Annales Univ. Sci. Budapest., Sect. Comp. 19 (2000) 187-197

LYAPUNOV TRANSFORMATION AND STABILITY OF DIFFERENTIAL EQUATION IN BANACH SPACES

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Abstract. Lyapunov transformation [1] conserves the stability of solutions of linear differential systems. vd-transformation in $\mathbb{R}^n$-space ([2]-[6]) is a generalization of Lyapunov transformation, it conserves, too, the stability of differential systems. In the article we will give the concept of vd-transformation in Banach space and apply it to study the stability of differential systems.

1. vd-transformation

Let E be a Banach space, G an open simple connected domain containing the origin O of E

$$H = G \times \mathbb{R} = \{\eta = (x,t) : \ x \in G, \ t \in \mathbb{R}\}.$$

Let us consider the continuous, monotone, strictly increasing function

$$v_0 = \quad \mathbb{R}^+ \to \mathbb{R}^+$$

for which

$$v_0(0) = 0; \qquad v_0(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Let be given a real function d of two variables

$$d: \ \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R},$$
$$(\gamma_1, \gamma_2) \to d(\gamma_1, \gamma_2),$$

satisfying the following conditions for all $\gamma > 0$, $\gamma_3 > \gamma_2 > \gamma_1 > 0$:

(d_1) $d(\gamma_2, \gamma_1) = -d(\gamma_1, \gamma_2)$;
(d_2) $d(\gamma_2, \gamma) > d(\gamma_1, \gamma)$;
(d_3) $d(\gamma_3, \gamma_2) + d(\gamma_2, \gamma_1) \geq d(\gamma_3, \gamma_1)$;
(d_4) $\bigcup_{\gamma \in \mathbb{R}^+} \{d(\gamma, \gamma_1)\} = \mathbb{R}$.

Suppose that l is a diffeomorphism from H to H

$$\begin{aligned} l:\ & H \longrightarrow H, \\ & \eta = (x,t) \longmapsto \eta' = (x', t') \end{aligned}$$

satisfying the following equalities

$$\begin{aligned} l(0,t) &= (0,t), \\ l(x,t) &= (x', t) \end{aligned}$$

for all $t \in \mathbb{R}$. It is easy to prove that $L = \{l\}$ is a group for the composition of maps.

Let v be a real function

$$\begin{aligned} v:\ & H^* \longrightarrow \mathbb{R}^+, \\ & \eta = (x,t) \longrightarrow v(\eta) = v_0(\|x\|) \end{aligned}$$

(where $H^* = G^* \times \mathbb{R} = (G \setminus \{0\} \times \mathbb{R})$.

Definition. The transformation $l \in L$ is called vd-transformation iff

$$\begin{aligned} \sup_{\eta \in H^*} |d\{v(\eta),\ v[l(\eta)]\}| &< +\infty, \\ \sup_{\eta' \in H^*} |d\{v(\eta'),\ v[l^{-1}(\eta')]\}| &< +\infty, \end{aligned} \tag{1}$$

i.e. l is vd-transformation iff l^{-1} is vd-transformation. Therefore the L_{vd}-set of vd-transformation is a subgroup of L.

Examples. 1. *Let be given* $v_0(x,t) = \|x\|$, $d_0(\gamma_1, \gamma_2) = \ln(\gamma_1/\gamma_2)$ *and* $l(x,t)$ *(with a fixed* t*) is a linear transformation having bounded partial derivative with respect to* t. *Then* l *is* $v_0 d_0$*-transformation if and only if it is a Lyapunov transformation [1].*

Proof. $l(x,t)$ is a linear homogeneous transformation for x iff

$$L(t) \in L(E); \quad l(x,t) = (L(t)x, t)$$

is a diffeomorphism, where

$$\sup_{(x,t)} ||D_2 l(x,t)|| < \infty \quad \Longleftrightarrow \quad \sup_t ||L(t)|| < \infty$$

($D_2 l(x,t)$ is the second partial derivative [7]). Then

$$l \in L_{v_0 d_0} \Longleftrightarrow \begin{cases} \sup\limits_{\eta \in H^*} \left| \ln \frac{||L(t)x||}{||x||} \right| < +\infty \\ \sup\limits_{\eta \in H^*} \left| \ln \frac{||L^{-1}(t)x||}{||x||} \right| < +\infty \end{cases} \Longleftrightarrow \begin{cases} \sup\limits_t ||L(t)|| < +\infty \\ \sup\limits_t ||L^{-1}(t)|| < +\infty \end{cases}$$

2. Let be given $v(x,t) = |x|^2$, $E = \mathbb{R}$,

$$d(\gamma_1, \gamma_2) = \begin{cases} \sqrt{\gamma_1} - \sqrt{\gamma_2} & \text{if} \quad \gamma_1 \cdot \gamma_2 \geq 1, \\ \frac{1}{\sqrt{\gamma_2}} - \frac{1}{\sqrt{\gamma_1}} & \text{if} \quad \gamma_1 \cdot \gamma_2 < 1. \end{cases}$$

All conditions $d_1) - d_4)$ are satisfied, it can be proved by immediate verification. Especially, here is the case when the inequality $d_3)$ holds strictly. For instance when $\gamma_1\gamma_3 < 1$, $\gamma_2\gamma_3 < 1$ (and therefore $\gamma_1\gamma_2 \geq 1$, where $\gamma_1 > \gamma_2 > \gamma_3 > 0$), we have

$$d(\gamma_1, \gamma_2) + d(\gamma_2, \gamma_3) - d(\gamma_1, \gamma_3) = \sqrt{\gamma_1} - \sqrt{\gamma_2} + \frac{1}{\sqrt{\gamma_3}} - \frac{1}{\sqrt{\gamma_2}} + \sqrt{\gamma_3} - \sqrt{\gamma_1} =$$

$$= \frac{\left(\sqrt{\gamma_2} - \sqrt{\gamma_3}\right)\left(1 - \sqrt{\gamma_2\gamma_3}\right)}{\sqrt{\gamma_2\gamma_3}} > 0.$$

Suppose

$$l(x,t) = \left(x + \frac{1}{2} \sin t \sin^2 x, \ \ t \right).$$

It is clear that $l \in L_{vd}$. Indeed,

$$\begin{vmatrix} \frac{\partial l_1}{\partial x} & \frac{\partial l_1}{\partial t} \\ \frac{\partial l_2}{\partial x} & \frac{\partial l_2}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 - \frac{1}{2}\sin t \sin 2x & \frac{1}{2}\cos t \sin^2 x \\ 0 & 1 \end{vmatrix} = 1 - \frac{1}{2}\sin t \sin 2x \neq 0,$$

this deduces the existence of differentiable $l^{-1}(x,t)$.

It is clear that $l(0,t) = (0,t)$, $l(x,t) = (y,t)$ and

$$(*) \qquad \sup_{x \neq 0} |d\{v(l(\eta)), \ v(\eta)\}| < +\infty.$$

In order to prove (*) we can immediately verify as follows

$$d\{v(l(x,t)),\ v(x,t)\} = \begin{cases} \left|x + \frac{1}{2}\sin t \sin^2 x\right| - |x| & \text{for } v(l(\eta))\cdot v(\eta) \ge 1, \\ \frac{1}{|x|} - \frac{1}{\left|x + \frac{1}{2}\sin t\sin^2 x\right|} & \text{for } v(l(\eta))\cdot v(\eta) < 1. \end{cases}$$

On the other hand, it is easy to find that

$$\left|\left|x + \frac{1}{2}\sin t \sin^2 x\right| - |x|\right| \le \frac{1}{2}\left|\sin t \sin^2 x\right|$$

and

$$\left|\frac{1}{|x|} - \frac{1}{\left|x + \frac{1}{2}\sin t\sin^2 x\right|}\right| \le \frac{\frac{1}{2}\left|\sin t\sin^2 x\right|}{|x|\left|x + \frac{1}{2}\sin t \sin^2 x\right|} \le$$

$$\le \frac{\frac{1}{2}\left|\sin t \sin^2 x\right|}{x^2\left|1 - \frac{1}{2}\left|\sin t \sin x\right|\right|} \le \frac{\sin^2 x}{x^2}.$$

Consequently,

$$\sup_{x\ne 0} \left|d\{v(l(\eta)),\ v(\eta)\}\right| < +\infty.$$

2. Properties of vd-transformation

Consider in Banach space E the differential equation

$$(2) \qquad \begin{cases} \frac{dx}{dt} = f(x,t), \\ f(0,t) \equiv 0. \end{cases}$$

We denote by $x(t;\xi)$ the solution of (2) satisfying the initial condition $x(t_0;\xi) =$ $= \xi$ and

$$\lambda = \lim_{\varepsilon\to 0^+} \sup_{\substack{\|\xi\|\le\varepsilon \\ t\ge t_0}} \|x(t;\xi)\|,$$

$$\lambda_1 = \lim_{\varepsilon\to 0^+} \sup_{\substack{\|\xi\|\le\varepsilon \\ t\ge t_0}} v(x(t;\xi)).$$

Proposition 1. $\lambda = 0 \iff \lambda_1 = 0.$

Proof. By continuity of v we immediately find that $\lim_{\xi\to 0} v(\xi) = 0$. Since $v(\|x\|)$ is monotone, strictly increasing

$$\lim_{v(\xi)\to 0} \xi = 0.$$

Hence

$$\lim_{k\to\infty} \xi_k = 0 \iff \lim_{k\to\infty} v(\xi_k) = 0. \tag{3}$$

We assume that $\lambda = 0$, then

$$\lim_{k\to\infty} \|x(t_k;\xi_k)\| = 0$$

for all sequences $\{\varepsilon_k\} \subset \mathbb{R}^+ : \varepsilon_k \to 0$; $\{\xi_k\} \subset E : \xi_k \to 0$ and $\{t_k\} \subset \mathbb{R} : t_k \geq t_0$. Because of (3) we have

$$\lim_{k\to\infty} \|x(t_k;\xi_k)\| = 0 \iff \lim_{k\to\infty} v(x(t_k;\xi_k)) = 0.$$

It follows that $\lambda = 0 \iff \lambda_1 = 0$.

Proposition 2. *vd-transformation conserves the stability of solution $x = = 0$ of differential equation (2).*

Proof. By vd-transformation

$$(x,t) \quad \to \quad l(x,t) = (y,t)$$

the equation (2) is transformed to

$$\frac{dy}{dt} = g(y,t). \tag{4}$$

By assumption the solution $x = 0$ of (2) is stable, that means

$$\lim_{\varepsilon\to 0+} \sup_{\substack{\|x_0\|\leq\varepsilon \\ t\geq t_0}} \|x(t;x_0)\| = 0 \iff \lim_{\varepsilon\to 0+} \sup_{\substack{v(x_0)\leq\varepsilon \\ t\geq t_0}} v[x(t;x_0)] = 0.$$

If this is false the solution $y = 0$ of (4) is unstable and then

$$\lim_{\varepsilon\to 0+} \sup_{\substack{v(y_0)\leq\varepsilon \\ t\geq t_0}} v[y(t;y_0)] > 0.$$

It means that there exists a positive number δ such that

$$(5)\qquad \exists\{\eta_n\}\subset E:\ \ \eta_n\to y_0;\quad \exists\{t_n\}\subset\mathbb{R}',\ \ \forall n\in\mathbb{N};\quad v[y(t_n;\eta_n)]\geq\delta.$$

By means of

$$v[x(t_n;\xi_n)]\to 0\quad \text{as}\quad n\to\infty,$$

where $(\xi_n,t_0)=l^{-1}(\eta_n,t_0)$, one could say

$$(6)\qquad v[x(t_n;\xi_n)]<\delta\qquad \forall n\in\mathbb{N}.$$

From (5) and (6) we deduce

$$|d\{v[x(t_n;\xi_n)],\ v[y(t_n;\eta_n)]\}|=d\{v[y(t_n;\eta_n)],\ v[x(t_n;\xi_n)]\}>$$

$$>d\{\delta,\ v[x(t_n;\xi_n)]\}\to+\infty\qquad \text{as}\quad n\to\infty.$$

Consequently

$$\sup|d\{v[x(t_n,\xi_n)],\ v[l(x(t_n,\xi_n))]\}|=+\infty,$$

that contradicts to the definition of d.

Proposition 3. *The vd-number*

$$\Omega^*vd\ x:=\lim_{t\to\infty}\frac{1}{t}\sup_{t_0\neq 0}d\{v[x(t_0+t)],\ v[x(t_0)]\}$$

is vd-invariant, i.e. $\Omega^*vd\ y=\Omega^*vd\ x$ *for all* $l\in L_{vd}$, $(y,t)=l(x,t)$.

Proof. We have

$$d\{v[y(t_0+t)],\ v[y(t_0)]\}=d\{v[l(x(t_0+t)),\ v[l(x(t_0))]\}=$$

$$=d\{v[x(t_0+t)],\ v[x(t_0)]\}+d\{v[l(x(t_0+t))],\ v[l(x(t_0))]\}-$$

$$-d\{v[x(t_0+t)],\ v[l(x(t_0))]\}+d\{v[x(t_0+t)],\ v[l(x(t_0))]-d\{v[x(t_0+t)],v[x(t_0)]\}\}$$

$$=d\{v[x(t_0+t)],\ v[x(t_0)]\}+A+B,$$

where

$$|A|=|d\{v[l(x(t_0+t),\ v[l(x(t_0))]\}-d\{v[x(t_0+t)],\ v[l(x(t_0))]\}\leq$$
$$\leq 2|d\{v[l(x(t_0+t)),\ v[x(t_0+t)]\},$$
$$|B|=|d\{v[l(x(t_0+t))],\ v[l(x(t_0))]\}-d\{v[x(t_0+t)],\ v[x(t_0)]\}|\leq$$
$$\leq 2|d\{v[l(x(t_0+t))],\ v[x(t_0+t)]\}|,$$

therefore A, B are bounded. Consequently,

$$\Omega^* vdy = \Omega^* vdx.$$

Proposition 4. *The vd - small number*

$$\overline{\Omega} vdx :=$$

$$:= \max\left\{ \overline{\lim_{t\to\infty}} \frac{1}{t} d\{v[x(t_0+t)],\ v[x(t_0)]\} - \lim_{t\to\infty} \frac{1}{t} d\{v[x(t_0+t)],\ v[x(t_0)]\} \right\}$$

is vd-invariant.

Proof. Because of

$$d\{v[y(t_0+t)],\ v[y(t_0)]\} = d\{v[x(t_0+t)],\ v[x(t_0)]\} + A + B$$

and A, B are bounded, we immediately find that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} d\{v[y(t_0+t)],\ v[y(t_0)]\} = \overline{\lim_{t\to\infty}} \frac{1}{t} d\{v[x(t_0+t)],\ v[x(t_0)]\}.$$

On the other hand

$$d\{v[y(t_0-t)],\ v[y(t_0)]\} = d\{v[x(t_0-t)],\ v[x(t_0)]\} + C + D,$$

where

$$\begin{aligned} |C| &= |d\{v[l(x(t_0-t))],\ v[l(x(t_0))]\} - d\{v[x(t_0-t)],\ v[l(x(t_0))]\}| \leq \\ &\leq 2|d\{v[l(x(t_0-t))],\ v[x(t_0-t)]\}|, \\ |D| &= |d\{v[x(t_0-t)],\ v[l(x(t_0))]\} - d\{v[x(t_0-t)],\ v[x(t_0)]\}| \leq \\ &\leq 2|d\{v[l(x(t_0))],\ v[x(t_0)]\}|, \end{aligned}$$

i.e. C, D are bounded. Therefore, the following equality is true

$$\lim_{t\to\infty} \frac{1}{t} d\{v[y(t_0-t)],\ v[y(t_0)]\} = \lim_{t\to\infty} \frac{1}{t} d\{v[x(t_0-t)],\ v[x(t_0)]\} \Rightarrow \overline{\Omega} vdy = \overline{\Omega} vdx.$$

3. Regular system

Definition. The transformation $y = L(t)x$ is a generalized Lyapunov one if

$$\chi[L(t)] = \chi[L^{-1}(t)] = 0. \tag{7}$$

Remark. By definition we immediately find that *generalized Lyapunov transformation conserves Lyapunov exponents.*

Theorem. *A necessary and sufficient condition that the system*

$$\frac{dx}{dt} = A(t)x, \tag{8}$$

where $A(t) \in C(t, \mathbb{R}^n)$, $x \in \mathbb{R}^n$, to be regular one ([1]) is that there exists a generalized Lyapunov transformation which carries the system (8) to the system with constant matrix

$$\frac{dx}{dt} = By. \tag{9}$$

Proof. Let $y = L(t)x$ be a generalized Lyapunov transformation, $X(t)$ a normal fundamental matrix of (8). It follows that $Y(t) = L(t)X(t)$ is a fundamental matrix of (9) and

$$\begin{aligned}
&\det Y(t) = \det L(t) \det X(t)\\
&\Leftrightarrow \det Y(t_0) \exp(t - t_0)\mathrm{Sp}B = \det L(t) \det X(t_0) \exp \int_{t_0}^{t} \mathrm{Sp}A(t_1)dt_1\\
&\Leftrightarrow \exp \int_{t_0}^{t} \mathrm{Sp}A(t_1)dt_1 = |C(t_0)||\det L^{-1}(t) \exp(t - t_0)\mathrm{Sp}B,
\end{aligned}$$

where

$$\begin{aligned}
&C(t_0) = \det[Y(t_0)X^{-1}(t_0)]\\
&\Rightarrow \frac{1}{t} \int_{t_0}^{t} \mathrm{Sp}A(t_1)dt_1 = \frac{1}{t} \ln |C(t_0)| + \frac{1}{t} \ln |\det L^{-1}(t)| + \left(1 - \frac{t_0}{t}\right) \mathrm{Sp}B\\
&\Rightarrow \overline{\lim_{t\to\infty}} \frac{1}{t} \int_{t_0}^{t} \mathrm{Sp}A(t_1)dt_1 = \mathrm{Sp}B + \chi[\det L^{-1}(t)].
\end{aligned}$$

Because of $\chi[L^{-1}(t)] = 0$ we have

$$\chi[\det L^{-1}(t)] \le n\chi[L^{-1}(t)] = 0.$$

Analogously from $\chi[L(t)] = 0$ it follows that

$$\chi[\det L(t)] \le 0.$$

On the other hand, since

$$\det L(t) \cdot \det L^{-1}(t) = 1,$$

the following holds

$$\chi[\det L(t)] + \chi[\det L^{-1}(t)] \ge 0.$$

Therefore $\chi[\det L(t)] = \chi[\det L^{-1}(t)] = 0$. It follows from these equalities that

$$\lim_{t\to\infty} \frac{1}{t} \ln |\det L^{-1}(t)| = 0$$

and finally

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \mathrm{Sp}A(t_1)dt_1 = \mathrm{Sp}B.$$

Since the Lyapunov transformation conserves Lyapunov exponents and the normality of X, Y, and

$$\sigma_X = \sigma_Y = \mathrm{Sp}B \Rightarrow \sigma_X = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \mathrm{Sp}A(t_1)dt_1,$$

i.e. the system (8) is regular.

Let the system (8) be regular. We will denote by $X(t)$ the fundamental normal matrix of (8) which has the exponent numbers $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$. Consider the Jordan matrix B in which $\lambda_1, \ldots, \lambda_n$ are the diagonal elements. Denoting by $Y(t)$ the fundamental normal matrix of the system (9) we constate that it has the column of same exponent numbers as (7) $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Putting $L(t) = Y(t)X^{-1}(t)$ we will prove that $y = L(t)x$ is a generalized Lyapunov transformation. Suppose that

$$Y(t) = \begin{bmatrix} y_{11}(t) & y_{12}(t) & & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & & y_{2n}(t) \\ \vdots & \vdots & & \vdots \\ y_{n1}(t) & y_{n2}(t) & & y_{nn}(t) \end{bmatrix},$$

$$X^{-1}(t) = \begin{bmatrix} x_{11}(t) & x_{12}(t) & & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & & x_{nn}(t) \end{bmatrix},$$

then $\chi[y^{(k)}] = \lambda_k$, where $y^{(k)} = colon(y_{1k}(t), \dots, y_{nk}(t))$. Because of the regularity of (7) we have $\chi[x^{(k)}] = -\lambda_k$, where $x^{(k)} = (x_{k1}, \dots, x_{kn})$. We consider now the diagonal matrix

$$\Delta = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

We find then

$$L(t) = Y(t)e^{-t\Delta}e^{t\Delta}X^{-1}(t) = \Phi(t) \cdot \Psi(t)$$

in which $\Phi(t) = Y(t)e^{-t\Delta}$, $\Psi(t) = e^{t\Delta}X^{-1}(t)$. It follows that

$$\chi[\Phi(t)] = \max_{j,k} \chi[y_{jk}e^{-\lambda_j t}] = 0,$$

$$\chi[\Psi(t)] = \max_{j,k} \chi[x_{jk}e^{\lambda_k t}] = 0.$$

Consequently

$$\chi[L(t)] \leq \chi[\Phi(t)] + \chi[\Psi(t)] = 0.$$

Analogously we can prove that $\chi[L^{-1}(t)] \leq 0$. But from $L(t) \cdot L^{-1}(t) = E$ we immediately find that $\chi[L(t)] + \chi[L^{-1}(t)] \geq 0$, i.e. $\chi[L(t)] + \chi[L^{-1}(t)] = 0$.

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(Received January 10, 1997)

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