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ON THE POSITIVITY OF ITERATIVE METHODS

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Abstract. In this paper we study the positivity of some vector sequences produced by given vector-iteration. In our investigation we apply the well-known power method (e.g. [5]). We give some sufficient conditions of the positivity of the generated vector sequence depending both on the initial vector and on the matrix of the iteration. Applying this result we formulate a sufficient condition of the power-positivity of a given quadratic matrix. Furthermore, we consider the numerical solution of the one dimensional heat conduction equation. Considering the results of [1] we give a condition that guaranties the positivity of the approximating vector sequence. Finally, we obtain some bounds for parameters of the discretization scheme. In the case of $n \geq 2$ we get a well-known sufficient condition, which was obtained by use of the Lorenz criterion ([4]).

In this paper we use the following notations:

$N_n := \{1, 2, ..., n\}$ is a set of indices; $S(R^{n \times n})$ is the class of the symmetric, real matrices of order n; $(\mathbf{A})_k$ is the k-th column of the matrix $\mathbf{A}$; $(\mathbf{v})_l$ is the l-th element of the vector $\mathbf{v}$; $\| \mathbf{v} \|_\infty$ denotes the maximum norm of the vector $\mathbf{v}$. We denote by λ_k $(k \in N_n)$ the eigenvalues of the matrix $\mathbf{A}$ $(\mathbf{A} \in S(R^{n \times n}))$ and we suppose that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ is valid. We shall say that an eigenvalue λ_r is dominant if $|\lambda_{r-1}| > |\lambda_r| > |\lambda_{r+1}|$ is fulfilled. It is obvious that we can choose orthonormal eigenvectors. These eigenvectors are denoted by $\mathbf{v_k}$ $(k \in N_n)$.

1. The power method

Lemma 1. *Let $\mathbf{A} \in S(R^{n \times n})$ be an arbitrary matrix with the eigenvalues and eigenvectors λ_k and $\mathbf{v_k}$ $(k \in N_n)$, respectively. Let $\mathbf{y}^{(0)} \in R^n$, $\mathbf{y}^{(0)} \neq 0$, be* *an arbitrary vector. Let us denote by* $\sigma \in N_n$ *that index for which* $(\mathbf{y}^{(0)}, \mathbf{v_r}) = 0$ *for every* $r < \sigma$ $(r \in N_n)$ *and* $(\mathbf{y}^{(0)}, \mathbf{v}_\sigma) \neq 0$. *If the eigenvalue* λ_σ *is positive and dominant then the procedure*

$$(1.1)\quad \mathbf{z}^{(i+1)} = \mathbf{A}\mathbf{y}^{(i)}; \qquad \mathbf{y}^{(i+1)} = \frac{\mathbf{z}^{(i+1)}}{\| \mathbf{z}^{(i+1)} \|_\infty}, \qquad i = 0, 1, 2, \ldots$$

is convergent and the vector sequence $\mathbf{y}^{(i)}$ *has the limit*

$$(1.2)\qquad \lim_{i\to\infty} \mathbf{y}^{(i)} = \operatorname{sign}\big((\mathbf{y}^{(0)}, \mathbf{v}_\sigma)\big) \frac{\mathbf{v}_\sigma}{\| \mathbf{v}_\sigma \|_\infty}\ .$$

Remark. The index σ depends both on the matrix $\mathbf{A}$ and on the vector $\mathbf{y}^{(0)}$, too. Since for an arbitrary $\mathbf{A} \in S(R^{n\times n})$ the vectors $(\mathbf{v_k})$ $(k \in N_n)$ form a basis in R^n so there exists such index $\sigma \in N_n$ for which $(\mathbf{y}^{(0)}, \mathbf{v}_\sigma) \neq 0$.

Proof. (Compare e.g. [5]) We can write the vector $\mathbf{y}^{(0)}$ in the basis $(\mathbf{v_k})$ in the form

$$(1.3)\qquad \mathbf{y}^{(0)} = \sum_{k=1}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k})\mathbf{v_k}.$$

From the iteration (1.1) it follows immediately that

$$(1.4)\qquad \mathbf{y}^{(i)} = \frac{\mathbf{A}^i\mathbf{y}^{(0)}}{\| \mathbf{A}^i\mathbf{y}^{(0)} \|_\infty}, \qquad i = 1, 2, \ldots.$$

Applying the formula (1.3) we have

$$\mathbf{A}^i\mathbf{y}^{(0)} = \mathbf{A}^i \left(\sum_{k=1}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k})\mathbf{v_k} \right) = \sum_{k=1}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k})\lambda_k^i \mathbf{v_k} =$$

$$(1.5)\qquad = \lambda_\sigma^i \Big((\mathbf{y}^{(0)}, \mathbf{v}_\sigma)\mathbf{v}_\sigma + \sum_{k=\sigma+1}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k}) \left(\frac{\lambda_k}{\lambda_\sigma}\right)^i \mathbf{v_k} \Big),$$

$$\| \mathbf{A}^i\mathbf{y}^{(0)} \|_\infty = \| \sum_{k=\sigma}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k})\lambda_k^i \mathbf{v_k} \|_\infty =$$

$$= |\lambda_\sigma^i| \, \| (\mathbf{y}^{(0)}, \mathbf{v}_\sigma)\mathbf{v}_\sigma + \sum_{k=\sigma+1}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k}) \left(\frac{\lambda_k}{\lambda_\sigma}\right)^i \mathbf{v_k} \|_\infty \, .$$

Using (1.5) the expression (1.4) can be rewritten in the form

$$\text{(1.6)} \qquad \mathbf{y}^{(i)} = \frac{\lambda_\sigma^i((\mathbf{y}^{(0)}, \mathbf{v}_\sigma)\mathbf{v}_\sigma + \sum_{k=\sigma+1}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k})(\frac{\lambda_k}{\lambda_\sigma})^i \mathbf{v_k})}{|\lambda_\sigma^i| \, \| (\mathbf{y}^{(0)}, \mathbf{v}_\sigma)\mathbf{v}_\sigma + \sum_{k=\sigma+1}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k})(\frac{\lambda_k}{\lambda_\sigma})^i \mathbf{v_k} \|_\infty}.$$

Finally, aproaching $i \to \infty$ we obtain the statement of the Lemma 1.

2. Application of the power method in vector-iteration

In general the power method (1.1) is used to obtain the eigenvector corresponding to the eigenvalue λ_σ. Further we will use this method to the investigation of vector sequences (1.1). For the sake of brevity we introduce the following definitions.

Definition. An arbitrary matrix $\mathbf{A} \in R^{n \times n}$ is called positive if all elements of the matrix are positive. In notation: $\mathbf{A} > 0$.

In a similar manner we can define and introduce the notion of a negative matrix. (Obviously, we can apply these definitions also to the vectors.)

Definition. An arbitrary $\mathbf{A} \in R^{n \times n}$ is called a power-positive matrix if there exists such natural number M that $\mathbf{A}^m > 0$ for all $m \geq M$ $(m \in N)$.

(Obviously any positive matrix is power-positive, too.)

Definition. Let $\{\alpha_m\}$ be any numerical, vector or matrix sequence. The sequence $\{\alpha_m\}$ is called quasi-positive (or quasi-negative) if there exists a natural number m_0 such that $\alpha_m > 0$ (or $\alpha_m < 0$) for every $m \geq m_0$ $(m \in N)$. (If $m_0 = 1$ then we call the sequence positive (or negative).)

Let $\mathbf{A} \in S(R^{n \times n})$ be an arbitrary matrix, $\mathbf{y}^{(0)} \neq 0 \in R^n$ an arbitrary vector and $l_0 \in N_n$ be a fixed index, respectively. We denote by $\eta = \eta(l_0, \mathbf{A}, \mathbf{y}^{(0)})$ the smallest index in N_n for which $(\mathbf{y}^{(0)}, \mathbf{v}_\eta) \neq 0$ and $(\mathbf{v}_\eta)_{l_0} \neq 0$. The value of η depends on the index l_0, the matrix $\mathbf{A}$ and the vector $\mathbf{y}^{(0)}$. It is easy to see that $\eta \geq \sigma$. However we remark that the index η may not exist for certain indices l_0. (For this case we shall give an example later.)

Lemma 2. *Let us consider the iteration*

$$\mathbf{y}^{(i+1)} = \mathbf{A}\mathbf{y}^{(i)} \quad ; \qquad i = 0, 1, 2, \ldots \tag{2.1}$$

where $\mathbf{A}$ *is a matrix from* $S(R^{n \times n})$ *and* $\mathbf{y}^{(0)} \neq 0$ *is an arbitrary vector. Let* $l_0 \in N_n$ *be a fixed index for which the* $\eta \in N_n$ *index exists. Furthermore, we suppose that the eigenvalue* λ_η *is positive and dominant. If* $(\mathbf{y}^{(0)}, \mathbf{v}_\eta)(\mathbf{v}_\eta)_{l_0} > 0$ *then the number sequence* $(\mathbf{y}^{(i)})_{l_0}$ *is quasi-positive and if* $(\mathbf{y}^{(0)}, \mathbf{v}_\eta)(\mathbf{v}_\eta)_{l_0} < 0$ *then it is quasi-negative, respectively.*

Proof. From the definition of the iteration (2.1) it follows directly that $\mathbf{y}^{(i)} = \mathbf{A}^i\mathbf{y}^{(0)}$. So, the sign of the elements of the vector $\mathbf{y}^{(i)}$ is identical with the sign of the elements of the vector

$$\mathbf{w}^{(i)} := \frac{\mathbf{A}^i\mathbf{y}^{(0)}}{\| \mathbf{A}^i\mathbf{y}^{(0)} \|_\infty}, \qquad i = 0, 1, 2, \ldots. \tag{2.2}$$

Corresponding to Lemma 1 if $i \to \infty$ then the vector sequence $\mathbf{w}^{(i)}$ $(i \in N)$ converges to its limit, that is

$$\lim_{i \to \infty} \mathbf{w}^{(i)} = \operatorname{sign}\big((\mathbf{y}^{(0)}, \mathbf{v}_\sigma)\big) \frac{\mathbf{v}_\sigma}{\| \mathbf{v}_\sigma \|_\infty}. \tag{2.3}$$

If $\eta = \sigma$ then the statement follows directly from the expression (2.3). If $\eta > \sigma$ then it can be seen from (2.3) that the numerical sequence $(\mathbf{w}^{(i)})_{l_0}$ converges to zero. In this case let us consider directly the values of $(\mathbf{y}^{(i)})_{l_0}$.

$$(\mathbf{y}^{(i)})_{l_0} = \sum_{k=\eta}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k})\lambda_k^i(\mathbf{v_k})_{l_0} = \tag{2.4}$$

$$= \lambda_\eta^i \left[(\mathbf{y}^{(0)}, \mathbf{v}_\eta)(\mathbf{v}_\eta)_{l_0} + \sum_{k=\eta+1}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k}) \left(\frac{\lambda_k}{\lambda_\eta}\right)^i (\mathbf{v_k})_{l_0} \right].$$

It can be seen that for $i \to \infty$ the multiplier $(\mathbf{y}^{(0)}, \mathbf{v}_\eta)(\mathbf{v}_\eta)_{l_0}$ determinates the sign of the elements of $(\mathbf{y}^{(i)})_{l_0}$. This completes the proof of the lemma.

Corollary. Let us suppose that $\mathbf{v}_\sigma > 0$. Then in the case of $(\mathbf{y}^{(0)}, \mathbf{v}_\sigma) > 0$ the vector sequence $\mathbf{y}^{(i)}$ is quasi-positive and in the case of $(\mathbf{y}^{(0)}, \mathbf{v}_\sigma) < 0$, it is quasi-negative, respectively.

Remark. If the index η does not exist (that is for every $r \in N_n$ we have $(\mathbf{y}^{(0)}, \mathbf{v_r}) = 0$ or $(\mathbf{v_r})_{l_0} = 0$) then

$$(2.5) \qquad (\mathbf{y}^{(i)})_{l_0} = \sum_{k=1}^{n} (\mathbf{y}^{(0)}, \mathbf{v_k}) \lambda_k^i (\mathbf{v_k})_{l_0} = 0, \qquad i = 0, 1, 2, \ldots$$

i.e. the l_0-th element of the vectors $\mathbf{y}^{(i)}$ is zero for every $i = 0, 1, 2, \ldots$.

We give now a condition for the power-positivity of a real, symmetric matrix.

Lemma 3. *Let* $\mathbf{A} \in S(R^{n \times n})$ *with a dominant and positive eigenvalue* λ_1. *If* $\mathbf{v}_1 > 0$ *then the matrix* $\mathbf{A}$ *is power-positive.*

Proof. $(\mathbf{A}^i)_k$, the k-th column of the matrix $\mathbf{A}^i$, can be written in the form $(\mathbf{A}^i)_k = \mathbf{A}^i \mathbf{e}_k$, where $\mathbf{e}_k$ denotes the k-th unit vector. Since $\lambda_1 > 0$ and $(\mathbf{e}_k, \mathbf{v}_1) > 0$ for any $k \in N_n$, we have iterations with the starting vectors $\mathbf{y}^{(0)} = \mathbf{e}_k$ $(k \in N_n)$. By applying the corollary of the previous lemma it is clear that $\mathbf{A}$ is power-positive.

3. Analysis of the numerical solution of the heat conduction equation

Let us consider the parabolic problem having the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2}, \qquad t > 0, \qquad \xi \in (0,1),$$

$$(3.1) \qquad u(0,t) = u(1,t) = 0, \qquad t \geq 0,$$

$$u(\xi, 0) = u_0(\xi), \qquad \xi \in [0,1].$$

The numerical solution of this problem can be obtained in every grid-point of an equidistant (τ, h) mesh by solving the following systems of linear algebraic equations (see e.g. [3])

$$(3.2) \qquad (\mathbf{E} + \theta\tau\mathbf{Q})\mathbf{y}^{(j+1)} = (\mathbf{E} - (1-\theta)\tau\mathbf{Q})\mathbf{y}^{(j)}, \; j = 0, 1, 2, \ldots .$$

Here τ and $h = \frac{1}{n+1}$ are the step-sizes of the discretization in the time and space variables, respectively; $\mathbf{E}$ denotes the unit matrix and $\mathbf{Q}$ is the uniformly continuant matrix $\frac{1}{h^2} tridiag[-1, 2, -1]$. The vector $\mathbf{y}^{(0)}$ is an approximation of the initial function $u_0(\xi)$. The parameter θ characterizing the discretization is a fixed number in $[0, 1]$. Introducing the notations $q := \frac{\tau}{h^2}$ and

$$z = \theta q; \qquad s = (1-\theta)q; \qquad p = 1 - 2q(1-\theta); \qquad x = \frac{1+2\theta q}{\theta q}, \tag{3.3}$$

the system of the linear algebraic equations (3.2) can be written in the following form

$$\mathbf{X_1}\mathbf{y}^{(j+1)} = \mathbf{X_2}\mathbf{y}^{(j)}, \qquad j = 0, 1, \dots . \tag{3.4}$$

Here the matrices

$$\begin{aligned} \mathbf{X_1} &= z \cdot tridiag[-1, x, -1], \\ \mathbf{X_2} &= tridiag[s, p, s] \end{aligned} \tag{3.5}$$

are symmetric, uniformly continuant matrices. If $\theta = 0$ then $\mathbf{X}_1 = \mathbf{E}$. Since $\mathbf{X_1}$ is invertable, so introducing the notation

$$\mathbf{K} := \mathbf{X_1^{-1}}\mathbf{X_2}$$

(3.4) can be rewritten in the form

$$\mathbf{y}^{(j+1)} = \mathbf{K}\mathbf{y}^{(j)}, \qquad j = 0, 1, 2, \dots . \tag{3.6}$$

We shall examine the following problem: under which conditions produces the iteration (3.6) a quasi-positive (quasi-negative) vector sequence? To this aim let us apply Lemma 2 checking that for the matrix $\mathbf{K}$ all conditions of the lemma are satisfied.

a) The matrix $\mathbf{K}$ is symmetric because it can be written in the form

$$\mathbf{K} = \frac{1}{z}[(xs + p)\mathbf{G} - s\mathbf{E}],$$

where the matrix $\mathbf{G}$ is a symmetric matrix ([1]).

b) The eigenvalues and eigenvectors of the matrix $\mathbf{K}$ are given by

$$\Lambda_k = 1 - \frac{\tau\omega_k}{1 + \theta\tau\omega_k}, \tag{3.7}$$

$$(\mathbf{v_k})_i = \sqrt{\frac{2}{n+1}} \sin\left(\frac{ik\pi}{n+1}\right), \qquad i, k \in N_n,$$

where $\omega_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi}{2(n+1)}\right)$ $(k \in N_n)$ are the eigenvalues of the matrix $\mathbf{Q}$ (see e.g. [2]).

Notice that it is doubtful that the indexing of the eigenvalues in (3.7) satisfies the conditions $|\Lambda_1| \geq |\Lambda_2| \geq \ldots \geq |\Lambda_n|$. But it is easy to see that

$$\Lambda_1 > \Lambda_2 > \ldots > \Lambda_n. \tag{3.8}$$

From the expression (3.7) it can be seen directly that the eigenvector $\mathbf{v_1}$ is positive. Now let $\mathbf{y}^{(0)}$ be such an initial vector for which $\sigma = 1$. For the quasi-positivity (or quasi-negativity) of the vector sequence (3.6) it is sufficient to show that the eigenvalue Λ_1 is positive and dominant. The inequality $\Lambda_1 > 0$ is assured under the condition

$$\frac{1-(1-\theta)\tau\omega_1}{1+\theta\tau\omega_1} > 0. \tag{3.9}$$

Substituting here the value of ω_1 we obtain the following inequality

$$q < \frac{1}{4(1-\theta)\sin^2\left(\frac{\pi}{2(n+1)}\right)}, \qquad \text{if } \theta \in [0,1]. \tag{3.10}$$

Notice that for $\theta = 1$ (3.10) holds for any q.

To ensure that Λ_1 is a dominant eigenvalue we require $\Lambda_1 > |\Lambda_n|$. If $\Lambda_n \geq 0$ then this condition is automatically fulfilled. In case of $\Lambda_n < 0$ we get the condition $\Lambda_1 > -\Lambda_n$. Due to

$$\omega_1 = \frac{4}{h^2}\sin^2\left(\frac{\pi}{2(n+1)}\right), \tag{3.11}$$

$$\omega_n = \frac{4}{h^2}\cos^2\left(\frac{\pi}{2(n+1)}\right),$$

the above requirement gives the following condition with respect to q:

$$4\theta(\theta-1)\sin^2\left(\frac{\pi}{n+1}\right)q^2 + 4\left(\theta-\frac{1}{2}\right)q + 1 > 0, \quad \text{if } \theta \in [0,1]. \tag{3.12}$$

In case $\theta = 1$, no condition arises.

We can summarize our results as follows:

Lemma 4. *If the parameters q and θ satisfy the conditions (3.10) and (3.12) and $\mathbf{y}^{(0)}$ is such an initial vector for which $\sigma = 1$ then the vector-sequence $\mathbf{y}^{(i)}$ defined by (3.6) is*

a) quasi-positive if $(\mathbf{y}^{(0)}, \mathbf{v_1}) > 0$,

b) quasi-negative if $(\mathbf{y}^{(0)}, \mathbf{v_1}) < 0$.

Remark. It was stated earlier that η may not exist to every index $l_0 \in N_n$ (see before Lemma 2). For example let the initial vector be $\mu\mathbf{v_2}$, $(0 \neq \mu \in R)$ and n an arbitrary odd natural number. Then in the case of $l_0 = \frac{n+1}{2}$ the index η satisfying the prescribed conditions does not exist. This is easy to see since $(\mathbf{v_2})_{l_0} = 0$ and $(\mu\mathbf{v_2}, \mathbf{v_k}) = 0$ if $k \neq 2$. Consequently for every $j = 0, 1, \ldots$ we have $\left(\mathbf{K}^j(\mu\mathbf{v_2})\right)_{l_0} = 0$.

Notice to Lemma 4 that $(\mathbf{y}^{(0)}, \mathbf{v}_1) > 0$ ($(\mathbf{y}^{(0)}, \mathbf{v}_1) < 0$) for any $0 \neq \mathbf{y}^{(0)} \geq 0$ $(0 \neq \mathbf{y}^{(0)} \leq 0)$ vectors since $\mathbf{v}_1 > 0$. Furthermore we consider the power-positivity of the matrix $\mathbf{K}$.

Lemma 5. *If the conditions (3.10) and (3.12) are fulfilled then the matrix* $\mathbf{K}$ *is power-positive.*

Proof. It is sufficient to show that the matrix $\mathbf{K}$ satisfies the conditions of Lemma 3. Since $\mathbf{v}_1 > 0$ the statement of the lemma is trivially valid.

Let us consider the conditions (3.10) and (3.12) in more detail. We want to obtain sufficient upper bounds for τ in terms of θ and $n+1$ (where $\theta \in [0, 1]$, $n > 0$) which are more practicable for use than that of (3.10) and (3.12). Since $\sin(\frac{\pi}{n+1}) < \frac{\pi}{n+1}$ for every $n > 0$ (3.10) leads to the condition

$$\tau \leq \frac{1}{\pi^2(1-\theta)}, \qquad \theta \in [0, 1]. \tag{3.15}$$

For $\theta = 0$ and $\theta = 1$ the expression (3.12) is linear in q, therefore we obtain the following upper bounds

$$(i) \qquad \tau < \frac{1}{2(n+1)^2} \qquad \text{if } \theta = 0,$$

$$(ii) \qquad \tau < \infty \qquad \text{if } \theta = 1.$$

Now suppose that $\theta \in (0, 1)$. If q is between the two roots of the quadratic expression (3.12) then (3.12) is satisfied. These roots are

$$q_{1,2} = \frac{2(\theta - \frac{1}{2}) \pm \sqrt{1 - 4\theta(1-\theta)\cos^2\frac{\pi}{n+1}}}{4\theta(1-\theta)\sin^2\frac{\pi}{n+1}}. \tag{3.16}$$

Since the absolute value of $2\theta - 1$ is smaller than the square root of the discriminant and since q is positive we obtain the bound

$$0 < q < \frac{2(\theta - \frac{1}{2}) + \sqrt{1 - 4\theta(1-\theta)\cos^2 \frac{\pi}{n+1}}}{4\theta(1-\theta)\sin^2 \frac{\pi}{n+1}}. \tag{3.17}$$

From (3.17) the following sufficient upper bounds can be derived for τ.

(*iii*) $$\tau \le \frac{2\theta - 1}{2\theta(1-\theta)\pi^2} \quad \text{if } \theta \in (0.5, 1),$$

(*iv*) $$\tau \le \frac{1}{\pi(n+1)} \quad \text{if } \theta = 0.5,$$

(*v*) $$\tau \le \frac{1}{2(1-\theta)(n+1)^2} \quad \text{if } \theta \in (0, 0.5).$$

Notice that the upper bounds (i)-(v) obtained from (3.12) are for any $\theta \in [0, 1)$ smaller than (3.15) which is obtained from (3.10). Therefore, the matrix $\mathbf{K}$ is power-positive for every n if the parameters τ and θ satisfy the adequate inequality from (i)-(v).

Finally we formulate a sufficient condition for the positivity of the matrix $\mathbf{K}$ having order greater one.

Lemma 6. *The matrix* $\mathbf{K}$ *of order greater than one is positive if the parameters* q *and* θ *satisfy one the following conditions*

$$\begin{aligned} &a) \quad q < 0.5, \quad \text{if } \theta = 0, \\ &b) \quad \text{for any } q, \quad \text{if } \theta = 1, \\ &c) \quad q < \frac{-1 + 2\theta + \sqrt{1 - \theta(1-\theta)}}{3\theta(1-\theta)}, \quad \text{if } \theta \in (0,1). \end{aligned} \tag{3.18}$$

Remark. *This means that under condition (3.18) the iteration (3.6) for* $n \ge 2$ *preserves the positivity.*

Proof. In the case $n = 2$ it can be seen that the condition (3.17) gives a stronger bound than (3.10). From this follows immediately that the matrix $\mathbf{K}$ of order greater than one is power-positive if

$$\begin{aligned} &q < 0.5 && \text{if } \theta = 0, \\ &\text{for any } q && \text{if } \theta = 1, \end{aligned}$$

and

$$q < \frac{-1 + 2\theta + \sqrt{1 - \theta(1 - \theta)}}{3\theta(1 - \theta)} \qquad \text{if } \theta \in (0, 1). \tag{3.19}$$

Using the results of [1] we know that the matrix **K** can contain nonpositive elements only on the main diagonal. But such **K** matrix cannot be power-positive. So, under the condition (3.19), the matrix **K** of order two is positive. It follows immediately from [1] that every matrix **K** of order greater than two is also positive. This completes the proof.

Remark. The bound (3.18) for $n \geq 2$ was obtained by Stoyan using the Lorenz criterion ([4]). Using the results of Faragó ([1]) this bound can also be derived.

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