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ON THE NUMERICAL SOLUTION OF A SYSTEM OF THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS BY SPLINE FUNCTIONS

Z. Ramadan (Cairo, Egypt)

Abstract. The purpose of this paper is to construct spline function approximations for solving the system of differential equations

$$y''' = f_1(x,y,y',z,z'), \quad z''' = f_2(x,y,y',z,z')$$

with $y^{(i)}(x_0) = y_0^{(i)}$ and $z^{(i)}(x_0) = z_0^{(i)}$, where $i = 0(1)2$.
The approximating functions used in the method are polynomial splines. It is shown that the method is a one-step method $O(h^{\alpha+r})$ in $y^{(i)}(x)$, $z^{(i)}(x)$, $i = 0(1)2$ and $O(h^{\alpha+r+3-q})$ in $y^{(q)}(x)$, $z^{(q)}(x)$ where $q = 3(1)r+3$, also shown that the method is stable.

1. Assumptions and procedures

Consider the system of differential equations

$$y''' = f_1(x,y,y',z,z'), \quad y^{(i)}(x_0) = y_0^{(i)}, \tag{1.1}$$

$$z''' = f_2(x,y,y',z,z'), \quad z^{(i)}(x_0) = z_0^{(i)}, \tag{1.2}$$

where $f_1, f_2 \in C^r([0,1] \times R^4)$, $i = 0(1)2$.

Let Δ be the partition

$$\Delta \; : \; 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1,$$

where $x_{k+1} - x_k = h < 1$ and $k = 0(1)n-1$.

Let L_1 and L_2 be the Lipschitz constants satisfied by the functions $f_1^{(q)}$, $f_2^{(q)}$ respectively, i.e.

$$\begin{aligned} &|f_i^{(q)}(x, y_1, y_1', z_1, z_1') - f_i^{(q)}(x, y_2, y_2', z_2, z_2')| \le \\ &\le L_i \left\{ |y_1 - y_2| + |y_1' - y_2'| + |z_1 - z_2| + |z_1' - z_2'| \right\}, \quad i = 1, 2 \end{aligned} \tag{1.3}$$

for all $(x, y_1, y_1', z_1, z_1')$, $(x, y_2, y_2', z_2, z_2')$ in the domain of definition of the functions $f_1^{(q)}, f_2^{(q)}$, where $q = 0(1)r$.

The functions $f_1^{(q)}$, $i = 1, 2$ and $q = 1(1)r$ are functions of x, y, y', z, z' only and they are given from the following algorithm.

Set $f_i^{(0)} = f_i\ (x, y, y', z, z')$ and if $f_i^{(q-1)}$ are defined, then

$$f_i^{(q)} = \frac{\partial f_i^{(q-1)}}{\partial x} + \frac{\partial f_i^{(q-1)}}{\partial y} y' + \frac{\partial f_i^{(q-1)}}{\partial y'} y'' + \frac{\partial f_i^{(q-1)}}{\partial z} z' + \frac{\partial f_i^{(q-1)}}{\partial z'} z''.$$

Then, we define the spline functions approximating $y(x)$ and $z(x)$ by $S_\Delta(x)$ and $\bar{S}_\Delta(x)$, where

(1.4)

$$\begin{aligned} S_\Delta(x) \equiv S_k(x) = S_{k-1}(x_k) + S_{k-1}'(x_k)(x - x_k) + S_{k-1}''(x_k)\frac{(x - x_k)^2}{2!} + \\ + \sum_{j=0}^{r} f_1^{(j)}\left[x_k, S_{k-1}(x_k), S_{k-1}'(x_k), \bar{S}_{k-1}(x_k), \bar{S}_{k-1}'(x_k)\right] \frac{(x - x_k)^{j+3}}{(j+3)!} \end{aligned}$$

and

(1.5)

$$\begin{aligned} \bar{S}_\Delta(x) \equiv \bar{S}_k(x) = \bar{S}_{k-1}(x_k) + \bar{S}_{k-1}'(x_k)(x - x_k) + \bar{S}_{k-1}'(x_k)\frac{(x - x_k)^2}{2!} + \\ + \sum_{j=0}^{r} f_2^{(j)}\left[x_k, S_{k-1}(x_k), S_{k-1}'(x_k), \bar{S}_{k-1}(x_k), \bar{S}_{k-1}'(x_k)\right] \frac{(x - x_k)^{j+3}}{(j+3)!}, \end{aligned}$$

where $S_{-1}^{(i)}(x_0) = y_0^{(i)}, \bar{S}_{-1}^{(i)}(x_0) = z_0^{(i)}, i = 0(1)2$.

By construction, it is clear that $S_\Delta(x), \bar{S}_\Delta(x) \in C^2([0, 1] \times R^4)$.

2. Error estimations and convergence

For all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, let the exact solution of (1.1) and (1.2) be written in the following forms

$$y(x) = \sum_{j=0}^{r+2} \frac{y_k^{(j)}}{j!}(x - x_k)^j + y^{(r+3)}(\xi_k)\frac{(x - x_k)^{r+3}}{(r+3)!} \tag{2.1}$$

and

$$z(x) = \sum_{j=0}^{r} \frac{z_k^{(j)}}{j!}(x - x_k)^j + z^{(r+3)}(\eta_k)\frac{(x - x_k)^{r+3}}{(r+3)!}, \tag{2.2}$$

where $\xi_k, \eta_k \in (x_k, x_{k+1})$ and $k = 0(1)n-1$.

Before we proceed to discuss the convergence of these spline approximants, we state first the following notations

$$\begin{aligned}
e(x) &= |y(x) - S_\Delta(x)|, \\
e_k &= |y_k - S_\Delta(x_k)|, \\
\bar{e}(x) &= |z(x) - \bar{S}_\Delta(x)|, \\
\bar{e}_k &= |z_k - \bar{S}_\Delta(x_k)|, \\
f_{1,k}^{(j)} &= f_1^{(j)}[x_k, S_{k-1}(x_k), S'_{k-1}(x_k), \bar{S}_{k-1}(x_k), \bar{S}'_{k-1}(x_k)], \\
f_{2,k}^{(j)} &= f_2^{(j)}[x_k, S_{k-1}(x_k), S'_{k-1}(x), \bar{S}_{k-1}(x_k), \bar{S}'_{k-1}(x_k)], \\
f_{1,k}^{*(j)} &= f_1^{(j)}[x_k, y_k, y'_k, z_k, z'_k], \\
f_{2,k}^{*(j)} &= f_2^{(j)}[x_k, y_k, y'_k, z_k, z'_k],
\end{aligned}$$

where $j = 0(1)r$ and $k = 0(1)n-1$.

Throughout this work we will consider the general subinterval

$$I_k = [x_k, x_{k-1}], \quad k = 0(1)n-1.$$

First, we estimate $|y(x) - S_k(x)|$. Using (1.4), (2.1), the Lipschitz condition (1.3) and the notations (2.3) we get

$$e(x) \le |y_k - S_{k-1}(x_k)| + \tag{2.4}$$

$$+ |y'_k - S'_{k-1}(x_k)| \cdot |x - x_k| + |y''_{k-1}(x_k) - S''_{k-1}(x_k)| \cdot \frac{|x - x_k|^2}{2!} +$$

$$+ \sum_{j=0}^{r-1} \left| y_k^{(j+3)} - f_{1,k}^{(j)} \right| \frac{|x - x_k|^{j+3}}{(j+3)!} + \left| y^{(r+3)}(\xi_k) - f_{1,k}^{(r)} \right| \frac{|x - x_k|^{r+3}}{(r+3)!} \leq$$

$$\leq e_k + he'_k + \frac{h^2}{2!} e''_k +$$

$$+ \sum_{j=0}^{r-1} \left| y_k^{(j+3)} - f_{1,k}^{(j)} \right| \frac{h^{j+3}}{(j+3)!} + \left| y^{(r+3)}(\xi_k) - f_{1,k}^{(r)} \right| \frac{h^{r+3}}{(r+3)!}.$$

If we let

$$P = \left| y_k^{(j+3)} - f_{1,k}^{(j)} \right|,$$

then, using (1.3) and (2.3), we get

$$P \leq L_1 (e_k + e'_k + \bar{e}_k + \bar{e}'_k) \tag{2.5}$$

Also, let

$$\hat{P} = \left| y^{(r+3)}(\xi_k) - f_{1,k}^{(r)} \right|,$$

then, using (1.3) and (2.3), we get

$$\hat{P} \leq \omega \left(y^{(r+3)}, h \right) + L_1 (e_k + e'_k + \bar{e}_k + \bar{e}'_k), \tag{2.6}$$

where $\omega(y^{(r+3)}, h)$ is the modulus of continuity of the function $y^{(r+3)}$.

Using (2.5) and (2.6) and noting that

$$\sum_{j=0}^{r-1} \frac{h^{j+2}}{(j+3)!} < e^h - 2 < e,$$

we can easily get

(2.7)

$$e(x) \leq (1 + c_0 h) e_k + c_0 h \bar{e}_k + (1 + c_0) h e'_k + c_0 h \bar{e}'_k + \frac{h^2}{2!} e''_k + \frac{h^{r+3}}{(r+3)!} \omega(y^{(r+3)}, h),$$

where $c_0 = L_1 \left(e + \frac{1}{(r+3)!} \right)$ is a constant independent of h.

In a similar manner, using (1.5), (2.2), the Lipschitz condition (1.3) and the notations (2.3), it can be easily shown that
(2.8)

$$\bar{e}(x) \leq c_1 he_k + (1+c_1 h)\bar{e}_k + c_1 he'_k + (1+c_1)h\bar{e}'_k + \frac{h^2}{2!}\bar{e}''_k + \frac{h^{(r+3)}}{(r+3)!}\omega(z^{(r+3)}, h),$$

where $\omega(z^{(r+3)}, h)$ is the modulus of continuity of the function $z^{(r+3)}$ and $c_1 = = L_2\left(e + \frac{1}{(r+3)!}\right)$ is a constant independent of h.

Now, we are going to estimate $|y'(x) - s'_k(x)|$ and $|z'(x) - \bar{S}'_k(x)|$.

Using (1.3)-(2.3) and noting that

$$\sum_{j=0}^{r} \frac{h^{j+1}}{(j+2)!} < e - 1 < e,$$

we can easily get

$$(2.9)\quad e'(x) \leq c_2 he_k + c_2 h\bar{e}_k + (1+c_2 h)e'_k + c_2 h\bar{e}'_k + he''_k + \frac{h^{r+2}}{(r+2)!}\omega(y^{(r+3)}, h)$$

and

$$(2.10)\quad \bar{e}'(x) \leq c_3 he_k + c_3 h\bar{e}_k + c_3 he'_k + (1+c_3 h)\bar{e}'_k + h\bar{e}''_k + \frac{h^{r+2}}{(r+2)!}\omega(z^{(r+3)}, h),$$

where $c_2 = L_1\left(e + \frac{1}{(r+2)!}\right)$ and $c_3 = L_2\left(e + \frac{1}{(r+2)!}\right)$ are constants independent of h.

We now estimate $|y''(x) - S''_k(x)|$ and $|z''(x) - \bar{S}''_k(x)|$.

Using equations (1.3)-(2.3) and utilizing the inequality

$$\sum_{j=0}^{r-1} \frac{h^j}{(j+1)!} < e$$

we can see that

$$(2.11)\quad e''(x) \leq c_4 he_k + c_4 h\bar{e}_k + c_4 he'_k + c_4 h\bar{e}'_k + e''_k + \frac{h^{r+1}}{(r+1)!}\omega(y^{(r+3)}, h)$$

and

$$(2.12)\quad \bar{e}''(x) \le c_5 h e_k + c_5 h \bar{e}_k + c_5 h e'_k + c_5 h \bar{e}'_k + \bar{e}''_k + \frac{h^{r+1}}{(r+1)!}\omega(z^{(r+3)}, h),$$

where $c_4 = L_1\left(e + \dfrac{1}{(r+1)!}\right)$ and $c_5 = L_2\left(e + \dfrac{1}{(r+1)!}\right)$ are constants independent of h.

To complete the convergence proof, we introduce the following definition of the matrix inequality

Definition 1. Let $A = [a_{i,j}]$, $B = [b_{i,j}]$ be two matrices of the same order, then we say that $A \le B$ iff

(i) $a_{i,j}$ and $b_{i,j}$ are nonnegative,

(ii) $a_{i,j} \le b_{i,j} \quad \forall i, j$.

In view of this definition and if we use the matrix notations

$$E(x) = \begin{pmatrix} e(x) & \bar{e}(x) & e'(x) & \bar{e}'(x) & e''(x) & \bar{e}''(x) \end{pmatrix}^T$$

and

$$E_k = (e_k \ \bar{e}_k \ e'_k \ \bar{e}'_k \ e''_k \ \bar{e}''_k)^T, \quad k = 0(1)n-1,$$

we can write the estimations (2.7)-(2.12) in the following form

$$(2.13)\qquad E(x) \le (I + hA)E_k + h^{r+1}\omega(h)B,$$

where

$$A = \begin{bmatrix} c_0 & c_0 & 1+c_0 & c_0 & 1/2! & 0 \\ c_1 & c_1 & c_1 & 1+c_1 & 0 & 1/2! \\ c_2 & c_2 & c_2 & c_2 & 1 & 0 \\ c_3 & c_3 & c_3 & c_3 & 0 & 1 \\ c_4 & c_4 & c_4 & c_4 & 0 & 0 \\ c_5 & c_5 & c_5 & c_5 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1/(r+3)! \\ 1/(r+3)! \\ 1/(r+2)! \\ 1/(r+2)! \\ 1/(r+1)! \\ 1/(r+1)! \end{bmatrix},$$

I is the identity matrix of order 6 and

$$\omega(h) = \max\left\{\omega(y^{(r+3)}, h),\ \omega(z^{(r+3)}, h)\right\}.$$

Next, we give the following definition of the matrix norm.

Definition 2. Let $T = [\tau_{ij}]$ be an $m \times n$ matrix, then we define

$$||T|| = \max_i \sum_{j=1}^{n} |\tau_{ij}|.$$

According to this definition, we get

$$||E(x)|| = \max\left\{e(x),\ \bar{e}(x),\ e'(x),\ \bar{e}'(x),\ e''(x),\ \bar{e}''sf(x)\right\}. \tag{2.14}$$

Since (2.13) is valid for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, then the following inequalities hold true

$$||E(x)|| \le (I + h||A||)||E_k|| + h^{r+1}\omega(h)||B||,$$
$$(1 + h||A||)||E_k|| \le (I + h||A||)^2||E_{k-1}|| + h^{r+1}\omega(h)||B||(1 + h||A||),$$
$$(1 + h||A||)^2||E_{k-1}|| \le (I + h||A||)^3||E_{k-2}|| + h^{r+1}\omega(h)||B||(1 + h||A||)^2,$$

....

$$(1 + h||A||)^k||E_1|| \le (I + h||A||)^{k+1}||E_0|| + h^{r+1}\omega(h)||B||(1 + h||A||)^k.$$

Adding L.H.S. and R.H.S. of these inequalities and noting that $||E_0|| = 0$, we get

$$||E(x)|| \le c_6 h^r \omega(h),$$

where $c_6 = \left(e^{||A||} - 1\right)\dfrac{||B||}{||A||}$ is a constant independent of h.

Thus using (2.14), we get

$$e^{(i)}(x) \le c_6 h^r \omega(h) = O(h^{\alpha+r}),$$

$$\bar{e}^{(i)}(x) \le c_6 h^r \omega(h) = O(h^{\alpha+r}), \tag{2.15}$$

where $i = 0(1)2$.

We are going to estimate $|y^{(q)}(x) - S_k^{(q)}(x)|$, where $q = 3(1)r+2$.

Using (1.3), (1.4), (2.1), (2.3), (2.5), (2.6) and (2.15), we get

$$|y^{(q)}(x) - S_k^{(q)}(x)| \le \sum_{j=q-3}^{r-1} |y_k^{(j+3)} - f_{1,k}^{(j)}|\frac{|x - x_k|^{j+3-q}}{(j+3-q)!} +$$
$$+ |y^{(r+3)}(\xi_k) - f_{1,k}^{(r)}|\frac{|x - x_k|^{r+3-q}}{(r+3-q)!} \le$$
$$\le c_7 h^{r+3-q}\omega(h) = O(h^{\alpha+r+3-q}),$$

where $c_7 = 4L_1c_6\left(e + \frac{1}{(r+3-q)!}\right) + \frac{1}{(r+3-q)!}$ is a constant independent of h.

Similarly, using (1.3), (1.5), (2.2), (2.3), (2.5), (2.6) and (2.15), it can be shown that

$$|z^{(q)}(x) - \bar{S}_k^{(q)}(x)| \le c_8 h^{r+3-q}\omega(h) = O(h^{\alpha+r+r-q}),$$

where $q = 3(1)r+2$ and $c_8 = 4L_2c_6\left(e + \frac{1}{(r+3-q)!}\right) + \frac{1}{(r+3-q)!}$ is a constant independent of h.

For the case $q = r+3$, we have

$$\begin{aligned}|y^{(r+3)}(x) - S_k^{(r+3)}| &= |y^{(r+3)}(x) - f_{1,k}^{(r)}| \le \\ &\le |y^{(r+3)} - y_k^{(r+3)}| + |f_{1,k}^{*(r)} - f_{1,k}^{(r)}| \le \\ &\le c_9\omega(h) = O(h^\alpha).\end{aligned}$$

Similarly,

$$|z^{(r+3)}(x) - \bar{S}_k^{(r+3)}| \le c_{10}\omega(h) = O(h^\alpha),$$

where $c_9 = 1 + 4L_1c_6$ and $c_{10} = 1 + 4L_2c_6$ are constants independent of h.

Thus, we have proved the following

Theorem 1. *Let $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ be the approximate solutions to problem (1.1)-(1.2) given by the equations (1.4)-(1.5), and let $f_1 f_2 \in C^r\left([x_0, x_n]\times \times R^4\right)$, then for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, we have*

$$\begin{aligned}|y^{(i)}(x) - S_k^{(i)}(x)| &\le Ch^r\omega(h),\\ |z^{(i)}(x) - \bar{S}_k^{(i)}(x)| &\le Ch^r\omega(h),\\ y^{(j)}(x) - S_k^{(j)}(x)| &\le Kh^{r+3-j}\omega(h)\end{aligned}$$

and

$$|z^{(j)}(x) - \bar{S}_k^{(j)}(x)| \le K^* h^{r+3-j}\omega(h),$$

where $i = 0(1)2$, $j = 3(1)r+3$, C, K and K^* are constants independent of h.

3. Stability of the method

The stability concept for a one-step method means that small perturbations in the initial data for the numerical method will result in small changes in the numerical values, independent of the grid size h of the numerical method.

To study the stability of the method given by (1.4)-(1.5), we change $S_\Delta(x)$ by $W_\Delta(x)$ and $\bar{S}_\Delta$ by $\overline{W}_\Delta(x)$, where
(3.1)

$$W_\Delta(x) \equiv W_k(x) = W_{k-1}(x_k) + W'_{k-1}(x_k)(x - x_k) + W''_{k-1}(x_k)\frac{(x-x_k)^2}{2!} +$$

$$+ \sum_{j=0}^{r} f_1^{(j)}\left\{x_k, W_{k-1}(x_k), W'_{k-1}(x_k), \overline{W}_{k-1}(x_k), \overline{W}'_{k-1}(x_k)\right\} \cdot \frac{|x-x_k|^{j+3}}{(j+3)!}$$

and
(3.2)

$$\overline{W}_\Delta(x) \equiv \overline{W}_k(x) = \overline{W}_{k-1}(x_k) + \overline{W}'_{k-1}(x_k)(x - x_k) + \overline{W}''_{k-1}(x_k)\frac{(x-x_k)^2}{2!} +$$

$$+ \sum_{j=0}^{r} f_2^{(j)}\left\{x_k, W_{k-1}(x_k), W'_{k-1}(x_k), \overline{W}_{k-1}(x_k), \overline{W}'_{k-1}(x_k)\right\} \cdot \frac{|x-x_k|^{j+3}}{(j+3)!},$$

where $W_{-1}^{(i)}(x_0) = y_{0^*(i)}$, $\overline{W}_{-1}^{(i)}(x_0) = z_0^{*(i)}$, $i = 0(1)2$.

We define the following notations

$$\varepsilon(x) = |W_\Delta(x) - S_\Delta(x)|, \quad \varepsilon_k = |W_\Delta(x_k) - S_\Delta(x_k)|,$$

$$\bar{\varepsilon}(x) = |\overline{W}_\Delta(x) - \bar{S}_\Delta(x)|, \quad \bar{\varepsilon}_k = |\overline{W}_\Delta(x_k) - \bar{S}_\Delta(x_k)|, \tag{3.3}$$

$$\hat{f}_{1,k}^{(j)} = f_1^{(j)}[x_k, W_{k-1}(x_k), W'_{k-1}(x_k), \overline{W}_{k-1}(x_k), \overline{W}'_{k-1}(x_k)]$$

and

$$\hat{f}_{2,k}^{(j)} = f_2^{(j)}[x_k, W_{k-1}(x_k), W'_{k-1}(x_k), \overline{W}_{k-1}(x_k), \overline{W}'_{k-1}(x_k)].$$

For all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, by using (1.4), (3.1), we get

$$|W_\Delta(x) - S_\Delta(x)| \le \tag{3.4}$$

$$\le |W_{k-1}(x_k) - S_{k-1}(x_k)| + |W'_{k-1}(x_k) - S'_{k-1}(x_k)||x - x_k| +$$

$$+|W''_{k-1}(x_k) - S''_{k-1}(x_k)|\frac{|x-x_k|^2}{2!} + \sum_{j=0}^{r} |\hat{f}^{(j)}_{1,k} - f^{(j)}_{1,k}| \cdot \frac{|x-x_k|^{j+3}}{(j+3)!}.$$

Now, let

$$\hat{V}_1 = \left|\hat{f}^{(j)}_{1,k} - f^{(j)}_{1,k}\right|. \tag{3.5}$$

Then, from (2.3), (3.3) and the Lipschitz condition (1.3), we get

$$\hat{V}_1 \leq L_1(\varepsilon_k + \varepsilon'_k + \bar{\varepsilon}_k + \bar{\varepsilon}'_k). \tag{3.6}$$

Thus, (3.4) gives

$$\varepsilon(x) \leq (1+d_0h)\varepsilon_k + d_0h\bar{\varepsilon}_k + (1+d_0)h\varepsilon'_k + d_0h\bar{\varepsilon}'_k + \frac{h^2}{2!}\varepsilon''_k, \tag{3.7}$$

where $d_0 = L_1 e$ is a constant independent of h.

In a similar manner, by using (1.4), (1.5), (3.1)-(3.3) and the Lipschitz condition (1.3), it can be shown that

$$\begin{aligned}
\bar{\varepsilon}(x) &\leq d_1h\varepsilon_k + (1+d_1h)\bar{\varepsilon}_k + d_1h\varepsilon'_k + (1+d_1)h\bar{\varepsilon}'_k + \frac{h^2}{2!}\bar{\varepsilon}''_k,\\
\varepsilon'(x) &\leq d_0h\varepsilon_k + d_0h\bar{\varepsilon}_k + (1+d_0h)\varepsilon'_k + d_0h\bar{\varepsilon}'_k + h\varepsilon''_k,\\
\bar{\varepsilon}'(x) &\leq d_1h\varepsilon_k + d_1h\bar{\varepsilon}_k + d_1h\varepsilon'_k + (1+d_1h)\bar{\varepsilon}'_k + h\bar{\varepsilon}''_k,\\
\varepsilon''(x) &\leq d_0h\varepsilon_k + d_0h\bar{\varepsilon}_k + d_0h\varepsilon'_k + d_0h\bar{\varepsilon}'_k + \varepsilon''_k
\end{aligned} \tag{3.8}$$

and

$$\bar{\varepsilon}''(x) \leq d_1h\varepsilon_k + d_1h\bar{\varepsilon}_k + d_1h\varepsilon'_k + d_1h\bar{\varepsilon}'_k + \bar{\varepsilon}''_k,$$

where $d_1 = L_2 e$, is a constant independent of h. If we put

$$\hat{E}(x) = \begin{pmatrix}\varepsilon(x) & \bar{\varepsilon}(x) & \varepsilon'(x) & \bar{\varepsilon}'(x) & \varepsilon''(x) & \bar{\varepsilon}''(x)\end{pmatrix}^T$$

and

$$\hat{E}_k = (\varepsilon_k \quad \bar{\varepsilon}_k \quad \varepsilon'_k \quad \bar{\varepsilon}'_k \quad \varepsilon''_k \quad \bar{\varepsilon}''_k)^T, \quad k = 0(1)n-1, \tag{3.9}$$

then, from (3.7)-(3.9), we get the following inequality

$$\hat{E}(x) \leq (I + h\hat{A})\hat{E}_k, \tag{3.10}$$

where

$$\hat{A} = \begin{bmatrix} d_0 & d_0 & 1+d_0 & d_0 & 1/2! & 0 \\ d_1 & d_1 & d_1 & 1+d_1 & 0 & 1/2! \\ d_0 & d_0 & d_0 & d_0 & 1 & 0 \\ d_1 & d_1 & d_1 & d_1 & 0 & 1 \\ d_0 & d_0 & d_0 & d_0 & 0 & 0 \\ d_1 & d_1 & d_1 & d_1 & 0 & 0 \end{bmatrix}$$

and I is the identity matrix of order 6.

Since (3.10) is valid for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, then the following inequalities hold true

$$||\hat{E}(x)|| \leq (1 + h||\hat{A}||)||\hat{E}k||,$$

$$(1 + h||\hat{A}||)||\hat{E}_k|| \leq (1 + h||\hat{A}||)^2||\hat{E}_{k-1}||,$$

$$(1 + h||\hat{A}||)^k||\hat{E}_1|| \leq (1 + h||\hat{A}||)^{k+1}||\hat{E}_0||.$$

Adding L.H.S. and R.H.S. of these inequalities, we can easily get

$$||E(x)|| \leq c_1||\hat{E}_0||, \tag{3.11}$$

where $c_1 = e^{||\hat{A}||}$ is a constant independent of h.

Applying Definition 2, we get

$$\varepsilon^{(i)}(x) \leq c_1||\hat{E}_0||$$

and

$$\bar{\varepsilon}^{(i)}(x) \leq c_1||\hat{E}_0||, \tag{3.12}$$

where $||\hat{E}_0|| = \max\{|y_0 - y_0^*|,\ |y_0' - y_0^{*\prime}|,\ y_0'' - y_0^{*\prime\prime}|,\ |z_0 - z_0^*|,\ |z_0' - z_0^{*\prime}|,\ |z_0'' - z_0^{*\prime\prime}|\}$ and $i = 0(1)2$.

We are going to estimate $|W_\Delta^{(q)}(x) - S_\Delta^{(q)}(x)|$, where $q = 3(1)r + 3$.

Using (1.4), (3.1), (3.6) and (3.11), we get

$$\left|W_\Delta^{(q)}(x) - S_\Delta^{(q)}(x)\right| \leq \sum_{j=q-3}^{r} \left|\hat{f}_{1,k}^{(j)} - f_{1,k}^{(j)}\right| \frac{|x - x_k|^{j+3-q}}{(j+3-q)!} \leq \tag{3.13}$$
$$\leq d^*||\hat{E}_0||,$$

where $d^* = 4L_1ec_1$ is a constant independent of h.

In a similar manner, using (1.5), (3.2) (3.11), it can be shown that

$$\left|\overline{W}_\Delta^{(q)}(x) - \bar{S}_\Delta^{(q)}(x)\right| \le \bar{d}^* \|\hat{E}_0\|, \tag{3.14}$$

where $\bar{d}^* = 4L_2ec_1$ is a constant independent of h and $q = 3(1)r + 3$.

Thus, we have proved the following

Theorem 2. *Let $(S_\Delta, \bar{S}_\Delta(x))$ given by (1.4)-(1.5) be the approximate solution to problem (1.1)-(1.2) with the initial conditions $y^{(i)}(x_0) = y_0^{(i)}$ and $z^{(i)}(x_0) = z_0^{(i)}$, and let $(W_\Delta(x), \overline{W}_\Delta(x))$ given by (3.1)-(3.2) be the approximate solution for the same problem with the initial conditions $y^{(i)}(x_0) = = y_0^{*(i)}$, $z^{(i)}(x_0) = z_0^{*(i)}$, $i = 0(1)2$, then the inequalities*

$$\left|W_\Delta^{(q)}(x) - S_\Delta^{(q)}(x)\right| \le \bar{c}\|\hat{E}_0\|$$

and

$$\left|\overline{W}_\Delta^{(q)}(x) - \bar{S}_\Delta^{(q)}(x)\right| \le \bar{k}\|\hat{E}_0\|$$

hold true for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n - 1$ and $q = 0(1)r + 3$, $r \in I^+$ where $\bar{c}, \bar{k}$ are constants independent of h and

$$\|\hat{E}_0\| = \max\{|y_0^{(i)} - y_0^{*(i)}|, |z_0^{(i)} - z_0^{*(i)}|\}, \quad i = 0(1)2.$$

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Z. Ramadan
Department of Mathematics
Faculty of Education
Ain Shams University
Cairo, Egypt