object of the present paper is to prove that if $P(z) = 1+A_1z+A_2z^2+A_3z^3+A_4z^4 \in \mathbb{R}[z] \setminus \mathbb{Q}[z]$ and $f \in \mathcal{A}^*$ satisfy the congruence relation

$$f(n)+A_1f(n+1)+A_2f(n+2)+A_3f(n+3)+A_4f(n+4) \equiv 0 \pmod 1$$

for every positive integer n, then $f(n)$ is identically zero. Another purpose of the authors was to discuss the problem from group theoretical aspects and to describe an algorithm verifying the conjecture for finite number of functions under a certain bound.

1. Introduction

An arithmetical function $f(n)$ is said to be *completely additive* if the relation $f(nm) = f(n)+f(m)$ holds for every positive integer n and m. Let $\mathcal{A}^*$ denote the class of all real-valued completely additive functions. Throughout this paper we apply the usual notations, i.e. $\mathcal{P}$ denotes the set of primes, $\mathbb{N}$ the set of positive integers, $\mathbb{Q}$ and $\mathbb{R}$ the fields of rational and real numbers, respectively.

Let $P(z) = 1 + A_1z + A_2z^2 + \ldots + A_kz^k$ $(k \geq 1)$ be a polynomial with real coefficients. Let E denote the operator $Ez_n := z_{n+1}$ in the linear space of infinite sequences. For the polynomial $P(z)$ we have

$$P(E)f(n) = f(n) + A_1f(n+1) + \ldots + A_kf(n+k).$$

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Conjecture I. (Kátai) *If* $f \in \mathcal{A}^*, P(z) \notin \mathbb{Q}[z]$ *and* $P(E)f(n) \equiv 0 \pmod 1$ *for all* $n \in \mathbb{N}$ *then* $f(n)$ *is identically zero.*

This conjecture was proved for the case of $k = 2$ (see [1]). Moreover, Kátai raised a more general question:

Conjecture II. (Kátai) *Let* $f_j \in \mathcal{A}^*$ $(j = 0, 1, 2, \dots, k)$. *Assume that*

$$\sum_{j=0}^{k} f_j(n+j) \equiv 0 \pmod 1$$

for all $n \in \mathbb{N}$. *Then* $f_j(n) \equiv 0 \pmod 1$ *for every* $n \in \mathbb{N}$ *and for every* j.

In [2] Kátai proved it for the case of $k = 3$. The idea can be extended to the Gaussian integers, as was done in [5, 7]. It was examined the *analogy* of the conjecture for three *completely additive* [4], as well as for two [3] and three [6] *additive* functions.

2. Group theoretical approach

We extend the domain of an arbitrary real valued completely additive function $f(n)$ for the set of positive rational numbers $\mathbb{Q}_*$ by $f(a/b) := f(a) - f(b)$. Let us do it for $f_0, f_1, f_2, \dots, f_k$ $(k \in \mathbb{N})$ and let us define the set

$$\Omega_k := \{ (a_0, a_1, a_2, \dots, a_k) \in \mathbb{Q}_*^{k+1} \quad \sum_{i=0}^{k} f_i(a_i) \equiv 0 \pmod 1 \}.$$

Performing the multiplications component-wise it is easy to see that $(\mathbb{Q}_*^{k+1}; \cdot)$ is an abelian group. Hence, it follows from the additivity that if $(a_0, a_1, \dots, a_k) \in \Omega_k$ and $(b_0, b_1, \dots, b_k) \in \Omega_k$ then $(a_0 b_0^{-1}, a_1 b_1^{-1}, \dots, a_k b_k^{-1}) \in \Omega_k$. Thus, the following assertion is obvious.

Assertion. $(\Omega_k; \cdot)$ *is an abelian group with respect to the above defined multiplication.*

In fact, we think that the next conjecture is true:

Conjecture III. *If* $(A, \cdot)$ *is a subgroup of the group* $(\mathbb{Q}_*^h; \cdot)$, *where* $h \in \mathbb{N}$ *and* $(n, n+1, \dots, n+h-1) \in A$ *for all* $n \in \mathbb{N}$, *then* $A = \mathbb{Q}_*^h$.

It is clear that the first conjecture follows from the second one and the second from the third one. It seems to be hard to prove each of them in general. The main result of this note is to prove that the first conjecture is true for $k = 4$.

3. The main theorem

Theorem. *Let $P(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4$ be a polynomial with real coefficients, $P(z) \notin \mathbb{Q}[z]$. If $f \in \mathcal{A}^*$ satisfies the congruence relation*

$$L_n := P(E)f(n) \equiv 0 \pmod 1 \tag{1}$$

for every $n \in \mathbb{N}$, then $f(n)$ is identically zero.

In order to prove the theorem we shall use an induction-like method.

3.1. Induction step

Lemma 1. *If (1) holds for every $n \in \mathbb{N}$ and $f(m) = 0$ is satisfied for every positive integer $m \leq 23$ then $f(n) = 0$ for every $n \in \mathbb{N}$.*

Proof. We prove the lemma indirectly. Let

$$\mathcal{K}_f := \{n \in \mathbb{N} \ : \ f(n) = 0\}.$$

Assume that $\mathcal{K}_f \neq \mathbb{N}$, i.e. there exists a smallest positive integer S such that

$$f(S) \neq 0. \tag{2}$$

By the assumption of the lemma it is pretty obvious that

$$S \in \mathcal{P} \tag{3}$$

and $S \geq 29$. First we shall prove that for each V satisfying the relations $V \in \{S+2, S+6, S+8, S+12\}$ and $V \equiv 1 \pmod 6$ we have

$$f(V) = 0 \quad \text{or} \quad f(V) \notin \mathbb{Q}. \tag{4}$$

Suppose that (4) does not hold. Then for some U for which $U \in \{S+2, S+ +6, S+8, S+12\}$ with $U \equiv 1 \pmod 6$, we obtain

$$f(U) \neq 0 \quad \text{and} \quad f(U) \in \mathbb{Q}. \tag{5}$$

By the fact $U \equiv 1 \pmod 6$ it is easily seen that

$$\{U \pm 1 - 6k, U + 2 - 6k, U + 3 - 6k, 2U \pm 2 \ : \ k = 0, 1\} \subseteq \mathcal{K}_f. \tag{6}$$

By using (1) and (6) we get that $L_{U-1} = A_1 f(U) \equiv 0 \pmod 1$, which with (5) implies that

$$A_1 \in \mathbb{Q}. \tag{7}$$

Let us observe that

$$f(U-2) \notin \mathbb{Q}, \tag{8}$$

since in the opposite case, by using (1) and (6) we get

$$L_{U-4} = A_2 f(U-2) + A_4 f(U) \equiv 0 \pmod 1,$$

$$L_{U-3} = A_1 f(U-2) + A_3 f(U) \equiv 0 \pmod 1,$$

$$L_{U-2} = \quad f(U-2) + A_2 f(U) \equiv 0 \pmod 1,$$

which with (5) and (7) would imply that $P(z) \in \mathbb{Q}[z]$. In virtue of (5), (8) and of the equation L_{U-2} it is obvious that

$$A_2 \notin \mathbb{Q}. \tag{9}$$

Since $3 \mid (2U+1)$ and $(2U+1)/3 < S$, we have $f(2U+1) = 0$, hence by using (5), (6), (7), (9) and the equation

$$L_{2U-2} = A_1 f(2U-1) + A_2 f(U) \equiv 0 \pmod 1$$

we obtain

$$A_1 \neq 0, \quad f(2U-1) \notin \mathbb{Q}. \tag{10}$$

Consider the equation

$$L_{2U-1} = f(2U-1) + A_1 f(U) + A_4 f(2U+3) \equiv 0 \pmod 1,$$

which with (5), (7) and (10) implies that

$$A_4 \neq 0, \quad A_4 f(2U+3) \notin \mathbb{Q}. \tag{11}$$

By using relations (5), (8), (10) and (11) it is easy to see that U, $U-2$, $2U-1$, $2U+3 \in \mathcal{P}$, consequently $U \equiv 4 \pmod 5$. Since $U \equiv 1 \pmod 6$, $U \equiv 4 \pmod 5$, $U \le S+12$ and $S \ge 29$ the following statements hold:

$$3 \mid 2U-5, \quad (2U-5)/3 < S, \quad 5 \mid 2U-3, \quad (2U-3)/5 < S.$$

It means that $f(2U-5) = f(2U-3) = 0$. Hence, using (5), (6) and the equations

$$L_{2U-6} = A_2 f(U-2) \equiv 0 \pmod 1,$$
$$L_{U-4} = A_2 f(U-2) + A_4 f(U) \equiv 0 \pmod 1$$

we have

(12) $$A_4 \in \mathbb{Q}.$$

On the other hand, using (5), (6), (7) and (10) the equation $L_{U-7} = A_1 f(U-\\-6) \equiv 0 \pmod 1$ gives that $f(U-6) \in \mathbb{Q}$, which with (11), (12) and by $L_{U-6} = f(U-6) + A_4 f(U-2) \equiv 0 \pmod 1$ implies $f(U-2) \in \mathbb{Q}$. This contradicts to (8). Thus, we have proved (4).

Now we are able to prove Lemma 1. We distinguish two cases:

(I) $S \equiv 1 \pmod 6$,
(II) $S \equiv 5 \pmod 6$.

Case (I). In this case we have

(13) $$\{S+1+6k, S+2+6k, S+3+6l, S+5+6l \,:\, k=0,1,2,3 \;\; l=0,1,2\} \subseteq \mathcal{K}_f,$$

and by virtue of the equations $L_{S-1}, L_{S-2}, L_{S-3}, L_{S-4}$ we get

$$A_1 f(S) \equiv A_2 f(S) \equiv A_3 f(S) \equiv A_4 f(S) \equiv 0 \pmod 1.$$

If $f(S) \in \mathbb{Q}$ then $A_j \in \mathbb{Q}$ $(j = 1,2,3,4)$ but this is a contradiction, since $P(z) \notin \mathbb{Q}[z]$. Then we have

(14) $$f(S) \notin \mathbb{Q}.$$

Observe that if $A_j \in \mathbb{Q}$ for some $1 \le j \le 4$ then $A_j = 0$. It means that

(15) $$A_j \notin \mathbb{Q} \;\text{ or }\; A_j = 0 \quad (j = 1,2,3,4).$$

Let $i = 0$. It follows from (14), (15) and from the equation $L_{S+6i} = f(S+\\+6i) + A_4 f(S+4+6i) \equiv 0 \pmod 1$ that

(16) $$A_4 \notin \mathbb{Q}, \quad f(S+4+6i) \ne 0.$$

The reader can readily verify that $f(S+6+6i) \neq 0$ since in the opposite case using the equation $L_{S+4+6i} = f(S+4+6i) \equiv 0 \pmod 1$ we get $f(S+4+6i) \in \in \mathbb{Q}$. This with (13), (15), (16) and by $L_{S+1+6i}, L_{S+2+6i}, L_{S+3+6i}$ implies $A_1 = = A_2 = A_3 = 0$. Hence the equation $L_{2S+4+12i}$ would infer $A_4 f(S+4+6i) \equiv 0 \pmod 1$, i.e. $A_4 \in \mathbb{Q}$ which is a contradiction. So we have $f(S+6+6i) \neq 0$. By similar arguments for $i = 1, 2$ and by using (4) we obtain $f(S+4) \neq 0$ and $f(S+6i) \neq 0$ $(i = 0, 1, 2, 3)$. It means that $S, S+4, S+6, S+12, S+18$ are primes which is impossible since $S \geq 29$ and one of these numbers is a multiple of 5. So we proved that Case (I) cannot occur.

Case (II). In this case it is no hard to see that

$$\{S+1+6k, S+3+6k, S+4+6k, S+5+6l \quad k = 0,1,2 \ \ l = 0,1\} \subseteq \mathcal{K}_f.$$

Let $i = 0$. Because of $P(z) \notin \mathbb{Q}[z]$ it is easily seen that $f(S+2+6i) \neq 0$ and by (4) we have

$$f(S+2+6i) \notin \mathbb{Q}. \tag{17}$$

Hence by using the equation

$$L_{S+2+6i} = f(S+2+6i) + A_4 f(S+6+6i) \equiv 0 \pmod 1$$

it follows $f(S+6+6i) \neq 0$. Suppose that $f(S+8+6i) = 0$. Then from L_{S+6+6i} we get $f(S+6+6i) \in \mathbb{Q}$, therefore by L_{S+3+6i}, L_{S+4+6i}, L_{S+5+6i} we infer $A_1, A_2, A_3 \in \mathbb{Q}$. Moreover, by the equations L_{S-1+6i}, L_{S+1+6i} and by (17) immediately follows that $A_1 = A_3 = 0$. Since $2S+8+12i \in \mathcal{K}_f$, $2S+10+12i \in \mathcal{K}_f$, the equation

$$L_{2S+8+12i} = A_4 f(2S+12+12i) \equiv A_4 f(S+6+6i) \equiv 0 \pmod 1$$

gives $A_4 \in \mathbb{Q}$ which is contradiction. So $f(S+8+6i) \neq 0$. Similarly, for $i = 1$ using (4) we obtain $S, S+2, S+6, S+8, S+14$ are primes, but this is impossible since $S \geq 29$ and one of these numbers is a multiple of 5. The proof of Lemma 1 is finished.

The proof of the theorem will be completed by proving the following

Lemma 2. *If* (1) *holds for every* $n \in \mathbb{N}$ *then* $f(m) = 0$ *for every positive integer* $m \leq 23$.

To verify this lemma we discuss the next conjecture which is a step-by-step approach of the second one.

3.2. Algorithm for the bounded case

Conjecture IV. *Let $f_0, f_1, \ldots, f_k \in \mathcal{A}^*$. Given any $e \in \mathbb{N}$ there exists a number $d \in \mathbb{N}$ such that if $L_n := \sum_{j=0}^{k} f_j(n+j) \equiv 0 \pmod 1$ holds for every $n \le d$ $(n \in \mathbb{N})$ then $f_j(n) \equiv 0 \pmod 1$ for every $n \le e$ $(j = 0, 1, \ldots, k)$.*

We introduce some notations.

Let us denote the n-tuple $\mathcal{L}_n := (n, n+1, \ldots, n+k) \in \Omega_k$. Let furthermore

$$\Gamma_k := \{(a_0, a_1, \ldots, a_k) \ : \ a_j \text{ are squarefree integers}, a_j > 1 \ (0 \le j \le k)\}.$$

For an arbitrary $\gamma = (a_0, a_1, \ldots, a_k) \in \Gamma_k$ we define the vector $\underline{v} = \underline{v}(\gamma)$ as follows. Let the prime decomposition of a_j be $a_j = p_1^{(j)} p_2^{(j)} \ldots p_{l_j}^{(j)}$, where the factors are written in ascending order. Then

$$\underline{v}(\gamma) := [f_0(p_1^{(0)}), \ldots, f_0(p_{l_0}^{(0)}), f_1(p_1^{(1)}), \ldots, f_1(p_{l_1}^{(1)}), \ldots, f_k(p_1^{(k)}), \ldots, f_k(p_{l_k}^{(k)})] =$$

$$= [v_1, v_2, \ldots, v_\mu], \quad \mu = l_0 + l_1 + \ldots + l_k.$$

Example. Let $k = 4, \gamma = (6, 30, 10, 15, 3)$.
Then $\underline{v} = [f_0(2), f_0(3), f_1(2), f_1(3), f_1(5), f_2(2), f_2(5), f_3(3), f_3(5), f_4(3)]$.
Let $\Delta_{\underline{v}}^{\mu} := \{[b_1, b_2, \ldots, b_\mu] \ \ \sum_{j=1}^{\mu} b_j v_j \equiv 0 \pmod 1, \ b_i \in \mathbf{Z}\}$. Let $\underline{b} = [b_1, \ldots, b_\mu] \in \Delta_{\underline{v}}^{\mu}$. Then

$$b_1 f_0(p_1^{(0)}) + \ldots + b_{l_0} f_0(p_{l_0}^{(0)}) + \ldots + b_\mu f_k(p_{l_k}^{(k)}) \equiv 0 \pmod 1,$$

which can be rewritten as

$$f_0(\beta_0) + f_1(\beta_1) + \ldots + f_k(\beta_k) \equiv 0 \pmod 1,$$

where $\beta_t = \prod_{k=1}^{l_t} (p_k^{(t)})^{b_{s+k}}$, $s = l_0 + \ldots + l_{t-1}$. Let $\underline{\beta}(= \underline{\beta}(\underline{b})) = (\beta_0, \beta_1, \ldots, \beta_k)$. Clearly, $\underline{\beta} \in \mathbb{Q}_*^k$. Let $\Delta_{\underline{v}}^{k} = \{\underline{\beta}(\underline{b}) \ \ \underline{b} \in \Delta_{\underline{v}}^{\mu}\}$.

Remarks.

1. It follows from our construction that there is a one-to-one correspondence between $\Delta_{\underline{v}}^{\mu}$ and $\Delta_{\underline{v}}^{k}$.

2. $\Delta_{\underline{v}}^{k} \subset \Omega_k$.

Example. Let $k, \gamma, \underline{v}$ as before. Suppose that $\underline{b} = (1,2,3,1,-2,-3,1,-2,3,1) \in \Delta_{\underline{v}}^{\mu}$. Then $\underline{\beta} = (18, \frac{24}{25}, \frac{5}{8}, \frac{125}{9}, 3) \in \Delta_{\underline{v}}^{k}$.

Algorithm *to verify Conjecture IV.*

Input k, $e \in I\!N$

1. Let γ be an arbitrary element from Γ_k and let $\underline{v}$ be the vector generated by γ as before.

2. By using the equations $L_1, L_2, \dots, L_i$ and the fact that $f_0, f_1, \dots, f_k \in \mathcal{A}^*$ express all the possible $F_1, F_2, \dots, F_j \in \Delta_{\underline{v}}^{k}$. The corresponding vectors $G_1, G_2, \dots, G_j \in \Delta_{\underline{v}}^{\mu}$ can be considered as rows of a matrix $M \in I\!R^{\mu \times \mu}$. Examine so many equations that the rank of the matrix M will be equal to μ.

3. Using Gaussian elimination over the integers it can be solved the linear equation $M\underline{v} \equiv 0 \pmod 1$. If the only solution is $\underline{v} \equiv 0 \pmod 1$ then go to the next step, otherwise go to the step 2, increase i and find a new matrix M or go to the step 1 and choose a new γ.

4. Investigate so many equations $L_1, L_2, \dots, L_i, \dots$ while all $f_j(p)$ can be expressed in the form $f_j(p) \equiv \sum_{l=1}^{\mu} b_l v_l \pmod 1$ $(p \le e, p \in \mathcal{P}, b_l \in \mathbf{Z}, 0 \le j \le k)$. Then the conjecture is true for k, e and d is equal to i, which is the number of the examined equations.

If the conjecture is true, the algorithm terminates.

Remark. For a given k, e the d is not unique, it depends on the selection of γ.

4. Examples

Implementing this algorithm the experiments with a simple Maple[1] program show the following results:

Example. Let $k = 4, \gamma = (210, 210, 210, 210, 2310)$. Then we have

$$F_1 = \frac{\mathcal{L}_9^5 \mathcal{L}_{11}^2 L_{12}^4 \mathcal{L}_{14}^5 \mathcal{L}_{17} \mathcal{L}_{21} \mathcal{L}_{23}^2 \mathcal{L}_{30}^2 \mathcal{L}_{32}^2 \mathcal{L}_{33} \mathcal{L}_{37} \mathcal{L}_{45}^2 \mathcal{L}_{54}^3 \mathcal{L}_{58} \mathcal{L}_{62}^4 \mathcal{L}_{91} \mathcal{L}_{141} \mathcal{L}_{143}}{\mathcal{L}_8 \mathcal{L}_{10}^6 \mathcal{L}_{13} \mathcal{L}_{15} \mathcal{L}_{16} \mathcal{L}_{18} \mathcal{L}_{22}^3 \mathcal{L}_{24}^4 \mathcal{L}_{25}^4 \mathcal{L}_{27} \mathcal{L}_{29} \mathcal{L}_{31}^4 \mathcal{L}_{44} \mathcal{L}_{47} \mathcal{L}_{48} \mathcal{L}_{70} \mathcal{L}_{74} \mathcal{L}_{92}^2 \mathcal{L}_{117} \mathcal{L}_{119}}$$

$$= \left(\frac{3^{19}7^5}{2^8 5^{12}}, \frac{3^{13}}{2^{12}}, \frac{2^{33}5^2 7^5}{3^{27}}, \frac{2^{10}3^3}{7^2}, \frac{3^{20}5}{2^5 7^7} \right),$$

[1] Maple is a registered trademark of Waterloo Maple Software.

$$F_2 = \frac{\mathcal{L}_8^3\mathcal{L}_{10}^3\mathcal{L}_{11}^3\mathcal{L}_{15}\mathcal{L}_{16}^2\mathcal{L}_{18}^2\mathcal{L}_{23}\mathcal{L}_{25}^2\mathcal{L}_{27}\mathcal{L}_{29}\mathcal{L}_{30}\mathcal{L}_{31}^4\mathcal{L}_{38}\mathcal{L}_{44}\mathcal{L}_{45}\mathcal{L}_{47}\mathcal{L}_{48}^3\mathcal{L}_{57}\mathcal{L}_{59}}{\mathcal{L}_9^3\mathcal{L}_{12}^2\mathcal{L}_{14}^8\mathcal{L}_{19}^2\mathcal{L}_{21}\mathcal{L}_{24}^2\mathcal{L}_{28}\mathcal{L}_{32}\mathcal{L}_{33}\mathcal{L}_{54}\mathcal{L}_{62}^4\mathcal{L}_{91}\mathcal{L}_{92}\mathcal{L}_{116}\mathcal{L}_{118}\mathcal{L}_{120}\mathcal{L}_{132}^2} \cdot$$

$$\cdot \frac{\mathcal{L}_{65}^2\mathcal{L}_{70}}{\mathcal{L}_{141}\mathcal{L}_{143}} = \left(\frac{5^{12}}{2^3 3^4 7^{10}}, \frac{2^{32}}{3^4 5^{15}}, \frac{3^{14}5^{10}}{2^{43}7^4}, \frac{2^{12}7^5}{3^{13}5^6}, \frac{5^4 7^7}{2^{21}3^8 11}\right),$$

$$F_3 = \frac{\mathcal{L}_9^2\mathcal{L}_{13}^2\mathcal{L}_{14}^4\mathcal{L}_{19}\mathcal{L}_{21}\mathcal{L}_{22}\mathcal{L}_{23}^2\mathcal{L}_{24}^2\mathcal{L}_{32}^3\mathcal{L}_{34}\mathcal{L}_{37}\mathcal{L}_{55}\mathcal{L}_{56}\mathcal{L}_{58}\mathcal{L}_{62}\mathcal{L}_{91}\mathcal{L}_{92}\mathcal{L}_{132}}{\mathcal{L}_{10}^3\mathcal{L}_{11}^3\mathcal{L}_{12}\mathcal{L}_{15}^2\mathcal{L}_{18}^3\mathcal{L}_{26}^2\mathcal{L}_{27}\mathcal{L}_{29}\mathcal{L}_{30}\mathcal{L}_{31}\mathcal{L}_{44}\mathcal{L}_{45}\mathcal{L}_{46}\mathcal{L}_{47}\mathcal{L}_{48}^2\mathcal{L}_{54}\mathcal{L}_{65}} \cdot$$

$$\cdot \frac{\mathcal{L}_{140}\mathcal{L}_{141}\mathcal{L}_{142}\mathcal{L}_{143}\mathcal{L}_{144}}{\mathcal{L}_{69}\mathcal{L}_{70}\mathcal{L}_{71}\mathcal{L}_{74}\mathcal{L}_{114}\mathcal{L}_{115}\mathcal{L}_{117}\mathcal{L}_{119}} = \left(\frac{2^{17}7^6}{3^{13}5^8}, \frac{5^5}{2^{14}3^2}, \frac{2^4 3^3 5}{7^6}, \frac{5^5}{3^4 7^4}, \frac{2^9 3^{14}}{5^5 7^3 11^5}\right),$$

$$F_4 = \frac{\mathcal{L}_9^2\mathcal{L}_{11}^2\mathcal{L}_{12}^6\mathcal{L}_{14}^3\mathcal{L}_{18}^2\mathcal{L}_{23}^2\mathcal{L}_{30}^3\mathcal{L}_{33}\mathcal{L}_{45}^3\mathcal{L}_{54}^5\mathcal{L}_{62}^4\mathcal{L}_{65}\mathcal{L}_{115}\mathcal{L}_{141}\mathcal{L}_{143}\mathcal{L}_{152}}{\mathcal{L}_8^2\mathcal{L}_{10}^6\mathcal{L}_{13}^2\mathcal{L}_{16}^2\mathcal{L}_{19}\mathcal{L}_{22}^3\mathcal{L}_{24}^4\mathcal{L}_{25}^6\mathcal{L}_{28}\mathcal{L}_{31}^4\mathcal{L}_{47}\mathcal{L}_{56}\mathcal{L}_{70}\mathcal{L}_{92}^3\mathcal{L}_{132}} =$$

$$= \left(\frac{3^{35}}{2^{17}5^{11}}, \frac{3^{11}\,5\cdot 7}{2^{10}}, \frac{2^{39}5^3 7^{10}}{3^{39}}, \frac{2^4 3^{10}}{5\ 7^2}, \frac{3^{14}11^6}{2^8 5^4 7^5}\right),$$

$$F_5 = \frac{\mathcal{L}_9\mathcal{L}_{14}\mathcal{L}_{18}\mathcal{L}_{21}\mathcal{L}_{30}\mathcal{L}_{33}\mathcal{L}_{45}\mathcal{L}_{48}\mathcal{L}_{54}\mathcal{L}_{62}^2\mathcal{L}_{91}\mathcal{L}_{115}\mathcal{L}_{121}\mathcal{L}_{141}\mathcal{L}_{143}\mathcal{L}_{168}}{\mathcal{L}_8\mathcal{L}_{10}^2\mathcal{L}_{11}\mathcal{L}_{13}\mathcal{L}_{15}\mathcal{L}_{16}\mathcal{L}_{22}^2\mathcal{L}_{24}\mathcal{L}_{25}^2\mathcal{L}_{28}\mathcal{L}_{31}^2\mathcal{L}_{39}\mathcal{L}_{44}\mathcal{L}_{47}\mathcal{L}_{56}\mathcal{L}_{60}\mathcal{L}_{70}\mathcal{L}_{92}} =$$

$$= \left(\frac{3^{11}7}{2^{13}5^6}, \frac{7^3}{2^{12}\ 3\ 5}, \frac{2^7 5^3}{3^{14}}, \frac{2^6 3^5}{5^5 7^4}, \frac{5^2 11^3}{2^{23}3}\right),$$

$$F_6 = \frac{\mathcal{L}_{11}^4\mathcal{L}_{12}^2\mathcal{L}_{18}^2\mathcal{L}_{23}^2\mathcal{L}_{26}\mathcal{L}_{27}\mathcal{L}_{30}^2\mathcal{L}_{31}^2\mathcal{L}_{42}\mathcal{L}_{45}^2\mathcal{L}_{48}^8\mathcal{L}_{54}^2\mathcal{L}_{74}\mathcal{L}_{114}\mathcal{L}_{117}\mathcal{L}_{119}\mathcal{L}_{174}}{\mathcal{L}_8\mathcal{L}_9^6\mathcal{L}_{10}\mathcal{L}_{13}^2\mathcal{L}_{14}^{10}\mathcal{L}_{19}\mathcal{L}_{22}^3\mathcal{L}_{24}^3\mathcal{L}_{25}^2\mathcal{L}_{32}\mathcal{L}_{37}\mathcal{L}_{55}\mathcal{L}_{56}\mathcal{L}_{58}\mathcal{L}_{62}^2\mathcal{L}_{85}\mathcal{L}_{92}^2} =$$

$$= \left(\frac{2^6 3^{19}}{5^3 7^9}, \frac{2^{15}7^{13}}{3^9 5^{15}}, \frac{5^{18}7^5}{2^{31}3^{15}}, \frac{3^{10}7^4}{2^{11}5^7}, \frac{5^6 7^2 11^2}{2^4 3^{15}}\right),$$

$$F_7 = \frac{\mathcal{L}_{11}^8\mathcal{L}_{12}\mathcal{L}_{15}\mathcal{L}_{16}\mathcal{L}_{18}^2\mathcal{L}_{23}^3\mathcal{L}_{26}\mathcal{L}_{29}\mathcal{L}_{30}^3\mathcal{L}_{31}^4\mathcal{L}_{45}^3\mathcal{L}_{47}\mathcal{L}_{48}^8\mathcal{L}_{54}^2\mathcal{L}_{65}\mathcal{L}_{70}\mathcal{L}_{74}}{\mathcal{L}_8^3\mathcal{L}_9^5\mathcal{L}_{13}^3\mathcal{L}_{14}^{13}\mathcal{L}_{19}\mathcal{L}_{21}\mathcal{L}_{22}^4\mathcal{L}_{24}^6\mathcal{L}_{25}\mathcal{L}_{32}^2\mathcal{L}_{33}\mathcal{L}_{34}\mathcal{L}_{37}\mathcal{L}_{55}\mathcal{L}_{58}\mathcal{L}_{60}\mathcal{L}_{62}^4\mathcal{L}_{92}^3} \cdot$$

$$\cdot \frac{\mathcal{L}_{114}\mathcal{L}_{117}\mathcal{L}_{119}\mathcal{L}_{182}}{\mathcal{L}_{132}\mathcal{L}_{141}\mathcal{L}_{143}} = \left(\frac{3^{11}5^5}{2^{20}7^{11}}, \frac{2^{40}7^6}{3^{14}5^{25}}, \frac{5^{15}7^5}{2^{61}3^8}, \frac{2\ 7^7}{3^9 5^{14}}, \frac{5^{11}7^2 11}{2^{35}3^{17}}\right),$$

$$F_8 = \frac{\mathcal{L}_9^2\mathcal{L}_{12}^2\mathcal{L}_{14}^3\mathcal{L}_{18}\mathcal{L}_{33}\mathcal{L}_{54}\mathcal{L}_{62}^3\mathcal{L}_{92}\mathcal{L}_{115}\mathcal{L}_{141}\mathcal{L}_{143}\mathcal{L}_{183}}{\mathcal{L}_{10}\mathcal{L}_{11}^2\mathcal{L}_{13}\mathcal{L}_{16}\mathcal{L}_{23}^2\mathcal{L}_{25}^2\mathcal{L}_{28}\mathcal{L}_{31}^3\mathcal{L}_{35}\mathcal{L}_{45}\mathcal{L}_{47}\mathcal{L}_{56}\mathcal{L}_{61}\mathcal{L}_{70}} =$$

$$= \left(\frac{2^3 3^{12}}{5^7 7}, \frac{3^4 5^6 7^2}{2^{23}}, \frac{2^{30}7}{3^{16}5^2}, \frac{3^{10}5^4}{2^{14}7^3}, \frac{2^{10}11^5}{5^7 7^3}\right),$$

$$F_9 = \frac{\mathcal{L}_8\mathcal{L}_{10}^5\mathcal{L}_{13}^3\mathcal{L}_{14}^5\mathcal{L}_{15}\mathcal{L}_{19}\mathcal{L}_{22}^5\mathcal{L}_{24}^9\mathcal{L}_{25}^3\mathcal{L}_{28}\mathcal{L}_{44}\mathcal{L}_{55}\mathcal{L}_{56}\mathcal{L}_{92}^3\mathcal{L}_{132}\mathcal{L}_{184}}{\mathcal{L}_{11}^8\mathcal{L}_{12}^2\mathcal{L}_{18}^2\mathcal{L}_{23}^3\mathcal{L}_{26}\mathcal{L}_{30}^3\mathcal{L}_{32}\mathcal{L}_{36}\mathcal{L}_{45}^3\mathcal{L}_{48}^5\mathcal{L}_{54}^3\mathcal{L}_{65}\mathcal{L}_{90}\mathcal{L}_{91}\mathcal{L}_{114}\mathcal{L}_{115}} =$$

$$= \left(\frac{2^{21}5^{4}7^{6}}{3^{23}}, \frac{5^{22}}{2^{18}3^{3}7^{6}}, \frac{2^{20}3^{23}}{5^{13}7^{5}}, \frac{3^{10}5^{11}}{2^{6}7^{8}}, \frac{2^{3}2^{7}7}{3^{5}5^{10}11^{2}}\right),$$

$$F_{10} = \frac{\mathcal{L}_8\mathcal{L}_{12}^2\mathcal{L}_{13}\mathcal{L}_{14}\mathcal{L}_{19}\mathcal{L}_{21}\mathcal{L}_{22}\mathcal{L}_{24}^3\mathcal{L}_{33}\mathcal{L}_{48}\mathcal{L}_{54}\mathcal{L}_{55}\mathcal{L}_{62}\mathcal{L}_{132}\mathcal{L}_{141}\mathcal{L}_{143}\mathcal{L}_{185}}{\mathcal{L}_9\mathcal{L}_{11}^4\mathcal{L}_{16}\mathcal{L}_{18}\mathcal{L}_{25}^2\mathcal{L}_{26}\mathcal{L}_{30}\mathcal{L}_{31}\mathcal{L}_{32}\mathcal{L}_{44}\mathcal{L}_{47}\mathcal{L}_{65}\mathcal{L}_{70}\mathcal{L}_{74}\mathcal{L}_{114}} =$$

$$= \left(\frac{2^{9}3^{7}7}{5^{5}}, \frac{5^{4}7^{6}}{2^{7}3^{5}}, \frac{2^{5}5^{5}7^{2}}{3^{7}}, \frac{2^{4}3^{16}5^{3}}{7^{9}}, \frac{2^{11}7^{5}}{3^{3}5^{4}}\right),$$

$$F_{11} = \frac{\mathcal{L}_9^4\mathcal{L}_{12}\mathcal{L}_{14}^9\mathcal{L}_{22}\mathcal{L}_{24}\mathcal{L}_{32}\mathcal{L}_{54}\mathcal{L}_{62}^3\mathcal{L}_{186}}{\mathcal{L}_{10}^3\mathcal{L}_{11}\mathcal{L}_{15}\mathcal{L}_{16}\mathcal{L}_{25}\mathcal{L}_{31}^4\mathcal{L}_{45}\mathcal{L}_{48}^5} =$$

$$= \left(\frac{3^{6}7^{9}}{2^{2}5^{7}}, \frac{3^{15}5^{16}}{2^{24}7^{7}}, \frac{2^{53}7^{2}}{3^{11}5^{10}}, \frac{3^{4}5^{7}}{2^{4}}, \frac{2^{8}3^{22}11^{3}}{5^{5}7^{8}}\right),$$

$$F_{12} = \frac{\mathcal{L}_8^2\mathcal{L}_9^5\mathcal{L}_{13}^2\mathcal{L}_{14}^{14}\mathcal{L}_{19}^2\mathcal{L}_{21}^2\mathcal{L}_{22}\mathcal{L}_{24}\mathcal{L}_{25}\mathcal{L}_{32}^3\mathcal{L}_{33}\mathcal{L}_{37}\mathcal{L}_{55}\mathcal{L}_{56}\mathcal{L}_{58}\mathcal{L}_{61}\mathcal{L}_{62}^4\mathcal{L}_{91}^2\mathcal{L}_{92}^2}{\mathcal{L}_{10}\mathcal{L}_{11}^2\mathcal{L}_{15}^2\mathcal{L}_{18}^4\mathcal{L}_{23}\mathcal{L}_{26}^2\mathcal{L}_{27}\mathcal{L}_{29}\mathcal{L}_{30}^3\mathcal{L}_{31}^4\mathcal{L}_{38}\mathcal{L}_{44}^2\mathcal{L}_{45}^2\mathcal{L}_{47}\mathcal{L}_{48}^7\mathcal{L}_{54}^2\mathcal{L}_{65}^2\mathcal{L}_{66}\mathcal{L}_{70}\mathcal{L}_{74}} \cdot$$

$$\cdot \frac{\mathcal{L}_{132}^2\mathcal{L}_{141}\mathcal{L}_{143}\mathcal{L}_{200}\mathcal{L}_{202}\mathcal{L}_{204}}{\mathcal{L}_{99}\mathcal{L}_{101}\mathcal{L}_{114}\mathcal{L}_{115}\mathcal{L}_{117}\mathcal{L}_{119}\mathcal{L}_{122}} = \left(\frac{2^{11}7^{17}}{3^{21}5^{7}}, \frac{3^{13}5^{13}}{2^{26}7^{5}}, \frac{2^{42}3^{7}}{5^{14}7^{3}}, \frac{2^{10}5^{8}}{3^{6}7^{2}}, \frac{2^{17}3^{26}}{5 \cdot 7^{11}11^{3}}\right),$$

$$F_{13} = \frac{\mathcal{L}_8^3\mathcal{L}_{10}^6\mathcal{L}_{11}^3\mathcal{L}_{13}\mathcal{L}_{16}\mathcal{L}_{18}^2\mathcal{L}_{22}^2\mathcal{L}_{24}\mathcal{L}_{25}^5\mathcal{L}_{27}\mathcal{L}_{29}\mathcal{L}_{31}^7\mathcal{L}_{44}\mathcal{L}_{47}^2\mathcal{L}_{48}^4\mathcal{L}_{49}\mathcal{L}_{65}\mathcal{L}_{70}^2\mathcal{L}_{74}\mathcal{L}_{84}}{\mathcal{L}_9^7\mathcal{L}_{12}^5\mathcal{L}_{14}^{12}\mathcal{L}_{19}\mathcal{L}_{21}^2\mathcal{L}_{23}^2\mathcal{L}_{30}^2\mathcal{L}_{32}^2\mathcal{L}_{33}^2\mathcal{L}_{37}\mathcal{L}_{41}\mathcal{L}_{45}^2\mathcal{L}_{54}^3\mathcal{L}_{55}\mathcal{L}_{58}\mathcal{L}_{62}^7\mathcal{L}_{91}} \cdot$$

$$\cdot \frac{\mathcal{L}_{92}^2\mathcal{L}_{114}\mathcal{L}_{117}\mathcal{L}_{119}\mathcal{L}_{205}}{\mathcal{L}_{102}\mathcal{L}_{132}\mathcal{L}_{141}^2\mathcal{L}_{143}^2} = \left(\frac{2^{6}5^{15}}{3^{2}6^{7}9}, \frac{2^{27}}{3^{16}5^{12}}, \frac{3^{35}5^{5}}{2^{7}7^{6}}, \frac{7^{8}}{2^{8}3^{19}5^{5}}, \frac{5^{3}7^{6}}{2^{4}3^{30}}\right),$$

$$F_{14} = \frac{\mathcal{L}_{12}^7\mathcal{L}_{18}\mathcal{L}_{20}\mathcal{L}_{23}^3\mathcal{L}_{26}^2\mathcal{L}_{30}^4\mathcal{L}_{31}^2\mathcal{L}_{33}^2\mathcal{L}_{44}\mathcal{L}_{45}^4\mathcal{L}_{48}^9\mathcal{L}_{53}\mathcal{L}_{54}^5\mathcal{L}_{90}\mathcal{L}_{115}}{\mathcal{L}_8^4\mathcal{L}_9^6\mathcal{L}_{10}\mathcal{L}_{11}^2\mathcal{L}_{13}^3\mathcal{L}_{14}^{11}\mathcal{L}_{16}\mathcal{L}_{21}^2\mathcal{L}_{22}^3\mathcal{L}_{24}\mathcal{L}_{25}^7\mathcal{L}_{27}\mathcal{L}_{28}\mathcal{L}_{43}\mathcal{L}_{47}^2\mathcal{L}_{56}\mathcal{L}_{62}^2\mathcal{L}_{70}^2\mathcal{L}_{84}} \cdot$$

$$\cdot \frac{\mathcal{L}_{141}^2\mathcal{L}_{143}^2\mathcal{L}_{213}\mathcal{L}_{215}}{\mathcal{L}_{91}\mathcal{L}_{92}^4\mathcal{L}_{106}\mathcal{L}_{142}} = \left(\frac{2^{12}3^{33}}{5^{5}7^{19}}, \frac{2^{6}7^{14}}{3^{10}5^{14}}, \frac{5^{22}7^{13}}{2^{31}3^{39}}, \frac{3^{29}}{2^{8}5^{11}7^{7}}, \frac{2^{3}, 7^{14}}{3^{20}5^{4}11^{3}}\right),$$

$$F_{15} = \frac{\mathcal{L}_9^3\mathcal{L}_{12}^7\mathcal{L}_{14}\mathcal{L}_{15}^2\mathcal{L}_{18}^2\mathcal{L}_{23}^2\mathcal{L}_{30}^3\mathcal{L}_{33}^3\mathcal{L}_{44}\mathcal{L}_{45}^3\mathcal{L}_{52}\mathcal{L}_{54}^6\mathcal{L}_{62}^2\mathcal{L}_{65}\mathcal{L}_{115}\mathcal{L}_{122}}{\mathcal{L}_8^3\mathcal{L}_{10}^4\mathcal{L}_{11}^2\mathcal{L}_{13}^2\mathcal{L}_{16}^3\mathcal{L}_{21}\mathcal{L}_{22}^2\mathcal{L}_{25}^7\mathcal{L}_{28}\mathcal{L}_{31}^2\mathcal{L}_{32}^2\mathcal{L}_{40}\mathcal{L}_{42}\mathcal{L}_{47}\mathcal{L}_{53}\mathcal{L}_{56}\mathcal{L}_{61}\mathcal{L}_{70}^2} \cdot$$

$$\cdot \frac{\mathcal{L}_{141}\mathcal{L}_{143}\mathcal{L}_{212}\mathcal{L}_{214}\mathcal{L}_{216}}{\mathcal{L}_{91}\mathcal{L}_{92}^3\mathcal{L}_{105}\mathcal{L}_{107}\mathcal{L}_{132}} = \left(\frac{3^{49}}{2^{17}5^{12}7^{7}}, \frac{5^{12}}{2^{4}3^{7}}, \frac{2^{32}5^{3}7^{12}}{3^{38}}, \frac{3^{28}}{2^{7}7^{9}}, \frac{2^{4}3^{2}7^{5}11^{4}}{5^{10}}\right),$$

$$F_{16} = \frac{\mathcal{L}_9^{12}\mathcal{L}_{12}^7\mathcal{L}_{13}\mathcal{L}_{14}^{24}\mathcal{L}_{19}^2\mathcal{L}_{21}^2\mathcal{L}_{23}^2\mathcal{L}_{24}^4\mathcal{L}_{32}^6\mathcal{L}_{33}^4\mathcal{L}_{34}^2\mathcal{L}_{37}^2\mathcal{L}_{54}^4\mathcal{L}_{55}^2\mathcal{L}_{58}^2\mathcal{L}_{62}^{10}\mathcal{L}_{91}^2\mathcal{L}_{132}^2}{\mathcal{L}_8^2\mathcal{L}_{10}^{11}\mathcal{L}_{11}^{10}\mathcal{L}_{15}^4\mathcal{L}_{16}^4\mathcal{L}_{18}^4\mathcal{L}_{25}^8\mathcal{L}_{26}^3\mathcal{L}_{27}^2\mathcal{L}_{29}^2\mathcal{L}_{31}^{11}\mathcal{L}_{44}^2\mathcal{L}_{46}\mathcal{L}_{47}^4\mathcal{L}_{48}^{11}\mathcal{L}_{65}^2\mathcal{L}_{69}\mathcal{L}_{70}^4\mathcal{L}_{71}} \cdot$$

$$\cdot \frac{\mathcal{L}_{140}\mathcal{L}_{141}^4\mathcal{L}_{142}\mathcal{L}_{143}^4\mathcal{L}_{217}}{\mathcal{L}_{74}^2\mathcal{L}_{108}\mathcal{L}_{114}^2\mathcal{L}_{117}^2\mathcal{L}_{119}^2} = \left(\frac{2^{6}3^{20}7^{24}}{5^{34}}, \frac{3^{20}5^{39}}{2^{73}7^{8}}, \frac{2^{120}7^{4}}{3^{45}5^{14}}, \frac{2^{5}3^{25}5^{17}}{7^{20}}, \frac{2^{28}3^{52}}{5^{22}7^{11}}\right),$$

$$F_{17} = \frac{\mathcal{L}_8^2\mathcal{L}_{10}^7\mathcal{L}_{13}\mathcal{L}_{15}\mathcal{L}_{19}\mathcal{L}_{22}^4\mathcal{L}_{24}^7\mathcal{L}_{25}^3\mathcal{L}_{29}\mathcal{L}_{31}^5\mathcal{L}_{44}\mathcal{L}_{48}\mathcal{L}_{92}^3\mathcal{L}_{132}\mathcal{L}_{234}}{\mathcal{L}_9^5\mathcal{L}_{11}^6\mathcal{L}_{12}^2\mathcal{L}_{14}^4\mathcal{L}_{18}\mathcal{L}_{21}\mathcal{L}_{23}^2\mathcal{L}_{30}^3\mathcal{L}_{32}\mathcal{L}_{45}^3\mathcal{L}_{46}\mathcal{L}_{54}^3\mathcal{L}_{62}^5\mathcal{L}_{65}\mathcal{L}_{76}\mathcal{L}_{91}\mathcal{L}_{116}} =$$

$$= \left(\frac{2^{23}5^7}{3^{21}7^6}, \frac{2^6 5^6}{3^{16}7^2}, \frac{3^{24}}{2^{38}7^4}, \frac{3^{10}5^4}{2^{14}7^8}, \frac{2^{20}7^{14}}{3^{22}5^8 11^5}\right),$$

$$F_{18} = \frac{\mathcal{L}_{10}\mathcal{L}_{11}^7\mathcal{L}_{16}^2\mathcal{L}_{18}^3\mathcal{L}_{23}\mathcal{L}_{25}^3\mathcal{L}_{26}\mathcal{L}_{27}\mathcal{L}_{30}\mathcal{L}_{31}^3\mathcal{L}_{38}\mathcal{L}_{45}\mathcal{L}_{47}^2\mathcal{L}_{48}^2\mathcal{L}_{65}^2\mathcal{L}_{70}^2\mathcal{L}_{74}\mathcal{L}_{114}\mathcal{L}_{242}}{\mathcal{L}_9\mathcal{L}_{12}^4\mathcal{L}_{13}\mathcal{L}_{14}^7\mathcal{L}_{19}^2\mathcal{L}_{22}^2\mathcal{L}_{24}^4\mathcal{L}_{33}^2\mathcal{L}_{37}\mathcal{L}_{54}\mathcal{L}_{55}\mathcal{L}_{62}^3\mathcal{L}_{92}\mathcal{L}_{120}\mathcal{L}_{132}^2\mathcal{L}_{141}^2\mathcal{L}_{143}^2} =$$

$$= \left(\frac{5^{11}}{2^{14}3^5 7^5}, \frac{2^{26}3^4}{5^{16}7^2}, \frac{3^{14}5^5}{2^{33}7^4}, \frac{2^2 7^{16}}{3^{23}5^9}, \frac{5^{11}}{2^{30}3^{27}}\right),$$

$$F_{19} = \frac{\mathcal{L}_9^2\mathcal{L}_{14}^6\mathcal{L}_{19}\mathcal{L}_{21}\mathcal{L}_{25}\mathcal{L}_{32}^2\mathcal{L}_{62}^2\mathcal{L}_{91}\mathcal{L}_{92}\mathcal{L}_{132}\mathcal{L}_{243}}{\mathcal{L}_{10}\mathcal{L}_{15}^2\mathcal{L}_{18}\mathcal{L}_{23}\mathcal{L}_{30}\mathcal{L}_{31}^2\mathcal{L}_{38}\mathcal{L}_{44}\mathcal{L}_{45}\mathcal{L}_{48}^3\mathcal{L}_{54}\mathcal{L}_{60}\mathcal{L}_{65}} =$$

$$= \left(\frac{2\cdot 7^8}{3^3 5^5}, \frac{3^8 5^7}{2^{13}7^3}, \frac{2^{20}3^2 7^2}{5^9}, \frac{2\cdot 5^6 7}{3^6}, \frac{3^{13}5\cdot 11}{2\cdot 7^6}\right),$$

$$F_{20} = \frac{\mathcal{L}_{10}^4\mathcal{L}_{13}\mathcal{L}_{15}\mathcal{L}_{22}^2\mathcal{L}_{24}^2\mathcal{L}_{25}^2\mathcal{L}_{28}\mathcal{L}_{31}^5\mathcal{L}_{44}\mathcal{L}_{48}^4\mathcal{L}_{56}\mathcal{L}_{60}\mathcal{L}_{92}\mathcal{L}_{244}}{\mathcal{L}_9^5\mathcal{L}_{11}\mathcal{L}_{12}^2\mathcal{L}_{14}^9\mathcal{L}_{18}\mathcal{L}_{21}\mathcal{L}_{27}\mathcal{L}_{32}\mathcal{L}_{45}\mathcal{L}_{54}^2\mathcal{L}_{61}\mathcal{L}_{62}^5\mathcal{L}_{91}\mathcal{L}_{115}\mathcal{L}_{121}} =$$

$$= \left(\frac{2^{15}5^8}{3^{18}7^9}, \frac{2^{15}7^5}{3^{17}5^{10}}, \frac{3^{14}5^9}{2^5 17^5}, \frac{3}{2^{15}5^4}, \frac{2^{16}7^8}{3^{23}5^2 11^6}\right),$$

$$F_{21} = \frac{\mathcal{L}_9^5\mathcal{L}_{11}^{11}\mathcal{L}_{12}^4\mathcal{L}_{14}\mathcal{L}_{18}^3\mathcal{L}_{21}\mathcal{L}_{23}^4\mathcal{L}_{26}\mathcal{L}_{30}^4\mathcal{L}_{32}\mathcal{L}_{38}\mathcal{L}_{45}^5\mathcal{L}_{48}\mathcal{L}_{54}^5\mathcal{L}_{61}\mathcal{L}_{62}^4\mathcal{L}_{65}^2\mathcal{L}_{91}}{\mathcal{L}_8\mathcal{L}_{10}^9\mathcal{L}_{13}^3\mathcal{L}_{15}\mathcal{L}_{19}^2\mathcal{L}_{20}\mathcal{L}_{22}^7\mathcal{L}_{24}^{13}\mathcal{L}_{25}^5\mathcal{L}_{28}\mathcal{L}_{31}^4\mathcal{L}_{44}\mathcal{L}_{52}\mathcal{L}_{55}\mathcal{L}_{56}}\cdot$$

$$\cdot\frac{\mathcal{L}_{114}\mathcal{L}_{115}\mathcal{L}_{158}\mathcal{L}_{245}}{\mathcal{L}_{79}\mathcal{L}_{92}^5\mathcal{L}_{122}\mathcal{L}_{132}^2} = \left(\frac{3^{36}7^3}{2^{46}5^9}, \frac{2^{10}3^{21}}{5^{18}\,7}, \frac{2^{20}5^{10}7^9}{3^{44}}, \frac{2^{32}7^{11}}{3^{17}5^{16}}, \frac{3^{29}5^{11}11^7}{2^{52}7^{16}}\right).$$

From these we can give a non-singular matrix M corresponding to the algorithm. Going to the step 4 the next chart shows the appropriate d values belonging to the different e-s.

e	1-97	98-137	138-179	180-191	192-239	240-269
d	245	299	411	721	788	954

e	270-419	420-431	432-439	440-599	600-659
d	1076	1674	1725	1765	2401

The attentive reader can observe that by this example Lemma 2 (even more) is proved and so the proof of the theorem is completed.

Without giving the exact vectors $F_1, F_2, \dots, F_j \in \Delta_{\underline{v}}^k$ we insert some computer tests verifying Conjecture IV for higher degree. For brevity let us denote $P_n := \prod a_j, a_j \in \mathcal{P}, a_j \leq n$.

Let $k = 5, \gamma = (P_{23}, P_{23}, P_{23}, P_{23}, P_{23}, P_{23})$. Then

e	1-109	110-179	180-191	192-229	230-307	308-313
d	624	815	853	960	1145	1240

e	314-397	398-419	420-431	432-439	440-457
d	1252	1587	1674	1726	1768

e	458-599	600-643
d	1833	2401

Let $k = 6, \gamma = (P_{37}, P_{41}, P_{41}, P_{43}, P_{43}, P_{43}, P_{43})$. Then

e	1-227	228-419	420-431	432-541	542-599
d	1336	1589	1674	2154	2164

Our method is clearly not appropriate for large number of completely additive functions, for large e or μ. Even if we could prove Conjecture IV for a given k to prove the induction step seems to be very hard.

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