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THE COEFFICIENTS OF DIFFERENTIATED EXPANSIONS OF DOUBLE AND TRIPLE ULTRASPHERICAL POLYNOMIALS

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Abstract. The tensor product of orthogonal ultraspherical (Gegenbauer) polynomials is used to approximate a function of more than one variable. Formulae expressing the coefficients of differentiated expansions of double and triple ultraspherical polynomials in terms of the coefficients of the original expansion are stated and proved. The special cases of double and triple Chebyshev polynomials are also considered. An application of how to use double ultraspherical polynomials for solving Poisson's equation inside a square subject to nonhomogeneous mixed boundary conditions is also noted.

1. Introduction

Spectral and pseudospectral methods have superior approximation properties if they are compared with other methods of discretization. It can be shown that if the eigenfunctions of a sigular Sturm-Liouville problem are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness of the function being expanded and not by any special boundary conditions satisfied by the function. Gottlieb and Orszag [5] have shown that if the functions of interest are infinitely differentiable, then the n-th coefficient a_n decreases faster than any finite power of $1/n$.

For the spectral method and its variants - the Galerkin and tau methods - explicit expressions for the expansion coefficients of the derivatives in terms of the original expansion coefficients of the solution are required.

A formula expressing the Chebyshev coefficients of the general order derivative of an infinitely differentiable function in terms of its Chebyshev coefficients is given by Karageorghis [6], and a corresponding formula for the Legendre coefficients is obtained by Phillips [7]. A more general formula - with its special cases - for ultraspherical coefficients is given in Doha [2].

Formulae expressing the coefficients of expansions of double and triple Chebyshev and Legendre polynomials in terms of the coefficients of the original expansions are given in Doha [3,4].

In the present paper we state and prove the corresponding formulae expressing the coefficients of expansions of double and triple ultraspherical polynomials which have been partially differentiated any number of times with respect to their variables in terms of the coefficients of the original expansions.

In Section 2 we give some properties of double ultraspherical polynomials and in Section 3 we describe how they are used to solve Poisson's equation in two variables inside a square subject to nonhomogeneous mixed boundary conditions with the tau method as a model problem. In Section 4 we state and prove the main results of the paper which are three expressions for the coefficients of general order partial derivatives of expansion in double ultraspherical polynomials in terms of the coefficients of original expansion, results for the Chebyshev polynomials of first and second kinds and for the Legendre polynomials are obtained as special cases. Extension to expansion in triple ultraspherical polynomials is also given in Section 5.

2. Some properties of double ultraspherical polynomials

The one-variable ultraspherical (Gegenbauer) polynomials associated with the real parameter $\alpha > -1/2$ (see [1]) are a sequence of polynomials $C_n^{(\alpha)}(x)$ $(n = 0, 1, 2, \ldots)$, each respectively of degree n.

For our present purposes it is convenient to standardize the ultraspherical polynomials so that $C_n^{(\alpha)}(1) = 1$ $(n = 0, 1, 2, \ldots)$. This is not the usual standartization, but it has the desirable properties that $C_n^{(0)}(x)$ is identical with Chebyshev polynomials of first kind $T_n(x)$, $C_n^{(1/2)}(x)$ is the Legendre polynomial $P_n(x)$, and $C_n^{(1)}(x)$ is equal to $\frac{1}{n+1}U_n(x)$, where $U_n(x)$ the Chebyshev polynomial of second kind. In this form the polynomials may be generated by using Rodrigue's formula

$$(1) \qquad C_n^{(\alpha)}(x) = \left(-\frac{1}{2}\right)^n \frac{\Gamma(\alpha + 1/2)}{\Gamma(n + \alpha + 1/2)}(1 - x^2)^{\frac{1}{2}-\alpha} D_x^n[(1 - x^2)^{n+\alpha-1/2}]$$

and are satisfying the orthogonality relation

$$\int_{-1}^{1}(1-x^2)^{\alpha-1/2}C_m^{(\alpha)}(x)C_n^{\alpha}(x)dx = \begin{cases} 0 & m \neq n,\ \alpha \neq 0; \\ \frac{\pi^{1/2}n!\Gamma(2\alpha)\Gamma(\alpha+1/2)}{(n+\alpha)\Gamma(\alpha)\Gamma(n+2\alpha)}, & m = n,\ \alpha \neq 0; \\ \pi, & m = n = 0,\ \alpha = 0; \\ \pi/2, & m = n \neq 0,\ \alpha = 0; \\ 0, & m \neq n,\ \alpha = 0. \end{cases}$$

The ultraspherical polynomials are eigenfunctions of the following singular Sturm-Liouville problem

$$(1-x^2)\Phi''(x) - (2\alpha+1)x\Phi'(x) + n(n+2\alpha)\Phi(x) = 0.$$

A consequence of this is that spectral accuracy can be achieved for expansions in ultraspherical polynomials.

Suppose we are given a function $u(x)$ which is infinitely differentiable in the closed interval $[-1,1]$. Then we can write

$$u(x) = \sum_{n=0}^{\infty} a_n C_n^{(\alpha)}(x)$$

and for the q-th derivative of $u(x)$

$$u^{(q)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} C_n^{(\alpha)}(x).$$

Doha [2] proved that

$$a_n^{(q)} = \frac{2^q(n+\alpha)\Gamma(n+2\alpha)}{(q-1)!n!}\times$$

$$\times\sum_{j=1}^{\infty}\frac{(j+q-2)!\Gamma(n+j+q+\alpha-1)(n+2+q-2)!}{(j-1)!\Gamma(n+j+\alpha)\Gamma(n+2j+q+2\alpha-2)}a_{n+2j+q-2}.$$

Now we define the double ultraspherical polynomials as

$$C_{mn}^{(\alpha)}(x,y) = C_m^{(\alpha)}(x)C_n^{(\alpha)}(y), \tag{2}$$

i.e. a product two one-variable ultraspherical polynomials, where $C_m^{(\alpha)}(x)$, $C_n^{(\alpha)}(y)$ are ultraspherical polynomials of degrees m and n in the variables x and y, respectively. These polynomials are satisfying the biorthogonality relation

$$\int_{-1}^{1}\int_{-1}^{1}[(1-x^2)(1-y^2)]^{\alpha-1/2}C_{ij}^{(\alpha)}(x,y)C_{kl}^{(\alpha)}(x,y)dxdy =$$

$$= \begin{cases} \frac{\pi i! j!}{(i+\alpha)(j+\alpha)\Gamma(i+2\alpha)\Gamma(j+2\alpha)}\left[\frac{\Gamma(2\alpha)\Gamma(\alpha+1/2)}{\Gamma(\alpha)}\right]^2, & i=k,\ j=l,\ \alpha\neq 0; \\ \pi^2, & i=j=k=l=0,\ \alpha=0; \\ \pi^2/4, & i=k\neq 0,\ j=l\neq 0,\ \alpha=0; \\ \pi^2/2, & i=k\neq 0,\ j=l=0 \text{ or } \\ & i=k=0,\ j=l\neq 0,\ \alpha=0; \\ 0, & \text{for all other values of } i,j,k,l. \end{cases}$$

It is worthy to note here that typical orthogonal polynomials - the double Chebyshev polynomials of first kind $T_{mn}(x,y)$ and of second kind $U_{mn}(x,y)$ and the double Legendre polynomials $P_{mn}(x,y)$ - are particular forms of the double ultraspherical polynomials. Namely, we have

$$T_{mn}(x,y) = C_{mn}^{(0)}(x,y) = T_m(x)T_n(y),$$
$$U_{mn}(x,y) = C_{mn}^{(1)}(x,y) = \frac{1}{(m+1)(n+1)}U_m(x)U_n(y),$$
$$P_{mn}(x,y) = C_{mn}^{(1/2)}(x,y) = P_m(x)P_n(y).$$

Let $u(x,y)$ be a continuous function defined on the square $S(-1\leq x,y\leq 1)$, and let it have continuous and bounded partial derivatives of any order with respect to its variables x and y. Then it is possible to express

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}C_m^{\alpha}(x)C_n^{(\alpha)}(y), \tag{3}$$

$$u^{(p,q)}(x,y) = D_x^p D_y^q u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y), \tag{4}$$

where $a_{mn}^{(p,q)}$ denote the ultraspherical expansion coefficients of $u^{(p,q)}(x,y)$ and $a_{mn}^{(0,0)} = a_{mn}$.

Using the expressions (see Doha [2])

$$(5) \qquad 2(m+\alpha)C_m^{(\alpha)}(x) = \frac{m+2\alpha}{m+1} D_x C_{m+1}^{(\alpha)}(x) - \frac{m}{m+2\alpha-1} D_x C_{m-1}^{(\alpha)}(x),$$

$$(6) \qquad 2(n+\alpha)C_n^{(\alpha)}(y) = \frac{n+2\alpha}{n+1} D_y C_{n+1}^{(\alpha)} - \frac{n}{n+2\alpha-1} D_y C_{n-1}^{(\alpha)}(y)$$

with the assumptions that

$$D_x \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p-1,q)} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y),$$

$$D_y \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q-1)} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y)$$

it is not difficult to derive the expressions

(7)
$$\frac{(m+2\alpha-1)}{2m(m+\alpha-1)} a_{m-1,n}^{(p,q)} - \frac{(m+1)}{2(m+\alpha+1)(m+2\alpha)} a_{m+1,n}^{(p,q)} = a_{mn}^{(p-1,q)} \qquad m,p \geq 1,$$

(8)
$$\frac{(n+2\alpha-1)}{2n(n+\alpha-1)} a_{m,n-1}^{(p,q)} - \frac{(n+1)}{2(n+\alpha+1)(n+2\alpha)} a_{m,n+1}^{(p,q)} = a_{mn}^{(p,q-1)} \qquad n,q \geq 1.$$

For computing purposes the equations (7) and (8) are not easy to use, since the coeeficients on the left hand sides are functions of m and n, respectively. To simplify the computing we define a related set of coefficients $b_{mn}^{(p,q)}$ by writing

$$(9) \qquad a_{mn}^{(p,q)} = \frac{(m+\alpha)(n+\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{m!n!} b_{mn}^{(p,q)}$$
$$m,n \geq 0, \qquad p,q = 0,1,2,\ldots$$

The equations (7) and (8) take the simpler forms

$$(10) \qquad b_{m-1,n}^{(p,q)} - b_{m+1,n}^{(p,q)} = 2(m+\alpha) b_{mn}^{(p-1,q)} \qquad m,p \geq 1,$$

$$(11) \qquad b_{m,n-1}^{(p,q)} - b_{m,n+1}^{(p,q)} = 2(n+\alpha) b_{mn}^{(p,q-1)} \qquad n,q \geq 1.$$

The repeated application of (10) keeping n and q fixed (see [8]) yields

$$b_{mn}^{(p,q)} = 2\sum_{i=1}^{\infty}(m+2i+\alpha-1)b_{m+2i-1,n}^{(p-1,q)} \qquad p \geq 1, \tag{12}$$

and the same with (11) keeping m and p fixed yields

$$b_{mn}^{(p,q)} = 2\sum_{j=1}^{\infty}(n+2j\alpha-1)b_{m,n+2j-1}^{(p,q-1)} \qquad q \geq 1. \tag{13}$$

3. The tau method for Poisson's equation in two variables

Consider Poisson's equation in the square $S(-1 \leq x, y \leq 1)$

$$D_x^2 u(x,y) + D_y^2 u(x,y) = f(x,y) \qquad -1 \leq x, y \leq 1, \tag{14}$$

subject to the nonhomogeneous mixed boundary conditions

$$\begin{cases} u + \alpha_1 D_x u = \gamma_1(y) & x = -1, \ -1 \leq y \leq 1, \\ u + \alpha_2 D_x u = \gamma_2(y) & x = 1, \ -1 \leq y \leq 1, \end{cases} \tag{15}$$

$$\begin{cases} u + \beta_1 D_y u = \delta_1(x) & y = -1, \ -1 \leq x \leq 1, \\ u + \beta_2 D_y u = \delta_2(x) & y = 1, \ -1 \leq x \leq 1 \end{cases} \tag{16}$$

and assume that both $u(x,y)$ and $f(x,y)$ are approximated by truncated double ultraspherical series

$$u(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{M} a_{mn} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y), \tag{17}$$

$$f(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{M} f_{mn} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y). \tag{18}$$

Assume also that the functions $\gamma_1(y)$, $\gamma_2(y)$, $\delta_1(x)$ and $\delta_2(x)$ have the following truncated ultraspherical expansions

$$\gamma_1(y) = \sum_{n=0}^{N} \gamma_n^{(1)} C_n^{(\alpha)}(y), \tag{19}$$

$$\gamma_2(y) = \sum_{n=0}^{N} \gamma_n^{(2)} C_n^{(\alpha)}(y), \tag{20}$$

$$\delta_1(x) = \sum_{m=0}^{M} \delta_m^{(1)} C_m^{(\alpha)}(x), \tag{21}$$

$$\delta_2(x) = \sum_{m=0}^{M} \delta_m^{(2)} C_m^{(\alpha)}(x), \tag{22}$$

then the ultraspherical tau equations for Poisson's equation (14) are given by

$$a_{mn}^{(2,0)} + a_{mn}^{(0,2)} = f_{mn}, \qquad 0 \le m \le M-2, \quad 0 \le n \le N-2, \tag{23}$$

while the boundary conditions (15) and (16) with (19)-(22) yield

$$\left.\begin{aligned} \sum_{m=0}^{M}(-1)^m \left[a_{mn} + \alpha_1 a_{mn}^{(1,0)}\right] &= \gamma_n^{(1)} \\ \sum_{m=0}^{M} \left[a_{mn} + \alpha_2 a_{mn}^{(1,0)}\right] &= \gamma_n^{(2)} \end{aligned}\right\} \qquad n = 0, 1, 2, \ldots, N, \tag{24}$$

$$\left.\begin{aligned} \sum_{n=0}^{N}(-1)^n \left[a_{mn} + \beta_1 a_{mn}^{(0,1)}\right] &= \delta_m^{(1)} \\ \sum_{n=0}^{N} \left[a_{mn} + \beta_2 a_{mn}^{(0,1)}\right] &= \delta_m^{(2)} \end{aligned}\right\} \qquad m = 0, 1, 2, \ldots, M. \tag{25}$$

The $2M + 2N + 4$ boundary conditions, given by (24) and (25), are not all linearly independent, there exist four linear relations among them. Thus, equations (23), (24) and (25) give $(M+1)(N+1)$ equations for the $(M+1)(N+1)$ unknowns a_{mn} $(0 \le m \le M,\ 0 \le n \le N)$.

The coefficients $a_{mn}^{(1,0)}$, $a_{mn}^{(0,1)}$, $a_{mn}^{(2,0)}$ and $a_{mn}^{(0,2)}$ of the first and second partial derivatives of the approximation $u(x,y)$ are related to the coefficients a_{mn} of

$u(x,y)$ by invoking (12) with $p=1$ and $p=2$, and (13) with $q=1$ and $q=2$, respectively. In the next section we show how the coefficients of arbitrary derivatives may be expressed in terms of the original expansion coefficients. This allows us to replace $a_{mn}^{(1,0)}$, $a_{mn}^{(0,1)}$, $a_{mn}^{(2,0)}$ and $a_{mn}^{(0,2)}$ in (24), (25) and (23) by explicit expressions in terms of the a_{mn}. In this way we can set up a linear system for a_{mn} $(0 \le m \le M,\ 0 \le n \le N)$ which may be solved using standard techniques.

4. Relations between the coefficients

The main result of this section is to prove the following

Theorem 1. *The coefficients* $b_{mn}^{(p,q)}$ *are related to the coefficients* $b_{mn}^{(0,q)}$, $b_{mn}^{(p,0)}$ *and the original coefficients* b_{mn} *by*

$$b_{mn}^{(p,q)} = \frac{2^p}{(p-1)!}\times \tag{26}$$

$$\times\sum_{i=1}^{\infty}\frac{(i+p-2)!}{(i-1)!}\frac{\Gamma(m+i+p+\alpha-1)}{\Gamma(m+i+\alpha)}(m+2i+p+\alpha-2)b_{m+2i+p-2,n}^{(0,q)} \qquad p\ge 1,$$

$$b_{mn}^{(p,q)} = \frac{2^p}{(q-1)!}\times \tag{27}$$

$$\times\sum_{j=1}^{\infty}\frac{(j+q-2)!}{(j-1)!}\frac{\Gamma(n+j+q+\alpha-1)}{\Gamma(n+j+\alpha)}(n+2j+q+\alpha-2)b_{m,n+2j+q-2}^{(p,0)} \qquad q\ge 1,$$

$$b_{mn}^{(p,q)} = \frac{2^{p+q}}{(p-1)!(q-1)!}\times \tag{28}$$

$$\times\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(j+q-2)!}{(i-1)!(j-1)!}\frac{\Gamma(m+i+p+\alpha-1)\Gamma(n+j+q+\alpha-1)}{\Gamma(m+i+\alpha)\Gamma(n+j+\alpha)}\times$$

$$\times(m+2i+p+\alpha-2)(n+2j+q+\alpha-2)b_{m+2i+p-2,n+2j+q-2} \qquad p,q\ge 1$$

for all $m,n\ge 0$.

In order to prove the theorem the following two lemmas are required.

Lemma 1.

$$\sum_{i=1}^{M}(m+2i+\alpha-1)\frac{(M-i+p-1)!\Gamma(m+i+M+p+\alpha-1)}{(M-i)!\Gamma(m+i+M+\alpha)}=$$

$$=\frac{1}{p}\frac{(M+p-1)!}{(M-1)!}\frac{\Gamma(m+M+p+\alpha)}{\Gamma(m+M+\alpha)}\qquad m,p\geq 1. \tag{29}$$

Lemma 2.

$$\sum_{j=1}^{N}(n+2j+\alpha-1)\frac{(N-j+q-1)!\Gamma(n+j+N+q+\alpha-1)}{(N-j)!\Gamma(n+j+N+\alpha)}=$$

$$=\frac{1}{q}\frac{(N+q-1)!\Gamma(n+N+q+\alpha)}{(N-1)!\Gamma(n+N+\alpha)}\qquad n,q\geq 1. \tag{30}$$

The interested reader is referred to Doha [2] for the proof of lemmas (29) and (30).

Proof of Theorem 1. Firstly we prove formula (26). For $p=1$ application of (12) with $p=1$ yields the required formula. Proceeding by induction, assuming that the relation is valid for p (keeping n and q fixed) we want to show that

(31)

$$b_{mn}^{(p+1,q)}=\frac{2^{p+1}}{p!}\sum_{i=1}^{\infty}\frac{(i+p-1)!}{(i-1)!}\frac{\Gamma(m+i+p+\alpha)}{\Gamma(m+i+\alpha)}(m+2i+p+\alpha-1)b_{m+2i+p-1,n}^{(0,q)}.$$

From (12), replacing p by $p+1$ and assuming the validity of (26) for p,

$$b_{mn}^{(p+1,q)}=\frac{2^{p+1}}{(p-1)!}\sum_{i=1}^{\infty}(m+2i+\alpha-1)\left\{\sum_{k=1}^{\infty}\frac{(k+p-2)!}{(k-1)!}\times\right. \tag{32}$$

$$\left.\times\frac{\Gamma(m+2i+k+p+\alpha-2)}{\Gamma(m+2i+k+\alpha-1)}(m+2i+2k+p+\alpha-3)b_{m+2i+2k+p-3,n}^{(0,q)}\right\}.$$

Let $i+k-1=M$, then (32) takes the form

$$b_{mn}^{(p+1,q)}=\frac{2^{p+1}}{(p-1)!}\sum_{M=1}^{\infty}\left[\sum_{\substack{i,k=1\\ i+k=M+1}}^{M}(m+2i+\alpha-1)\frac{(k+p-2)!}{(k-1)!}\times\right.$$

$$\times \frac{\Gamma(m+2i+k+p+\alpha-2)}{\Gamma(m+2i+k+\alpha-1)}(m+2M+p+\alpha-1)b^{(0,q)}_{m+2M+p-1,n}\Bigg],$$

which may also be written as

$$b^{(p+1,q)}_{mn} = \frac{2^{p+1}}{(p-1)!}\sum_{M=1}^{\infty}\left\{\sum_{i=1}^{M}(m+2i+\alpha-1)\frac{(M-i+p-1)!}{(M-i)!}\times\right.$$

$$\left.\times\frac{\Gamma(m+M+i+p+\alpha-1)}{\Gamma(m+M+i+\alpha)}(m+2M+p+\alpha-1)\right\} b^{(0,q)}_{m+2M+p-1,n}.$$

Application of lemma (29) to the second series yields equation (31) and the proof of formula (26) is complete.

It can be also shown that formula (27) is true by following the same procedure with (13), keeping m and p fixed. Formula (28) is obtained immediately by substituting (26) into (27) or (27) into (26). This completes the proof of Theorem 1.

Now the substitution of (26), (27) and (28) into (9) gives the relations between the coefficients $a^{(p,q)}_{mn}$, $a^{(0,q)}_{mn}$, $a^{(p,0)}_{mn}$ and a_{mn} as

$$a^{(p,q)}_{mn} = \frac{2^p(m+\alpha)\Gamma(m+2\alpha)}{(p-1)!m!}\times \tag{33}$$

$$\times\sum_{i=1}^{\infty}\frac{(i+p-2)!\Gamma(m+i+p+\alpha-1)(m+2i+p-2)!}{(i-1)!\Gamma(m+i+\alpha)\Gamma(m+2i+p+2\alpha-2)}a^{(0,q)}_{m+2i+p-2,n} \qquad p\geq 1,$$

$$a^{(p,q)}_{mn} = \frac{2^q(n+\alpha)\Gamma(n+2\alpha)}{(q-1)!n!}\times \tag{34}$$

$$\times\sum_{j=1}^{\infty}\frac{(j+q-2)!\Gamma(n+j+q+\alpha-1)(n+2j+q-2)!}{(j-1)!\Gamma(n+j+\alpha)\Gamma(n+2j+q+2\alpha-2)}a^{(p,0)}_{m,n+2j+q-2} \qquad q\geq 1,$$

$$a^{(p,q)}_{mn} = \tag{35}$$

$$= \frac{2^{p+q}(m+\alpha)(n+\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{(p-1)!(q-1)!m!n!}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(j+q-2)!}{(i-1)!(j-1)!}\times$$

$$\frac{\Gamma(m+i+p+\alpha-1)\Gamma(n+j+q+\alpha-1)(m+2i+p-2)!(n+2j+q-2)!}{\Gamma(m+i+\alpha)\Gamma(n+j+\alpha)\Gamma(m+2i+p+2\alpha-2)\Gamma(n+2j+q+2\alpha-2)}\times$$

$$\times a_{m+2i+p-2,n+2j+q-2} \qquad p,q \geq 1.$$

In particular, the special cases for the "bivariate" Chebyshev polynomials of the first and second kinds may be obtained directly by taking $\alpha = 0, 1$ respectively, and for the "bivariate" Legendre polynomials by taking $\alpha = 1/2$. These are given as corollaries to the previous theorem.

Corollary 1. *If*

$$u(x,y) = \sum_{n=0}^{\infty} {\sum_{m=0}^{\infty}}'' a_{mn} T_m(x) T_n(y), \tag{36}$$

$$u^{(p,q)}(x,y) = \sum_{n=0}^{\infty} {\sum_{m=0}^{\infty}}'' a_{mn}^{(p,q)} T_m(x) T_n(y), \tag{37}$$

then the coefficients $a_{mn}^{(p,q)}$ *are related to* $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ *and* a_{mn} *by*

$$a_{mn}^{(p,q)} = \frac{2^p}{(p-1)!} \times \tag{38}$$

$$\times \sum_{i=1}^{\infty} \frac{(i+p-2)!}{(i-1)!} \frac{(m+i+p-2)!}{(m+i-1)!} (m+2i+p-2) a_{m+2i+p-2,n}^{(0,q)} \qquad p \geq 1,$$

$$a_{mn}^{(p,q)} = \frac{2^q}{(q-1)!} \times \tag{39}$$

$$\times \sum_{j=1}^{\infty} \frac{(j+q-2)!}{(j-1)!} \frac{(n+j+q-2)!}{(n+j-1)!} (n+2j+q-2) a_{m,n+2j+q-2}^{(p,0)} \qquad q \geq 1,$$

$$a_{mn}^{(p,q)} = \frac{2^{p+q}}{(p-1)!(q-1)!} \times \tag{40}$$

$$\times \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{(i+p-2)!(j+q-2)!(m+i+p-2)!(n+j+q-2)!}{(i-1)!(j-1)!(m+i-1)!(n+j-1)!} \times$$

$$\times (m+2i+p-2)(n+2j+q-2) a_{m+2i+p-2,n+2j+q-2} \qquad p,q \geq 1$$

for all $m,n \geq 0$.

Note that the double primes in (36) and (37) indicate that the first term is $1/4a_{00}$, a_{m0} and a_{0n} are to be taken as $1/2a_{m0}$ and $1/2a_{0n}$ for $m,n > 0$, respectively.

Proof. Formulae (38), (39) and (40) are obtained directly by simply setting $\alpha = 0$ in (33), (34) and (35) respectively, noting that $\alpha\Gamma(2\alpha) = 1/2\Gamma(2\alpha+1)$ which equals $1/2$ as α tends to zero.

It is worthy to note that formulae (38), (39) and (40) are in agreement with those obtained by Doha [3].

Corollary 2. *If*

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} U_m(x) U_n(y),$$

$$u^{(p,q)}(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}^{(p,q)} U_m(x) U_n(y),$$

then the coefficients $A_{mn}^{(p,q)}$ *are related to the coefficients* $A_{mn}^{(0,q)}$, $A_{mn}^{(p,0)}$ *and* A_{mn} *by*

(41)
$$A_{mn}^{(p,q)} = \frac{2^p(m+1)}{(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!(m+i+p-1)!}{(i-1)!(m+i)!}A_{m+2i+p-2,n}^{(0,q)} \qquad p \geq 1,$$

(42)
$$A_{mn}^{(p,q)} = \frac{2^q(n+1)}{(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-1)!(n+j+q-1)!}{(j-1)!(n+j)!}A_{m,n+2j+q-2}^{(p,0)} \qquad q \geq 1,$$

$$A_{mn}^{(p,q)} = \frac{2^{p+q}(m+1)(n+1)}{(p-1)!(q-1)!}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(j+q-2)!(m+i+p-1)!}{(i-1)!(j-1)!(m+i)!}\times$$

(43)
$$\times\frac{(n+j+q-1)!}{(n+j)!}A_{m+2i+p-2,n+2j+q-2} \qquad p,q \geq 1$$

for all $m, n \geq 0$.

Proof. Simply set $\alpha = 1$ in (33), (34) and (35), noting that

$$A_{mn} = \frac{a_{mn}}{(m+1)(n+1)}, \qquad A_{mn}^{(p,q)} = \frac{a_{mn}^{(p,q)}}{(m+1)(n+1)}.$$

Corollary 3. *If*

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn} P_m(x) P_n(y),$$

$$u^{(p,q)}(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)} P_m(x)P_n(y),$$

then the coefficients $a_{mn}^{(p,q)}$ *are related to the coefficients* $a_{mnn}^{(0,p)}$, $a_{mn}^{(p,0)}$ *and* a_{mn} *by*

(44)
$$a_{mn}^{(p,q)} =$$
$$= \frac{2^{p-1}(2m+1)}{(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!\Gamma(m+i+p-1/2)}{(i-1)!\Gamma(m+i+1/2)} a_{m+2i+p-2,n}^{(0,q)} =$$
$$= \frac{(2m+1)}{2^{p-2}(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!(2m+2i+2p-3)!(m+i)!}{(i-1)!(2m+2i)!(m+i+p-2)!} a_{m+2i+p-2,n}^{(0,q)}$$
$$p \geq 1,$$

(45)
$$a_{mn}^{(p,q)} =$$
$$= \frac{2^{q-1}(2n+1)}{(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!\Gamma(n+j+q-1/2)}{(j-1)!\Gamma(n+j+1/2)} a_{m,n+2j+q-2}^{(p,0)} =$$
$$= \frac{(2n+1)}{2^{q-2}(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!(2n+2j+2q-3)!(n+j)!}{(j-1)!(2n+2j)!(n+j+q-2)!} a_{m,n+2j+q-2}^{(p,0)}$$
$$q \geq 1,$$

(46)
$$a_{mn}^{(p,q)} = \frac{2^{p+q-2}(2m+1)(2n+1)}{(p-1)!(q-1)!}\times$$
$$\times\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(j+q-2)!\Gamma(m+i+p-1/2)\Gamma(n+j+q-1/2)}{(i-1)!(j-1)!\Gamma(m+i+1/2)\Gamma(n+j+1/2)}\times$$
$$\times a_{m+2i+p-2,n+2j+q-2} = \frac{(2m+1)(2n+1)}{2^{p+q-4}(p-1)!(q-1)!}\times$$
$$\times\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(j+q-2)!(2m+2i+2p-3)!(2n+2j+2q-3)!}{(i-1)!(j-1)!(2m+2i)!(2n+2j)!}\times$$
$$\times\frac{(m+i)!(n+j)!}{(m+i+p-2)!(n+j+q-2)!} a_{m+2j+p-2,n+2j+q-2} \qquad p,q \geq 1$$

for all $m, n \geq 0$.

Proof. Simply set $\alpha = 1/2$ in (33), (34) and (35), noting that

$$\frac{\Gamma(m+i+p-1/2)}{\Gamma(m+i+1/2)} = \frac{1}{2^{2p-3}} \frac{(2m+2i+2p-3)!(m+i)!}{(2m+2i)!(m+i+p-2)!}.$$

It is worthy to be mentioned here the formulae (44), (45) and (46) are in agreement with those stated and proved in [4].

5. Extension to triple ultraspherical series expansions

Let $u(x, y, z)$ be a continuous function defined on the cube $C(-1 \leq \leq x, y, z \leq 1)$, and let it have continuous and bounded partial derivatives of any order with respect to its variables x, y and z. Then it is possible to express

$$u(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} a_{lmn} C_l^{(\alpha)}(x) C_m^{(\alpha)}(y) C_n^{(\alpha)}(z),$$

$$u^{(p,q,r)}(x, y, z) =$$

$$= D_x^p D_y^q D_z^r u(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} a_{lmn}^{(p,q,r)} C_l^{(\alpha)}(x) C_m^{(\alpha)}(y) C_n^{(\alpha)}(z).$$

Further, let

$$a_{lmn}^{(p,q,r)} = \frac{(l+\alpha)(m+\alpha)(n+\alpha)\Gamma(l+2\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{l!m!n!} b_{lmn}^{(p,q,r)}$$

$$l, m, n \geq 0, \qquad p, q, r = 0, 1, 2, \dots, \tag{47}$$

then it is not difficult to show that

$$b_{l-1,m,n}^{(p,q,r)} - b_{l+1,m,n}^{(p,q,r)} = 2(l+\alpha) b_{lmn}^{(p-1,q,r)} \qquad p \geq 1,$$

$$b_{l,m-1,n}^{(p,q,r)} - b_{l,m+1,n}^{(p,q,r)} = 2(m+\alpha) b_{lmn}^{(p,q-1,r)} \qquad q \geq 1,$$

$$b_{l,m,n-1}^{(p,q,r)} - b_{l,m,n+1}^{(p,q,r)} = 2(n+\alpha) b_{lmn}^{(p,q,r-1)} \qquad r \geq 1,$$

which, in turn, yield

$$b_{lmn}^{(p,q,r)} = 2\sum_{i=1}^{\infty}(l+2i+\alpha-1)b_{l+2i-1,m,n}^{(p-1,q,r)} \qquad p \geq 1, \tag{48}$$

$$b_{lmn}^{(p,q,r)} = 2\sum_{j=1}^{\infty}(m+2j+\alpha-1)b_{l,m+2j-1,n}^{(p,q-1,r)} \qquad q \geq 1, \tag{49}$$

$$b_{lmn}^{(p,q,r)} = 2\sum_{k=1}^{\infty}(n+2k+\alpha-1)b_{l,m,n+2k-1}^{(p,q,r-1)} \qquad r \geq 1. \tag{50}$$

Now we state a theorem which is considered as an extension of Theorem 1 of Section 4.

Theorem 2. *The coefficients* $b_{lmn}^{(p,q,r)}$ *are related to the coefficients with superscripts (0,q,r), (p,0,r), (p,q,0), (0,0,r), (0,q,0), (0,0,p) and* b_{lmn} *by*

$$b_{lmn}^{(p,q,r)} = \frac{2^p}{(p-1)!}\times \tag{51}$$

$$\times\sum_{i=1}^{\infty}\frac{(i+p-2)!\Gamma(l+i+p+\alpha-1)}{(i-1)!\Gamma(l+i+\alpha)}(l+2i+p+\alpha-2)b_{l+2i+p-1,m,n}^{(0,q,r)} \qquad p \geq 1,$$

$$b_{lmn}^{(p,q,r)} = \frac{2^q}{(q-1)!}\times \tag{52}$$

$$\times\sum_{j=1}^{\infty}\frac{(j+q-2)!\Gamma(m+j+q+\alpha-1)}{(j-1)!\Gamma(m+j+\alpha)}(m+2j+q+\alpha-2)b_{l,m+2j+q-1,n}^{(p,0,r)} \qquad q \geq 1,$$

$$b_{lmn}^{(p,q,r)} = \frac{2^r}{(r-1)!}\times \tag{53}$$

$$\times\sum_{k=1}^{\infty}\frac{(k+r-2)!\Gamma(n+k+r+\alpha-1)}{(k-1)!\Gamma(n+k+\alpha)}(n+2k+r+\alpha-2)b_{l,m,n+2k+r-1}^{(p,q,0)} \qquad r \geq 1,$$

$$b_{lmn}^{(p,q,r)} = \frac{2^{p+q}}{(p-1)!(q-1)!}\times \tag{54}$$

$$\times\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(i+p-2)!(j+q-2)!\Gamma(l+i+p+\alpha-1)\Gamma(m+j+q+\alpha-1)}{(i-1)!(j-1)!\Gamma(l+i+\alpha)\Gamma(m+j+\alpha)}\times$$

$$\times(l+2i+p+\alpha-2)(m+2j+q+\alpha-2)b^{(0,0,r)}_{l+2i+p-1,m+2j+q-1,n} \qquad p,q\geq 1,$$

(55)
$$b^{(p,q,r)}_{lmn}=\frac{2^{p+r}}{(p-1)!(r-1)!}\times$$
$$\times\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{(i+p-2)!(k+r-2)!\Gamma(l+i+p+\alpha-1)\Gamma(n+k+r+\alpha-1)}{(i-1)!(k-1)!\Gamma(l+i+\alpha)\Gamma(n+k+\alpha)}\times$$
$$\times(l+2i+p+\alpha-2)(n+2k+r+\alpha-2)b^{(0,q,0)}_{l+2i+p-1,m,n+2k+r-1} \qquad p,r\geq 1,$$

(56)
$$b^{(p,q,r)}_{lmn}=\frac{2^{q+r}}{(q-1)!(r-1)!}\times$$
$$\times\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{(j+q-2)!(k+r-2)!\Gamma(m+j+q+\alpha-1)\Gamma(n+k+r+\alpha-1)}{(j-1)!(k-1)!\Gamma(m+j+\alpha)\Gamma(n+k+\alpha)}\times$$
$$\times(m+2j+q+\alpha-2)(n+2k+r+\alpha-2)b^{(p,0,0)}_{l,m+2j+q-1,n+2k+r-1} \qquad q,r\geq 1,$$

(57)
$$b^{(p,q,r)}_{lmn}=\frac{2^{p+q+r}}{(p-1)!(q-1)!(r-1)!}\times$$
$$\times\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{(i+p-2)!(j+q-2)!(k+r-2)!}{(i-1)!(j-1)!(k-1)!}\times$$
$$\times\frac{\Gamma(l+i+p+\alpha-1)\Gamma(m+j+q+\alpha-1)\Gamma(n+k+r+\alpha-1)}{\Gamma(l+i+\alpha)\Gamma(m+j+\alpha)\Gamma(n+k+\alpha)}\times$$
$$\times(l+2i+p+\alpha-2)(m+2j+q+\alpha-2)(n+2k+r+\alpha-2)\times$$
$$\times b_{l+2i+p-1,m+2j+q-1,n+2k+r-1} \qquad p,q,r\geq 1.$$

Outlines of the proof. Formulae (51) can be proved by induction on p, (52) by induction on q and (53) by induction on r, respectively. Substituting (51) into (52) and (53), and substituting (52) into (53) give formulae (54), (55) and (56). Formula (57) is obtained by substituting (53) into (54). This completes the proof.

The explicit formulae relate the coefficients $a^{(p,q,r)}_{lmn}$ with those with superscripts $(0,q,r)$, $(p,0,r)$, $(p,q,0)$, $(0,0,r)$, $(0,q,0)$, $(p,0,0)$ and the original coefficients a_{lmn} can be obtained by using formula (47) with the formulae (51)-(57).

The formulae corresponding to expansions in triple Chebyshev polynomials of the first and second kinds and triple Legendre polynomials may be obtained - as special cases - by taking $\alpha = 0, 1$ and $1/2$ respectively in formulae (47) with (51)-(57).

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