Annales Univ. Sci. Budapest., Sect. Comp. 19 (2000) 199-210

ON THE LATTICE OF WAITING TIMES

G. Zbăganu (Bucharest, Roumania)

Let $(\Omega, \mathbf{K}, P)$ be a probabilized space and let $(E, \mathbf{E})$ be a measurable one. Let $(X_n)_{n\leq 1}$ be a sequence of i.i.d. random variables and let $p = P \circ X_n^{-1}$ be their distribution on E. Let us also denote by q the quantity $1 - p$: $q(A) :=$ $:= 1 - p(A)$, $\forall A \in \mathbf{E}$.

Let $A \in \mathbf{E}$ be such that $p(A) := P(X_n \in A) > 0$. For any such set A we shall consider the random variable given by

$$T(A)(\omega) = \min\{n \geq 1 \mid X_n(\omega) \in A\}$$

and we shall denote by $\mathcal{T}$ the set of all such waiting times.

The purpose of this note is to study the lattice generated by $\mathcal{T}$. In the sequel, the relations between sets and random variables should be understood as occuring only almost surely; for instance $T(A) \leq T(B)$ $(mod\ P)$ a.s.o.

1. The distribution of $T(A)$

This is classical, studied in all the handbooks of probability theory (e.g. [2]): it is the geometrical one given by $P(T(A) = n) = p(A)q(A)^{n-1}$. Therefore its generating function is

$$\varphi_{T(A)}(x) = E(x^{T(A)}) = \frac{p(A)x}{1 - q(A)x}, \tag{1.1}$$

the expectation is

$$E(T(A)) = \frac{1}{p(A)}, \tag{1.2}$$

the tail probability is

$$P(T(A) > t) = q(A)^t \tag{1.3}$$

for any positive integer t, and its variance is

$$Var(T(A)) = ET(A)^2 - (ET(A))^2 = \frac{q(A)}{p^2(A)}. \tag{1.4}$$

Moreover, $T(A)$ has all the moments of order n finite, that is $T(A) \in \in \bigcap\limits_{p\geq 1} L^p(\Omega, \mathbf{K}, P)$.

2. $\mathcal{T}$ is an inferior semilattice

Actually, the following identity holds:

$$T(A) \wedge T(B) = T(A \cup B). \tag{2.1}$$

Indeed, $\{T(A) \wedge T(B) > n\} = \{X_1 \notin A,\ X_1 \notin B,\ X_2 \notin A,\ X_2 \notin B, \ldots, X_n \notin \notin A,\ X_n \notin B\} = \{X_1 \notin A \cup B,\ X_2 \notin A \cup B, \ldots, X_n \notin A \cup B\} = \{T(A \cup B) > n\}$. It means that the minimum of a finite family of waiting times $T(A_j)$ $1 \leq j \leq n$ is the waiting time $T(A_1 \cup \ldots \cup A_n)$, that is it is itself a member of $\mathcal{T}$.

Moreover, it is clear that

$$A \subset B \Leftrightarrow T(A) \geq T(B), \tag{2.2}$$

$$T(\Omega) = 1, \tag{2.3}$$

$$T(A) \wedge T(A^c) = 1. \tag{2.4}$$

As a consequence of (2.1) the lattice generated by $\mathcal{T}$ is

$$Lattice(\mathcal{T}) = \{T(A_1) \vee \ldots \vee T(A_n) \mid n \geq 1,\ A_1, \ldots, A_n \in \mathbf{E}\}. \tag{2.5}$$

3. The distribution and the expectation of the maximum

Let as before $A_1, \ldots, A_n \in \mathbf{E}$ and $T = T(A_1) \vee \ldots \vee T(A_n)$.

Lemma 3.1. *The generating function of T is*

$$\varphi_T(x) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k}} \frac{p\left(\bigcup\limits_{j\in J} A_j\right) x}{1 - xq\left(\bigcup\limits_{j\in J} A_j\right)} \tag{3.1}$$

and, as a consequence

$$ET = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k}} \frac{1}{p\left(\bigcup_{j \in J} A_j\right)}. \tag{3.2}$$

In the particular case when the sets $A_1, \ldots, A_n$ are disjoint we get the formulas

$$\varphi_T(x) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k}} \frac{x \sum_{j \in J} p(A_j)}{1 - x\left(1 - \sum_{j \in J} p(A_j)\right)} \tag{3.3}$$

and

$$ET = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k}} \frac{1}{\sum_{j \in J} p(A_j)}. \tag{3.4}$$

Proof. Clearly $P(T > n) = P(\exists\ 1 \le j \le n$ *such that* $T(A_j) > n) =$ $= P\left(\bigcup_{1 \le j \le n} \{T(A_j) > n\}\right)$ and then, by Poincaré's formula we get

$$\begin{aligned}
P(T > n) &= \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k}} P\left(\bigcap_{j \in J} \{T(A_j) > n\}\right) = \\
&= \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k}} P\left(\bigwedge_{j \in J} T(A_j) > n\right) = \\
&= \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k}} P\left(T\left(\bigcup_{j \in J} A_j\right) > n\right),
\end{aligned}$$

therefore by subtracting

$$P(T = n) =$$

$$= P(T > n-1) - P(T > n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k}} P\left(T\left(\bigcup_{j \in J} A_j\right) = n\right).$$

Apply eventually (1.1) and (1.2).

We are going now to answer the question : let $p_j = p(A_j)$, $1 \le j \le n$. Suppose that the sets $(A_j)_{1\le j\le n}$ are disjoint and let $E(p_1,\ldots,p_n) = ET$. **How should be the numbers $p_1,\ldots,p_n$ such that ET be minimum?**

Remark first that the domain of E is the set $S = \{\mathbf{p} = (p_1,\ldots,p_n) \in \in [0,1)^n \mid p_1 + \ldots + p_n \le 1\}$ and that $E : S \to [1,\infty)$ is continuous and symmetrical, i.e. $E(\mathbf{p}) = E(p_{\sigma(1)},\ldots,p_{\sigma(n)})$ for any permutation σ of the set $\{1,\ldots,n\}$. We are going to use the following result:

Lemma 3.2. (see [3]) *Let $f : [0,\infty)^n \to \Re$ be a continuous symmetric function. Suppose that*

$$f(p_1,\ldots,p_n) \ge f\left(\frac{p_1+p_2}{2},\ \frac{p_1+p_2}{2},\ p_3,\ldots,p_n\right) \qquad \forall \mathbf{p} \in [0,\infty)^n. \tag{3.5}$$

Then $f(\mathbf{p}) \ge f\left(\frac{s}{n},\frac{s}{n},\ldots,\frac{s}{n}\right)$, where $s = p_1 + \ldots + p_n$.

Proposition 3.3.

$$E(p_1,\ldots,p_n) \ge E\left(\frac{s}{n},\frac{s}{n},\ldots,\frac{s}{n}\right) = \frac{n}{s}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right),$$

where $s = p_1 + \ldots + p_n$, therefore the answer to our question is: ET is minimum when $p_1 = p_2 = \ldots = p_n = 1/n$.

In order to apply Lemma 3.2, let us compute the difference

$$D(\mathbf{p}) := E(\mathbf{p}) - E(p,p,p_3,\ldots,p_n)$$

with $p = \dfrac{p_1+p_2}{2}$. We get

Lemma 3.4. *The following equality holds*

$$D(\mathbf{p}) = \tag{3.6}$$

$$f(0) - \sum_{3\le j\le n} f(p_j) + \sum_{\substack{3\le j_1,j_2\le n\\ j_1\ne j_2}} f(p_{j_1}+p_{j_2}) - \ldots = \sum_{k=0}^{n-2}(-1)^k \sum_{\substack{J\subset\{3,4,\ldots,n\}\\ |J|=k}} f\left(\sum_{j\in J} p_j\right)$$

with $f :\ [0,\infty)$ given by

$$f(x) = \frac{1}{p_1+x} + \frac{1}{p_2+x} - \frac{2}{\frac{p_1+p_2}{2}+x}. \tag{3.7}$$

Proof. If one replaces $(p_1, p_2, \ldots, p_n)$ with $(p, p, p_3, \ldots, p_n)$, then in (3.4) the sum $\sum\limits_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k}} \frac{1}{\sum\limits_{j \in J} p_j}$ becomes

$$\sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k,\ J \supset \{1,2\} \text{ or } J^c \supset \{1,2\}}} \frac{1}{\sum\limits_{j \in J} p_j} + 2 \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J|=k-1,\ J^c \supset \{1,2\}}} \frac{1}{\frac{p_1+p_2}{2} + \sum\limits_{j \in J} p_j}.$$

After doing the difference the first term disappears.

For instance for $n = 3$ one gets $D(\mathbf{p}) = f(0) - f(p_3)$; for $n = 4$ the formula (3.6) becomes $D(\mathbf{p}) = f(0) - f(p_3) - f(p_4) + f(p_3 + p_4)$; for $n = 5$ one gets $D(\mathbf{p}) = f(0) - f(p_3) - f(p_4) - f(p_5) + f(p_3 + p_4) + f(p_3 + p_5) + f(p_4 + p_5) - f(p_3 + p_4 + p_5)$ and so on. If one examines these quantities one sees that they can be expressed using the difference operators Δ defined as

$$\Delta_h f(x) = f(x) - f(x + h) \tag{3.8}$$

as follows: for $n = 3$ $D(\mathbf{p}) = \Delta_{p_3} f(0)$; for $n = 4$ $D(\mathbf{p})$ can be expressed by the "multiplication" $D(\mathbf{p}) = \Delta_{p_3} \Delta_{p_4} \Delta_{p_5} f(0)$ a.s.o. By induction over n one easily checks that (3.6) becomes

$$D(\mathbf{p}) = \Delta_{p_3} \Delta_{p_4} \ldots \Delta_{p_n} f(0). \tag{3.9}$$

Now, the difference operators are classical and they have been studied for hundreds of years, beginning with Newton. The reader can find a study of their properties in [1]. However, we did not see the following formula which the reader can easily check by induction over n.

Lemma 3.5. *The following equality holds*

$$D(\mathbf{p}) = \int_0^{p_3} \int_{t_3}^{t_3+p_4} \ldots \int_{t_{n-1}}^{t_{n-1}+p_n} (-1)^n f^{(n-2)}(t_n) dt_n dt_{n-1} \ldots dt_3, \tag{3.10}$$

where $f^{(n)}$ is the n-th derivative of f.

Now we are going to check that (3.5) holds, i.e. that $D(\mathbf{p}) \geq 0$.

Lemma 3.6. *The function f given by (3.7) has the property that*

$$(-1)^n f^{(n)}(x) \geq 0 \qquad \forall x \geq 0. \tag{3.11}$$

Proof. It is better to write $f(x) = (x+2a)^{-1} + (x+2b)^{-1} - 2(a+b+x)^{-1}$ with $a = p_1/2$, $b = p_2/2$. Then

$$(3.12)\quad (-1)^n f^{(n)}(x) = n!((x+2a)^{-n-1} + (x+2b)^{-n-1} - 2(x+a+b)^{-n-1}).$$

Now this quantity is nonnegative due to convexity reasons: the function $\varphi(a) = (x+2a)^{-n-1}$ is convex for any $x \geq 0$ fixed, hence

$$\frac{\varphi(a)+\varphi(b)}{2} - \varphi\left(\frac{a+b}{2}\right) \geq 0.$$

Now we can prove Proposition 3.3. As the assumptions of Lemma 3.2 are fulfilled, the first inequality is clear. Let us compute

$$E\left(\frac{s}{n}, \frac{s}{n}, \ldots, \frac{s}{n}\right) = \frac{n}{s}\left(C_n^1 - \frac{C_n^2}{2} + \frac{C_n^3}{4} - \ldots\right).$$

If one considers the derivative of the function $x \mapsto xC_n^1 - \frac{x^2C_n^2}{2} + \frac{x^3C_n^3}{4} - \ldots$ which is $(1-(1-x)^n)/x$ one can see that

$$C_n^1 - \frac{C_n^2}{2} + \frac{C_n^3}{4} - \ldots = \int_0^1 \frac{1-(1-x)^n}{x} dx;$$

making the change of variable $x := 1 - x$ one gets the result

$$C_n^1 - \frac{C_n^2}{2} + \frac{C_n^3}{3} - \ldots = \int_0^1 \frac{1-x^n}{1-x} dx = 1 + \frac{1}{2} + \ldots + \frac{1}{n}.$$

Now we shall point out a similar result for the tail probabilities $P(T > t)$.

Proposition 3.7. *(i) Let $A_1, \ldots, A_n$ sets from E and $T = T(A_1) \vee T(A_2) \vee \vee \ldots T(A_n)$. Then*

$$(3.13)\qquad P(T > t) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{J \subset \{1,2,\ldots,n\} \\ |J| = k}} q^t\left(\bigcup_{j \in J} A_j\right).$$

(ii) Suppose that the sets $(A_j)_{1 \leq j \leq n}$ are disjoint and $p(A_j) = p_j$. Let $s = p_1 + p_2 + \ldots + p_n$ and denote the probability $P(T > t)$ by $r_t(p_1, \ldots, p_n)$ with $t \geq n$. Then

$$(3.14)\qquad r_t(p_1, \ldots, p_n) \geq$$

$$\geq r_t\left(\frac{s}{n}, \frac{s}{n}, \ldots, \frac{s}{n}\right) = C_n^1\left(1-\frac{s}{n}\right)^t - C_n^2\left(1-\frac{2s}{n}\right)^t + C_n^3\left(1-\frac{3s}{n}\right)^t - \ldots$$

Proof. (i) Apply (1.3) and (3.1).

(ii) The trick will be the same as in the proof of Proposition 3.3, except that in this case the difference $D(\mathbf{p}) = r_t(p_1, \ldots, p_n) - r_t(p, p, p_3, \ldots, p_n)$ (with $p = (p_1 + p_2)/2$) is equal to

$$D(\mathbf{p}) = \Delta_{p_3}\Delta_{p_4}\ldots\Delta_{p_n} g_t(0) \tag{3.15}$$

with

$$g_t(x) = (1-p_1-x)^t + (1-p_2-x)^t - 2(1-p-x)^t. \tag{3.16}$$

As

$$(-1)^j g_t^{(j)}(x) =$$

$$= t(t-1)\ldots(t-j+1)[(1-p_1-x)^{t-j} + (1-p_2-x)^{t-j} - 2(1-p-x)^{t-j}] =$$

$$= \varphi(1-p_1) + \varphi(1-p_2) - 2\varphi\left(\frac{(1-p_1)+(1-p_2)}{2}\right)$$

with $\varphi(u) = (u-x)^{t-j}$ a convex function for any $j < t+2$ it follows that $D(\mathbf{p}) \geq 0$ (use formula (3.10)) and that settles the first assumption of (ii). As about the second equality in (3.14), it immediately follows from (3.13).

About the variance of T: we do not believe that it is possible to find a nice formula for it. To see what happens, consider the case of two sets A and B. The generating function is

$$\varphi := \varphi_{T(A)\vee T(B)} = \varphi_{T(A)} + \varphi_{T(B)} - \varphi_{T(A\cup B)}. \tag{3.17}$$

Then $Var(T) = \varphi''(1) + \varphi'(1) - (\varphi'(1))^2$. Doing the computation one gets

$$Var(T) = Var(T(A)) + Var(T(B)) - Var(T(A\cup B)) - \tag{3.18}$$

$$-2\left(\frac{1}{p(A)} - \frac{1}{p(A\cup B)}\right)\left(\frac{1}{p(B)} - \frac{1}{p(A\cup B)}\right).$$

Now compare this formula with $ET = \frac{1}{p(A)} + \frac{1}{p(B)} - \frac{1}{p(A\cup B)}$. If $p(A) = a$, $p(B) = b$ with $a \leq b$ and $p(A\cup B) = x$ then it is easy to see that ET is minimum when x is minimum and maximum when x is maximum (hence $x = (a+b)\wedge 1$). In other words, if we want ET to be the least we should have the inclusion $A \subset B$ and if we want it to be the greatest then $p(A\cup B)$ should be as great as possible. This is not true in the case of the variance: nor the maximum, neither the minimum are attained in these extreme situations. For example, if $p(A) = p(B) = 0.25$ then $Var(T(A) \vee T(B))$ is maximum for $p(A \cup B) = \frac{6}{17}$ and not for $p(A \cup B) = 0.5$. If $a = 0.5$ and $b = 0.75$ then, unlike the case of expectations, $Var(T(A) \vee T(B))$ is maximum (equal to 2) for $A \subset B$ and minimum for $A \cup B = E$ (equal to $2 - \frac{2}{9}$), as the reader can check doing some tedious elementary computations.

We do not know a result similar to Proposition 3.3 holds. Even in the case $n = 2$ the computations are not very simple, not to mention greater n. In other words *we do not know when the variance of T is minimum.* At least we can prove

Proposition 3.8. $Var(T(A) \vee T(B)) \geq Var(T(A)) \wedge Var(T(B))$.

Proof. Let a, b, x as before. Then $Var(T(A)) \wedge Var(T(B)) = \dfrac{1-b}{b^2}$. Let

$$g(x) = Var(T(A) \vee T(B)) - Var(T(A)) \wedge Var(T(B)) =$$

$$= \frac{1-a}{a^2} - \frac{1-x}{x^2} - \frac{2(x-a)(x-b)}{abx^2},$$

$g : [b, (a+b) \wedge 1] \to \Re$. As g is a function of the form $g(x) = A + \dfrac{B}{x} - \dfrac{3}{x^2}$, its derivative has at most one zero on the interval $[b, (a+b) \wedge 1]$. It follows that there are only two situations: either g increases and then decreases or g is monotonous. Be as it be,

$$\min g = g(b) \wedge g((a+b) \wedge 1). \tag{3.19}$$

As $g(b) = \dfrac{1-a}{a^2} - \dfrac{1-b}{b^2} \geq 0$, all we have to check is that $g((a+b) \wedge 1) \geq 0$.

Case 1. $0 < a \leq b$, $a + b \geq 1 \Rightarrow b \geq 0.5$, then $(a+b) \wedge 1 = 1$, $g(1) = = \dfrac{1-a}{a}\left(\dfrac{1}{a} - \dfrac{2(1-b)}{b}\right)$. Now $g(1) \geq 0 \Leftrightarrow a \leq \dfrac{b}{2(1-b)}$. But $a \leq b$ and $b \leq \dfrac{b}{2(1-b)} \Leftrightarrow 0.5$, which is true.

Case 2. $0 < a \leq b$, $a + b \leq 1 \Rightarrow a \leq 0.5$, then $(a+b) \wedge 1 = a + b \Rightarrow \Rightarrow g(a+b) = \dfrac{1-a}{a^2} - \dfrac{3-t}{t^2}$ with $t = a + b \geq 2a$. We have to check that $\dfrac{3-t}{t^2} \leq \dfrac{1-a}{a^2}$ for all $t \in [2a, 1]$. As the function $t \mapsto \dfrac{3-t}{t^2}$ is decreasing it is enough to check that $\dfrac{3-2a}{4a^2} \leq \dfrac{3-a}{a^2} \Leftrightarrow 3 - 2a \leq 4 - 4a \Leftrightarrow 2a \leq 1$, which is true.

4. The case of only two sets: correlation between $T(A)$ and $T(B)$

We shall be concerned now with the joint distribution of the random vector $(T(A),\ T(B))$.

Lemma 4.1.

$$P(T(A)=i,\ T(B)=j)=\begin{cases} p(A)p(B\setminus A)q(A\cup B)^{j-1}q(A)^{i-j-1} & \text{if } i>j,\\ p(B)p(A\setminus B)q(A\cup B)^{i-1}q(B)^{j-i-1} & \text{if } i<j,\\ p(A\cap B)q(A\cap B)^{i-1} & \text{if } i=j.\end{cases}$$

Proof. Very easy and therefore left to the reader.

Proposition 4.2. *The following equalities hold:*

$$E(T(A)T(B))=\frac{p(A)+p(B)-p(A)p(B)}{p(A)p(B)p(A\cup B)}, \tag{4.1}$$

$$cov(T(A),T(B)):= \tag{4.2}$$

$$:=E(T(A)T(B))-E(T(A))E(T(B))=\frac{p(A\cap B)-p(A)p(B)}{p(A)p(B)p(A\cup B)},$$

$$\rho(T(A),T(B))=\frac{cov(T(A),T(B))}{\sqrt{Var(T(A))Var(T(B))}}=\frac{p(A\cap B)-p(A)p(B)}{p(A)p(B)\sqrt{q(A)q(B)}}. \tag{4.3}$$

Proof. The only tiresome computation is (4.1). First the reader should compute the series

$$s_1(x,y)=\sum_{i,j\geq 1, i>j} ijx^{j-1}y^{i-j-1} \quad \text{and} \quad s_2(x)=\sum_{i=1}^{\infty} i^2x^{i-1}, \tag{4.4}$$

to establish that

(4.5)

$$s_1(x,y)=\frac{2}{(1-x)^3(1-y)}+\frac{y}{(1-x)^2(1-y)^2},\quad s_2(x)=\frac{2}{(1-x)^3}-\frac{1}{(1-x)^2}$$

which further implies
(4.6)
$$E(T(A)T(B);\ T(A) > T(B)) = p(B \setminus A)\left(\frac{2}{p(A \cup B)^3} + \frac{q(A)}{p(A)p(A \cup B)^2}\right),$$

(4.7)
$$E(T(A)T(B);\ T(A) < T(B)) = p(A \setminus B)\left(\frac{2}{p(A \cup B)^3} + \frac{q(B)}{p(B)p(A \cup B)^2}\right),$$

$$(4.8)\quad E(T(A)T(B);\ T(A) = T(B)) = p(A \cap B)\left(\frac{2}{p(A \cup B)^3} - \frac{1}{p(A \cup B)^2}\right).$$

Adding (4.6), (4.7) and (4.8) one gets

$$E(T(A)T(B)) =$$

$$= \frac{2(p(B \setminus A) + p(A \setminus B) + p(A \cap B))}{p(A \cup B)^3} + \frac{\frac{p(B\setminus A)q(A)}{p(A)} + \frac{p(A\setminus B)q(B)}{p(B)} - p(A \cap B)}{p(A \cup B)^2} =$$

$$= \frac{2 + \frac{p(B\setminus A)q(A)}{p(A)} + \frac{p(A\setminus B)q(B)}{p(B)} - p(A \cap B)}{p(A \cup B)^2}.$$

Let $x = p(A),\ y = p(B),\ z = p(A \cap B)$. Then

$$(4.9)\qquad E(T(A)T(B)) =$$

$$= \frac{2xy - xyz + (y - z)(y - xy) + (x - z)(x - xy)}{xy(x + y - z)^2} = \frac{x + y - xy}{xy(x + y - z)}$$

which is exactly (4.1).

There is something interesting with the random variables from $\mathcal{T}$: as in the normal case *they are independent iff they are noncorrelated*, i.e. their correlation coefficient is equal to 0.

Proposition 4.2. *(Bounds on the correlation coefficient)*

(i) The correlation coefficient between $T(A)$ and $T(B)$ satisfies the inequalities

$$(4.10)\qquad -0.5 \le \rho(T(A),\ T(B)) \le 1.$$

(ii) $T(A)$ and $T(B)$ are noncorrelated iff they are independent. Precisely

$$(4.11)\qquad \rho(T(A),\ T(B)) = 0 \Leftrightarrow p(A \cap B) = p(A)p(B) \Leftrightarrow$$

$\Leftrightarrow$ *A and B are independent (with respect to the probability p)* $\Leftrightarrow$ *T(A) and T(B) are independent.*

Proof. (i) The right bound in (4.10) is attained if $A = B$. We shall prove the left inequality and seek the case in which the equality is attained. Let $x = p(A)$, $y = p(B)$, $a = p(A \cap B)$. Suppose that a is fixed. Then we consider ρ as a function

$$\rho(x, y) = \frac{a - xy}{(x + y - a)\sqrt{(1 - x)(1 - y)}}.$$

The domain of ρ is the set $D_a\{(x, y) \mid x, y \geq a,\ x + y \leq a + 1\}$ (because $p(A), p(B) \geq p(A \cap B)$ and $p(A \cup B) = x + y - a \leq 1$). The set D_a is symmetric for any $0 \leq a < 1$ and the function ρ is again symmetric (clearly $\rho(x, y) = \rho(y, x)$). Suppose the sum $x + y = s$ is fixed. Denoting $u = 1 - x$, $v = 1 - y$, $t^2 = uv$ we see that we can write $\rho(x, y) = A(B/t - t)$. This function is decreasing in t, that is why for any fixed s the function is minimum when t is maximum $\Leftrightarrow$ t^2 is maximum $\Leftrightarrow 1 - x = 1 - y \Leftrightarrow x = y$. Consequently $\rho(x, y) \geq \rho(x, x) = \dfrac{a - x^2}{(2x - a)(1 - x)}$. Denote this function by $g(x)$. The domain of g is the interval $[a, (a + 1)/2]$. One checkes that the derivative $g' \leq 0$, hence the minimum of g is attained for $x = (a + 1)/2$. Consequently we get that $\rho(x, y) \geq g\left(\dfrac{a + 1}{2}\right) = \dfrac{a - 1}{2}$. We conclude that $\rho(x, y) \geq -0.5$ and the bound is attained iff $a = 0$ (i.e. the sets A and B are disjoint) and $p(A) = p(B) = 1/2$.

(ii) From (4.3) one gets $\rho(x, y) = 0$ iff $p(A \cap B) = p(A)p(B) \Leftrightarrow A$ and B are independent with respect to the probability $p = P^0(X_n)^{-1}$. Looking at the relations from Lemma 1 one sees that in this case $T(A)$ and $T(B)$ are independent. Conversely, if $T(A)$ and $T(B)$ are independent, they are noncorrelated, too.

References

[1] **Aigner M.**, *Combinatorial theory,* Springer, 1979.

[2] **Feller W.**, *An introduction to probability theory and its applications,* John Wiley & Sons, 1957.

[3] **Zbăganu G.**, Some entropy-like indices and their connection with the metrization of σ-algebras, *Studii şi cercetări,* **38** (1) (1986), 76-88.

[4] **Mărgăritescu E.,** Optimal properties for some Bonferroni-type inequalities, *Rev. Roumaine de Math. Pures et Appl.,* **35** (6) (1990), 541-548.

(Received September 14, 1997)

G. Zbăganu
Centre for Mathematical Statistics
Calea 13 septembrie 13.
Ro-76100 Buçuresti
Roumania