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ERROR OF AN ARBITRARY ORDER FOR THE APPROXIMATE SOLUTION OF SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH SPLINE FUNCTIONS I.

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Abstract. In this paper we introduce a method for approximating the solution of the system of nonlinear second order differential equations $y" = f_1(x, y, z)$, $z" = f_2(x, y, z)$ with $y(x_0) = y_0$, $y'(x_0) = y'(x_0)$, $z(x_0) = z_0$ and $z'(x_0) = z'_0$. We use spline functions which are not necessarily polynomial spline for finding the approximate solution. The method is a one-step method $O(h^{2m+\alpha})$ in $y^{(i)}(x)$ and $z^{(i)}(x)$, where $i = 0(1)2$, $0 < \alpha \leq 1$ and m is an arbitrary positive integer which equals the number of iteration processes describing the spline functions defined in the method, assuming that $f_1, f_2 \in C([0,1] \times R^2)$. It is also shown that the method is stable.

1. Assumptions and procedures

Consider the system of nonlinear second order differential equations

$$y" = f_1(x, y, z), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{1}$$

$$z" = f_2(x, y, z), \quad z(x_0) = z_0, \quad z'(x_0) = z'_0, \tag{2}$$

where $f_1, f \in C([0,1] \times R^2)$.

Let $\Delta : \ 0 = x_0 < x_1 < x_2 \ldots < x_k < x_{k+1} < \ldots < x_n = 1$ be the partition of the interval $[0,1]$, where $x_{k+1} - x_k = h < 1$ and $k = 0(1)n-1$.

Let L_i be the Lipschitz constant satisfied by the functions f_i, where $i = 1$ and 2, i.e.

$$|f_i(x, y_1, z_1) - f_i(x, y_2, z_2)| \le L_i\{|y_1 - y_2| + |z_1 - z_2|\} \tag{3}$$

for all (x, y_1, z_1) and (x, y_2, z_2) in the domain of definition of the functions f_1 and f_2. Choosing the arbitrary positive integer m, for any $x \in [x_k, x_{k+1}]$, $k = 0(1)n - 1$, we define the spline functions approximating the solutions $y(x)$ and $z(x)$ by $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ as follows

$$\begin{aligned} S_\Delta(x) \equiv S_k^{[m]}(x) = & S_{k-1}^{[m]}(x_k) + S'^{[m]}_{m-1}(x_k)(x - x_k) + \\ & + \int_{x_k}^{x}\int_{x_k}^{t_1} f_1[u_1, S_k^{[m-1]}(u_1), \bar{S}_k^{[m-1]}(u_1)]du_1 dt_1 \end{aligned} \tag{4}$$

and

$$\begin{aligned} \bar{S}_\Delta(x) \equiv \bar{S}_k^{[m]}(x) = & \bar{S}_{k-1}^{[m]}(x_k) + \bar{S}'^{[m]}_{k-1}(x_k)(x - x_k) + \\ & + \int_{x_k}^{x}\int_{x_k}^{x} f_2[u_1, S_k^{[m-1]}(u_1), \bar{S}_k^{[m-1]}(u_1)]du_1 dt_1, \end{aligned} \tag{5}$$

where $S_{-1}^{[m]}(x_0) = y_0$, $S'^{[m]}_{-1}(x_0) = y'_0$, $\bar{S}_{-1}^{[m]}(x_0) = z_0$ and $\bar{S}'^{[m]}_{-1}(x_0) = z'_0$.

In equations (4) and (5) we use the following m iteration processes. For any $x \in [x_k, x_{k+1}]$, $k = 0(1)n - 1$, $j = 1(1)m$, $x_k \le u_m \le t_m \le u_{m-1} \le t_{m-1} \le \le \dots \le u_{m-j+1} \le t_{m-j+1} \le \dots \le u_1 \le t_1 \le x \le x_{k+1}$

$$\begin{aligned} S_k^{[j]}(x) = & S_{k-1}^{[m]}(x_k) + S'^{[m]}_{k-1}(x_k)(x - x_k) + \\ & + \int_{x_k}^{x}\int_{x_k}^{t_{m-j+1}} f_1[u_{m-j+1}, S_k^{[j-1]}(u_{m-j+1}), \bar{S}'^{[j-1]}_k(u_{m-j+1})]du_{m-j+1}dt_{m-j+1}, \end{aligned}$$

$$\begin{aligned} \bar{S}_k^{[j]}(x) = & \bar{S}_{k-1}^{[m]}(x_k) + \bar{S}'^{[m]}_{k-1}(x_k)(x - x_k) + \\ & + \int_{x_k}^{x}\int_{x_k}^{t_{m-j+1}} f_2[u_{m-j+1}, S_k^{[j-1]}(u_{m-j+1}), \bar{S}_k^{[j-1]}(u_{m-j+1})]du_{m-j+1}dt_{m-j+1}, \end{aligned}$$

$$
(6)\qquad
\begin{aligned}
S_k^{[0]}(x) &= S_{k-1}^{[m]}(x_k) + S'^{[m]}_{k-1}(x_k)(x - x_k) + \frac{N_k}{2}(x - x_k)^2, \\
\bar{S}_k^{[0]}(x) &= \bar{S}_{k-1}^{[m]}(x_k) + \bar{S}'^{[m]}_{k-1}(x_k)(x - x_k) + \frac{\bar{N}_k}{2}(x - x_k)^2, \\
N_k &= f_1[x_k, S_{k-1}^{[m]}(x_k), \bar{S}_{k-1}^{[m]}(x_k)], \\
\bar{N}_k &= f_2[x_k, S_{k-1}^{[m]}(x_k), \bar{S}_{k-1}^{[m]}(x_k)].
\end{aligned}
$$

By this construction, it is clear that $S_\Delta(x)$ and $\bar{S}_\Delta(x) \in C^1([0,1] \times R^2)$. We give the flow-chart diagram of the method.

The flow-chart diagram of the method:

Start

↓

Read

m, n

↓

k=0

$J = 1$

$x = x_k$

↓

Read

$S_{k-1}^{[m]}(x_k), \quad \bar{S}_{k-1}^{[m]}(x_k)$

$S'^{[m]}_{k-1}(x_k), \quad \bar{S}'^{[m]}_{k-1}(x_k)$

B

N_k

$\bar{N}_k$

↓

A

↓

$S_k^{[0]}(x), \quad \bar{S}_k^{[0]}(x)$

↓

$$S_k^{[j]}(x), \quad \bar{S}_k^{[j]}(x)$$

$J = J + 1$

↓

No if $J = m$

B

↑

$k = k + 1$

↑

Yes ↓

if $k = n - 1$ No

Yes ↓

Write

$$S_k(x) = S_k^{[m]}(x)$$
$$\bar{S}_k(x) = \bar{S}_k^{[m]}(x)$$

↓

Stop

2. Error estimations and convergence

For the purpose of error estimations, we write the exact solution in the following forms

(7)

$$y(x) \equiv y^{[m]}(x) = y_k + y_k'(x - x_k) + \int_{x_k}^{x}\int_{x_k}^{t_1} f_1[u_1, y^{[m-1]}(u_1), z^{[m-1]}(u_1)]du_1 dt_1,$$

(8)

$$z(x) \equiv z^{[m]}(x) = z_k + z_k'(x - x_k) + \int_{x_k}^{x}\int_{x_k}^{t_1} f_2[u_1, y^{[m-1]}(u_1), z^{[m-1]}(u_1)]du_1 dt_1.$$

We use the following notations: $y^{(i)}(x_k) = y_k^{(i)}$ and $z^{(i)}(x_k) = z_k^{(i)}$ where $i = 0$ and 1, also the following iterations are defined: for $x_k \leq u_m \leq t_m \leq u_{m-1} \leq \leq t_{m-1} \ldots \leq u_{m-j+1} \leq t_{m-j+1} \leq \ldots \leq u_1 \leq t_1 \leq x \leq x_{k+1}$, we have

(9)

$$y^{[j]}(x) = y_k + y_k'(x - x_k) + \\ + \int_{x_k}^{x} \int_{x_k}^{t_{m-j+1}} f_1[u_{m-j+1}, y^{[j-1]}(u_{m-j+1}), z^{[j-1]}(u_{m-j+1})] du_{m-j+1} dt_{m-j+1},$$

$$z^{[j]}(x) = z_k + z_k'(x - x_k) + \\ + \int_{x_k}^{x} \int_{x_k}^{t_{m-j+1}} f_2[u_{m-j+1}, y^{[j-1]}(u_{m-j+1}), z^{[j-1]}(u_{m-j+1})] du_{m-j+1} dt_{m-j+1},$$

$$y^{[0]}(x) = y_k + y_k'(x - x_k) + \frac{y''(\xi_k)(x - x_k)^2}{2},$$

$$z^{[0]}(x) = z_k + z_k'(x - x_k) + \frac{z''(\eta_k)(x - x_k)^2}{2},$$

where $\xi_k, \eta_k \in (x_k, x_{k+1})$. The functions $y''(\xi_k)$ and $z''(\eta_k)$ have moduli of continuity $\omega(y'', h)$ and $\omega(z'', h)$ where $\omega(f, h) = \sup\limits_{|x_1 - x_2| < h} |f(x_1) - f(x_2)|$. Moreover, we use the following notations

$$\begin{aligned} e^{(i)}(x) &= |y^{(i)}(x) - s_\Delta^{(i)}(x)|; \quad e_k^{(i)} = |y_k^{(i)} - s_\Delta^{(i)}(x_k)|, \\ \bar{e}^{(i)}(x) &= |z^{(i)}(x) - \bar{s}_\Delta^{(i)}(x)|; \quad \bar{e}_k^{(i)} = |z_k^{(i)} - \bar{s}_\Delta^{(i)}(x_k)|, \end{aligned} \tag{10}$$

where $i = 0$ and 1.

Lemma 1. *Let α and B be positive real numbers, $\{A_i\}_{i=1}^m$ is a sequence satisfying $A_1 \geq 0$ and $A_i \leq \alpha + BA_{i+1}$ for $i = 1(1)m - 1$, then*

$$A_1 \leq B^{m-1} A_m + \alpha \sum_{i=0}^{m-2} B^i. \tag{11}$$

Proof. Since $A_i \leq \alpha + BA_{i+1}$ for $i = 1(1)m - 1$, then

$$A_1 \leq \alpha + BA_2 \Rightarrow A_1 \leq \alpha + BA_2$$

$$A_2 \leq \alpha + BA_3 \Rightarrow BA_2 \leq \alpha B + B^2 A_3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_{m-1} \leq \alpha + BA_m \Rightarrow B^{m-2} A_{m-1} \leq \alpha B^{m-2} + B^{m-1} A_m.$$

By successive substitutions in these inequalities we get

$$A_1 \leq B^{m-1} A_m + \alpha \sum_{i=0}^{m-2} B^i.$$

Lemma 2. *Let α and B be nonnegative real numbers, $B \neq 1$, $\{A_i\}_{i=0}^{k}$ is a sequence satisfying $A_0 \geq 0$ and $A_{i+1} \leq \alpha + BA_i$ for $i = 0(1)k$, then*

$$A_{k+1} \leq B^{k+1} A_0 + \frac{\alpha[B^{k+1} - 1]}{B - 1}. \tag{12}$$

The proof is similar to that of Lemma 1.

Definition 1. For any $u_j \in [x_k, x_{k+1}]$, $j = 1(1)m$ and $k = 0(1)n-1$ we define the operator $T_k(u_j)$ by

$$T_k(u_j) = \left\{ \left(y^{[m-j]}(u_j) - s_k^{[m-j]}(u_j) \right) + \left(z^{[m-j]}(u_j) - \bar{s}_k^{[m-j]}(u_j) \right) \right\}, \tag{13}$$

whose norm is defined by

$$\|T_{k,j}\| = \max_{x_k \leq u_j \leq x_{k+1}} \left\{ |y^{[m-j]}(u_j) - s_k^{[m-j]}(u_j)| + |z^{[m-j]}(u_j) - \bar{s}_k^{[m-j]}(u_j)| \right\}.$$

Lemma 3. *For any $u_j \in [x_j, x_{k+1}]$, $j = 1(1)m$ and $k = 0(1)n-1$ if $T_k(u_j)$ satisfies (13), then*

$$\|T_{k,m}\| \leq \left[1 + \frac{(L_1 + L_2)}{2} \right] (e_k + \bar{e}_k) + e'_k + \bar{e}'_k + h^2 \omega(h) \tag{14}$$

and

$$\|T_{k,1}\| \leq a(e_k + \bar{e}_k) + b(e'_k + \bar{e}'_k) + ch^{2m} \omega(h), \tag{15}$$

where

$$a = \sum_{i=0}^{m} \left(\frac{L_1 + L_2}{2} \right)^i, \quad b = \sum_{i=0}^{m-1} \left(\frac{L_1 + L_2}{2} \right)^i \quad \text{and } C = \left(\frac{L_1 + L_2}{2} \right)^{m-1}$$

are constants independent of h.

Proof. By using (9), (6), (3) and (10), we get

$$\left|y^{[0]}(u_m) - s_k^{[0]}(u_m)\right| \le |y_k - s_{k-1}^{[m]}(x_k)| +$$
$$+|y'_k - s'^{[m]}_{k-1}(x_k)||u_m - x_k| + |y''_k(\xi_k) - N_k|\frac{|u_m - x_k|^2}{2!} \le$$
$$\le |y_k - s_k^{[m]}(x_k)| + |y'_k - s_k'^{[m]}(x_k)||u_m - x_k| + \frac{V_1|u_m - x_k|^2}{2!},$$

where

$$v_1 = |y''(\xi_k) - N_k| \le |y''(\xi_k) - y''_k| + |y''_k - N_k| \le$$
$$\le \omega(y'', h) + L_1\{|y_k - s_{k-1}^{[m]}(x_k)| + |z_k - \bar{s}_{k-1}^{[m]}(x_k)|\} \le$$
$$\le \omega(h) + L_1(e_k + \bar{e}_k).$$

Hence,
(i)

$$\max_{x_k \le u_m \le x_{k+1}} |y^{[0]}(u_m) - s_k^{[0]}(u_m)| \le e_k + he'_k + \frac{L_1 h^2}{2}(e_k + \bar{e}_k) + \frac{h^2}{2}\omega(h) \le$$
$$\le e + e'_k + \frac{L_1}{2}(e_k + \bar{e}_k) + \frac{h^2}{2}\omega(h).$$

Similarly, we obtain the inequality

$$(ii) \qquad \max_{x_k \le u_m \le x_{k+1}} |z^{[0]}(u_m) - \bar{s}_k^{[0]}(u_m)| \le \bar{e}_k + \bar{e}'_k + \frac{L_2}{2}(e_k + \bar{e}_k) + \frac{h^2}{2}\omega(h).$$

By adding (i) and (ii), we can prove

$$||T_{k,m}|| \le \left[1 + \frac{(L_1 + L_2)}{2}\right] (e_k + \bar{e}_k) + e'_k\bar{e}'_k + h^2\omega(h).$$

To prove (15), we compute $||T_{k,j}||$ using (9), (6), (3) and (10). Then we obtain

$$|y^{[m-j]}(u_j) - s_k^{[m-j]}(u_j)| \le |y_k - s_{k-1}^{[m]}(x_k)| + |y'_k - s'^{[m]}_{k-1}(x_k)||u_j - x_k| +$$
$$+ L_1 \int_{x_k}^{x} \int_{x_k}^{t_{j+1}} f_1\{|y^{[m-j-1]}(u_{j+1}) - s_k^{[m-j-1]}(u_{j+1})| +$$
$$+ |z^{[m-j-1]}(u_{j+1}) - \bar{s}_k^{[m-j-1]}(u_{j+1})|\}du_{j+1}dt_{j+1}.$$

Thus
(*iii*)

$$\max_{x_k \le u_j \le x_{k+1}} |y^{[m-j]}(u_j) - s_k^{[m-j]}(u_j)| \le e_k + he'_k +$$

$$+L_1 \int_{x_k}^{x} \int_{x_k}^{t_{j+1}} \max_{x_k \le u_{j+1} \le x_{k+1}} \{|y^{[m-j-1]}(u_{j+1}) - s_k^{[m-j-1]}(u_{j+1})| +$$

$$+|z^{[m-j-1]}(u_{j+1}) - \bar{s}_k^{[m-j-1]}(u_{j+1})|\} du_{j+1} dt_{j+1} \le e_k + e'_k + \frac{L_1 h^2}{2} ||T_{k,j+1}||.$$

In a similar manner

$$(iv) \qquad \max_{x_k \le u_j \le x_{k+1}} |z^{[m-j]}(u_j) - \bar{s}_k^{[m-j]}(u_j)| \le \bar{e}_k + \bar{e}'_k + \frac{L_2 h^2}{2} ||T_{k,j+1}||.$$

Adding (iii) and (iv), we get

$$||T_{k,j}|| \le (e_k + \bar{e}_k + e'_k + \bar{e}'_k) + \frac{(L_1 + L_2)}{2} h^2 ||T_{k,j+1}||.$$

Using (11) and (14), we get

$$||T_{k,1}|| \le \left(\frac{L_1 + L_2}{2}\right) h^{2m-2} ||T_{k,m}|| + (e_k + \bar{e}_k + e'_k + \bar{e}'_k) \cdot$$

$$\cdot \sum_{i=0}^{m-2} h^{2i} \left(\frac{L_1 + L_2^i}{2}\right) \le \left(\frac{L_1 + L_2}{2}\right)^{m-1} h^{2m-2} \left\{ \left[1 + \frac{L_1 + L_2}{2} \cdot \right.\right.$$

$$\left.\left. \cdot (e_k - \bar{e}_k) + e'_k + \bar{e}'_k + h^2 \omega h) \right\} + \right.$$

$$+ (e_k + \bar{e}_k + e'_k + \bar{e}'_k) \sum_{i=0}^{m-2} h^{2i} \left(\frac{L_1 + L_2}{2}\right)^i \le$$

$$\le (e_k + \bar{e}_k) \sum_{i=0}^{m} \left(\frac{L_1 + L_2}{2}\right)^i + (e'_k + \bar{e}'_k) \sum_{i=0}^{m-1} \left(\frac{L_1 + L_2}{2}\right)^i +$$

$$+ \left(\frac{L_1 + L_2}{2}\right)^{m-1} h^{2m} \omega(h).$$

Hence

$$||T_{k,1}|| \le a(e_k + \bar{e}_k) + b(e'_k + \bar{e}'_k) + c h^{2m} \omega(h),$$

where

$$a=\sum_{i=0}^{m}\left(\frac{L_1+L_2}{2}\right)^i,\ b=\sum_{i=0}^{m-1}\left(\frac{L_1+L_2}{2}\right)^i \text{ and } C=\left(\frac{L_1+L_2}{2}\right)^{m-1}$$

are constants independent of h.

Lemma 4. *Let $e^{(i)}(x)$ and $\bar{e}^{(i)}(x)$ be defined as in (10), where $i=0$ and 1, then there exist constants d_1, d_2, d_3 and c_1 independent of h such that the following inequalities hold true*

$$e(x)\le(1+d_1h)e_k+d_1h\bar{e}_k+d_2he'_k+d_3h\bar{e}'_k+c_1h^{2m+2}\omega(h), \tag{16}$$

$$\bar{e}(x)\le\bar{d}_1he_k+(1+\bar{d}_1h)\bar{e}_k+\bar{d}_2he'_k+\bar{d}_3h\bar{e}'_k+\bar{c}_1h^{2m+2}\omega(h), \tag{17}$$

$$e'(x)\le d_4he_k+d_4h\bar{e}_k+(1+d_5h)e'_k+d_5h\bar{e}'_k+c_2h^{2m+1}\omega(h) \tag{18}$$

and

$$\bar{e}'(x)\le\bar{d}_4he_k+\bar{d}_4h\bar{e}_k+\bar{d}_5he'_k+(1+\bar{d}_5h)\bar{e}'_k+\bar{c}_2h^{2m+1}\omega(h). \tag{19}$$

Proof. Using (7), (8), (4), (5), (3), (10) and (15), we get

$$\begin{aligned}
e(x)&=|y(x)-s_k(x)|\equiv|y^{[m]}(x)-s_k^{[m]}(x)|\le\\
&\le|y_k-s_{k-1}^{[m]}(x_k)|+|y'_k-s'^{[m]}_{k-1}(x_k)||x-x_k|+\\
&+L_1\int_{x_k}^{x}\int_{x_k}^{t_1}\{|y^{[m-1]}(u_1)-s_k^{[m-1]}(u_1)|+\\
&+|z^{[m-1]}(u_1)-\bar{s}_k^{[m-1]}(u_1)|\}du_1dt_1\le\\
&\le e_k+he'_k+L_1\int_{x_k}^{x}\int_{x_k}^{t_1}\max_{x_k\le u_1\le x_{k+1}}\{|y^{[m-1]}(u_1)-s_k^{[m-1]}(u_1)|+\\
&+|z^{[m-1]}(u_1)-\bar{s}_k^{[m-1]}(u_1)|\}du_1dt_1\le\\
&\le e_k+he'_k+L_1\|\Gamma_{k,1}\|\frac{h^2}{2}\le\\
&\le(1+d_1h)e_k+d_1h\bar{e}_k+d_2he'_k+d_3h\bar{e}'_k+c_1h^{2m+2}\omega(h),
\end{aligned}$$

where $d_1 = \frac{L_1 a}{2}$, $d_3 = \frac{L_1 b}{2}$, $d_2 = 1 + d_3$ and $c_1 = \frac{L_1 c}{2}$ are constants independent of h. Hence the lemma follows.

Let

$$\begin{aligned} E(x) &= [e(x)\bar{e}(x)e'(x)\bar{e}'(x)]^T, \\ E_k &= [e_k \bar{e}_k e'_k \bar{e}'_k]^T \qquad \text{and} \\ C &= [c\bar{c}_1 c_2 \bar{c}_2]^T, \end{aligned}$$

where T stands for the transpose. The initial conditions imply that $E_0 = = [0\ 0\ 0\ 0]^T$. From Lemma 4 we write

$$E(x) \leq (I + hA)E_k + ch^{2m+1}\omega(h), \tag{20}$$

where I is the unit matrix of order 4 and

$$A = \begin{bmatrix} d_1 & d_1 & d_2 & d_3 \\ \bar{d}_1 & \bar{d}_1 & \bar{d}_2 & \bar{d}_3 \\ d_4 & d_4 & d_5 & d_4 \\ \bar{d}_4 & \bar{d}_4 & \bar{d}_5 & \bar{d}_5 \end{bmatrix}.$$

Now, we give the definition of the matrix norm.

Definition. Let $T = [t_{ij}]$ be an $m \times n$ matrix, then we define

$$||T|| = \max_i \sum_j |t_{ij}|.$$

Using this definition equation (2) becomes

$$||E(x)|| \leq (1 + h||A||)||E_k|| + ||c||h^{2m+1}\omega(h).$$

This inequality is true for any $x \in [0, 1]$. Setting $x = x_{k+1}$ we get

$$||E_{k+1}|| \leq (1 + h||A||)||E_k|| + ||c||h^{2m+1}\omega(h),$$

then using (12) and noting that $||E_0|| = 0$, we get

$$\begin{aligned} ||E(x)|| &\leq ||c||h^{2m+1}\omega(h)\frac{[(1 + h||A||)^{k+1} - 1]}{1 + h||A|| - 1} \leq \\ &\leq \frac{||c||}{||A||}h^{2m}\omega(h)(e^{||A||} - 1) \leq Bh^{2m}\omega(h), \end{aligned} \tag{21}$$

where $B = \frac{\|c\|}{\|A\|}(e^{\|A\|} - 1)$ is a constant independent of h. Using (7), (8), (4), (5), (3), (15) and (21) it is easy to prove that

$$|y''(x) - s_k''(x)| \leq B_1 h^{2m}\omega(h)$$

and

$$|z''(x) - \bar{s}''_k(x)| \leq B_2 h^{2m}\omega(h),$$

where B_1 and B_2 are constants independent of h. Thus, we proved the following

Theorem 1. *Let $y(x)$ and $z(x)$ be the exact solutions to the problem (1)-(2). If $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ are the approximate solutions, given in (4)-(5), then the inequalities*

$$|y^{(q)}(x) - s_\Delta^{(q)}(x)| \leq B_3 h^{2m}\omega(h)$$

and

$$|z^{(q)}(x) - \bar{s}_\Delta^{(q)}(x)| \leq B_4 h^{2m}\omega(h)$$

hold true for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$ and $q = 0(1)2$, where B_3 and B_4 are constants independent of h.

3. Stability of the method

The stability concept for a one step method means that "small changes in the starting values only produce bounded changes in the numerical approximation provided by the method". To study the stability of the method, given in (4-5), we change $s_\Delta(x)$ by $\omega_\Delta(x)$ and $\bar{s}_\Delta(x)$ by $\bar{w}_\Delta(x)$, where

(22)
$$w_\Delta(x) \equiv w_k^{[m]}(x) = w_{k-1}^{[m]}(x_k) + w'^{[m]}_{k-1}(x_k)(x - x_k) + \\ + \int_{x_k}^{x}\int_{x_k}^{t_1} f_1[u_1, w_k^{[m-1]}(u_1),\ w_k^{[m-1]}(u_1)dy_1 dt_1,$$

(23)
$$\bar{w}_\Delta(x) \equiv w_k^{[m]}(x) = w_{k-1}^{[m]}(x_k) + w'^{[m]}_{k-1}(x_k)(x - x_k) + \\ + \int_{x_k}^{x}\int_{x_k}^{t_1} f_2[u_1, w_k^{[m-1]}(u_1),\ \bar{w}_k^{[m-1]}\bar{w}_k(u_1)]du_1 dt_1,$$

where
$$w_{-1}^{[m]}(x_0) = y_0^*,\ w'^{[m]}_{-1}(x_0) = y'^*_0,\ \bar{w}_{-1}^{[m]}(x_0) = z_0^*$$

and $\bar{w}'^{[m]}_{-1}(x_0) = z'^*_0$. In equations (22) and (23) we use the following m iteration processes. For any $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, $j = 1(1)m$, $x_k \le u_m \le t_m \le$ $\le u_{m-1} \le t_{m-1} \le \dots \le u_{m-j+1} \le t_{m-j+1} \le \dots \le u_1 \le t_1 \le x \le x_{k+1}$,

$$w_k^{[j]}(x) = w_{k-1}^{[m]}(x_k) + w'^{[m]}_{k-1}(x_k)(x - x_k) +$$
$$+ \int_{x_k}^{x} \int_{x_k}^{t_{m-j+1}} f_1[u_{m-j+1},\ w_k^{[j-1]}(u_{m-j+1}), \bar{w}_k^{[j-1]}(u_{m-j+1})] du_{m-j+1} dt_{m-j+1},$$
$$\bar{w}_k^{[j]}(x) = \bar{w}_{k-1}^{[m]}(x_k) + \bar{w}'^{[m]}_{k-1}(x_k)(x - x_k) +$$
$$+ \int_{x_k}^{x} \int_{x_k}^{t_{m-j+1}} f_2[u_{m-j+1}, w_k^{[j-1]}(u_{m-j+1}), \bar{w}_k^{[j-1]}(u_{m-j+1})] du_{m-j+1} dt_{m-j+1},$$

(24)
$$\begin{aligned} w_k^{[0]}(x) &= w_{k-1}^{[m]}(x_k) + w'^{[m]}_{k-1}(x_k)(x - x_k) + \frac{M_k}{2}(x - x_k)^2, \\ \bar{w}_k^{[0]}(x) &= \bar{w}_{k-1}^{[m]}(x_k) + \bar{w}'^{[m]}_{k-1}(x_k)(x - x_k) + \frac{\bar{M}_k}{2}(x - x_k)^2, \\ M_k &= f_1[x_k, w_{k-1}^{[m]}(x_k), \bar{w}_{k-1}^{[m]}(x_k)], \\ \bar{M}_k &= f_2[x_k, w_{k-1}^{[m]}(x_k), \bar{w}_{k-1}^{[m]}(x_k)]. \end{aligned}$$

Moreover, the following notations are used

(25)
$$\begin{aligned} e^*(x) &= |w_\Delta(x) - S_\Delta(x)|;\ e_k^* = |w_\Delta(x_k) - S_\Delta(x_k)|, \\ e'^*(x) &= |w'_\Delta(x) - S'_\Delta(x)|;\ e'^*_k = |w'_\Delta(x_k) - S'_\Delta(x_k)|, \\ \bar{e}^*(x) &= |\bar{w}_\Delta(x) - \bar{S}_\Delta(x)|;\ \bar{e}_k^* = |\bar{w}(x_k) - \bar{S}_\Delta(x_k)|, \\ \bar{e}'^*(x) &= |\bar{w}'_\Delta(x) - \bar{S}'_\Delta(x)|;\ \bar{e}'^*_k = |\bar{w}'_\Delta(x_k) - \bar{S}'_k(x_k)|. \end{aligned}$$

Definition 2. For any $u_j \in [x_k, x_{k+1}]$, $j = 1(1)m$ and $k = 0(1)n-1$, we define the operator $T_k^*(u_j)$ as

(26)
$$T_k^*(u_j) = \{(w_k^{[m-j]}(u_j) - S_k^{[m-j]}(u_j)) + (\bar{w}_k^{[m-j]}(u_j) - \bar{S}_k^{[m-j]}(u_j))\}$$

with the norm

$$||T^*_{k,j}|| = \max_{x_k \le u_j \le x_{k+1}} \{|w_k^{[m-j]}(u_j) - S_k^{[m-j]}(u_j)| + |\bar{w}_k^{[m-j]}(u_j) - \bar{S}_k^{[m-j]}(u_j)|\}.$$

Lemma 5. *For any* $u_j \in [x_k, x_{k+1}]$, $j = 1(1)m$ *and* $k = 0(1)n-1$, *if* $T^*_k(u_j)$ *satisfies (26) then*

$$||T^*_{k,m}|| \le \left[1 + \frac{L_1 + L_2}{2}\right] (e^*_k + \bar{e}^*_k) + (e'^*_k + \bar{e'}^*_k), \tag{27}$$

$$||T^*_{k,1}|| \le a(e^*_k + \bar{e}^*_k) + b(e'^*_k + \bar{e'}^*_k). \tag{28}$$

The proof is similat to that of Lemma 3.

Lemma 6. Let $e^{*(i)}(x)$ *and* $\bar{e}^{*(i)}(x)$ *be defined as in (25), where* $i = 0$ *and* 1, *then there exist constants* d_1, d_2 *and* d_3 *independent of* h *such that the following inequalities hold true*

$$e^*(x) \le (1 + d_1 h)e^*_k + d_1 h\bar{e}^*_k + d_2 h e'^*_k + d_3 h\bar{e}^*_k, \tag{29}$$

$$\bar{e}^*(x) \le \bar{d}_1 h e^*_k + (1 + \bar{d}_1 h)\bar{e}^*_k + \bar{d}_2 h e'^*_k + \bar{d}_3 h\bar{e}'^*_k, \tag{30}$$

$$e'^*(x) \le d_4 h e^*_k + d_4 h\bar{e}^*_k + (1 + d_5 h)e'^*_k + d_5 h\bar{e}'^*_k \tag{31}$$

and

$$\bar{e}'^*(x) \le \bar{d}_4 h e^*_k + \bar{d}_4 h\bar{e}'^*_k + \bar{d}_5 h e'_k + (1 + \bar{d}_5 h)\bar{e}'^*_k. \tag{32}$$

The proof is similar to that of Lemma 4.

Now, let $E^*(x) = [e^*(x)\bar{e}^*(x)\bar{e}'^*(x)\bar{e}'^*(x)]^T$ and $E^*_k = [e^*_k\bar{e}^*_k\bar{e}'^*_k\bar{e}'^*_k]^T$. Then, we write $E^*(x)$ in the form

$$E^*(x) \le (I + hA)E^*_k, \tag{33}$$

where I and A are the matrices defined in Lemma 4.

Let $||E^*(x)|| = ||E^*(x)||_\infty$, then (33) becomes

$$||E^*(x)|| \le (1 + h||A||)||E^*_k||.$$

Using the same technique, used for deriving inequality (21), we get

$$(34)\qquad ||E^*(x)|| \le (1 + h||A||)^{k+1}||E_0|| \le e^{||A||}||E_0|| = B_5||E_0||,$$

where $B_5 = e^{||A||}$ is a constant independent of h. Using (4), (5), (22), (23), (3), (28) and (34), we get

$$|w_k''(x) - S_k''(x)| \le B_6||E_0^*||$$

and

$$|\bar{w}_k''(x) - \bar{S}_k''(x)| \le B_7||E_0^*||,$$

where B_6 and B_7 are constants independent of h.

Theorem 2. *If $S_\Delta(x)$ and $\bar{S}_\Delta(x)$, given in (4-5), are the approximate solutions of the problem (1-2) with initial values $y^{(i)}(x_0) = y_0^{(i)}$ and $z^{(i)}(x_0) = = z_0^{(i)}$ and if $w_\Delta(x)$ and $\bar{w}_\Delta(x)$, given in (22-23) are the other solutions for the problem with initial $y^{(i)}(x_0) = y_0^{*(i)}$ and $z^{(i)}(x_0) = z_0^{*(i)}$, where $i = 0$ and 1, then the following inequalities*

$$|w_\Delta^{(q)}(x) - S_\Delta^{(q)}(x)| \le B_8||E_0^*||$$

and

$$|\bar{w}_\Delta^{(q)}(x) - \bar{S}_\Delta^{(q)}(x)| \le B_9||E_0^*||$$

hold true for any $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$ and all $q = 0(1)2$, where B_8 and B_9 are constants independent of h.

$$||E_0^*|| = \max\{|y_0 - y_0^*|,\ |y_0' - y'^*_0|,\ |z_0 - z_0^*|,\ |z_0' - z'^*_0|\}.$$

Numerical example. Consider the following system of differential equations

$$y" = y + z - \bar{e}^*, \quad y(0) = 1,\ y'(0) = 0,$$
$$z" = y + z - e^x, \quad y(0) = 1,\ z'(0) = 0.$$

The method is tested by using the above example in the interval $[0, 1]$ with step size $h = 0 - 1$ for different values of m. The exact solutions are

$$y(x) = e^x - x \quad \text{and} \quad z(x) = \bar{e}^x + x.$$

The tabulated results are calculated at $x = 1$. To test the stability of the method, we solve the above with different initial values

$$y(0) = 1.000001, \quad y'(0) = 0.000001,$$
$$z(0) = 1.000001 \quad \text{and} \quad z'(0) = 0.000001.$$

The errors

ERROR	$M=1$	$M=3$	$M=5$
e	$1.312826E-05$	$2.925854E-12$	$7.499005E-16$
e'	$4.821804E-05$	$1.116665E-11$	$9.159340E-16$
e''	$3.801709E-04$	$1.716113E-10$	$1.443290E-15$
$\bar{e}$	$1.312826E-05$	$2.925882E-12$	$7.771561E-16$
$\bar{e}'$	$4.811804E-05$	$1.116673E-11$	$1.0130785E-15$
$\bar{e}''$	$3.801709E-04$	$1.716113E-10$	$1.422473E-15$

Stability of the method

Diff	$M=1$	$M=3$	$M=5$
e^*	$3.546324E-06$	$3.546482E-06$	$3.546482E-06$
e'^*	$4.914292E-06$	$4.914781E-06$	$4.914781E-06$
e''^*	$7.089711E-06$	$7.092965E-06$	$7.092964E-06$
$\bar{e}^*$	$3.546324E-06$	$3.546482E-06$	$3.546482E-06$
$\bar{e}'^*$	$4.914292E-06$	$4.914781E-06$	$4.914781E-06$
$\bar{e}''^*$	$7.089711E-06$	$7.092965E-06$	$7.092965E-06$

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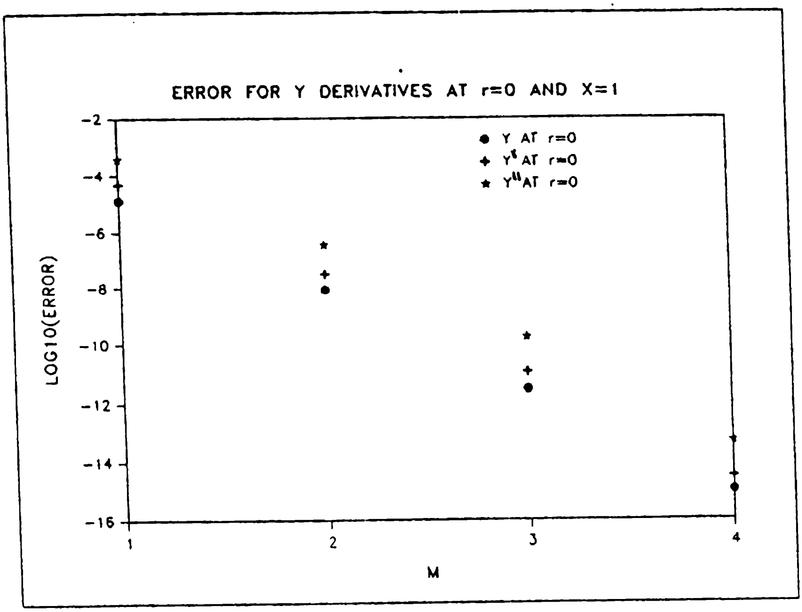

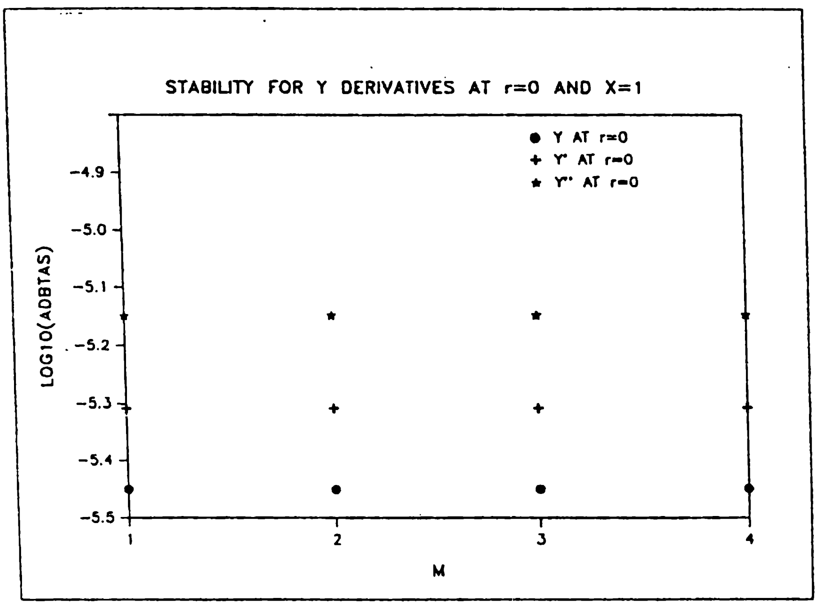

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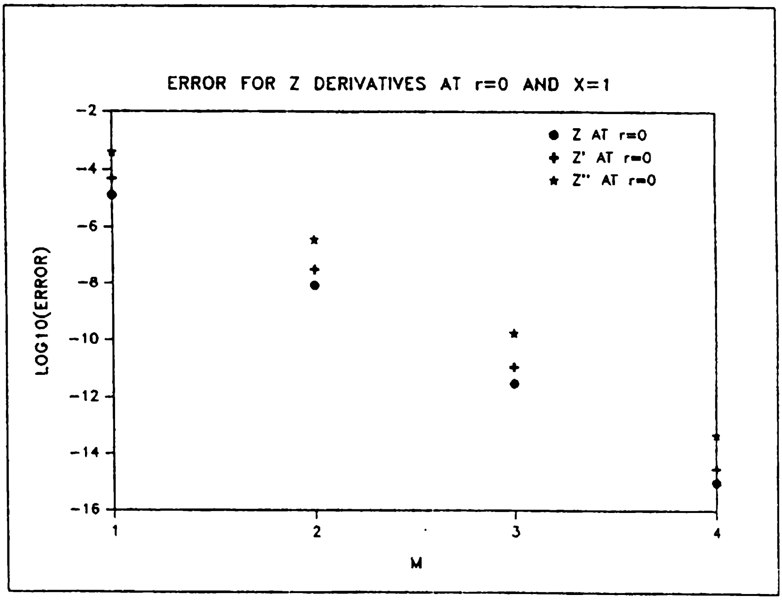

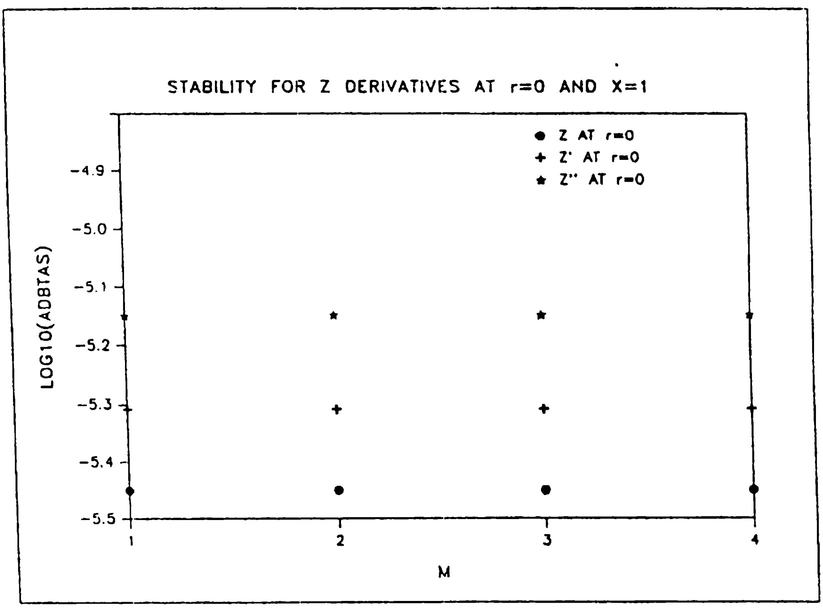

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