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NONNEGATIVITY OF THE NUMERICAL SOLUTION OF ONE–DIMENSIONAL HEAT–CONDUCTION EQUATION WITH VARIABLE COEFFICIENT

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1. Introduction

In this paper we study the nonnegativity of the numerical solution of one-dimensional heat-conduction problem with special polynomial heat-diffusion coefficient. We also consider this problem with any strictly positive function, which can be well approximated by piecewise-linear functions. The problem is considered with constant diffusion coefficient in [3], [5]. In this paper, using linear finite elements for space discretization and the one-step method for the time discretization, we give a new sufficient condition of the nonnegativity for the approximation function.

2. Formulation of the problem

We consider the linear parabolic problem in one dimension having the form

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) = 0, \quad 0 < x < L, \quad t > 0, \tag{2.1}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le L, \tag{2.2}$$

$$u(0,t) = u(L,t) = 0, \tag{2.3}$$

where $(L \in R^+)$,

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m, \quad a_0, a_1, a_2, \ldots, a_m \ge 0. \tag{2.4}$$

Under some natural conditions (i.e. the initial function u_0 is sufficiently smooth and nonnegative) the solution of the problem (2.1)-(2.3) is nonnegative on the whole domain $\Omega \times R_0^+$; [4], [7], where $\Omega = [0, L] \subset R$, R_0^+ nonnegative real numbers. This property plays an important part in applications, i.e. (2.1)-(2.3) describes the process of heat conduction. Our goal is to formulate the conditions of conservation of this property to the Galerkin type's numerical solutions. We apply to the space-discretization the standard one-dimensional linear FEM's basis functions and one-step method for time discretization. We give some sufficient conditions for the conservation of the nonnegativity of the fully discretized scheme.

The weak form of the problem (2.1)-(2.3) is

$$\int_0^L \left(\frac{\partial u}{\partial t} v + p(x) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \right) dx = 0; \quad \forall v \in H_0^1(\Omega). \tag{2.5}$$

Let us divide the domain Ω into the subdomains $\Omega_i = [x_{i-1}, x_i]$, where $x_i = ih$, $i = 1, 2, \dots, n$, $h = \dfrac{L}{n}$. Denote by $\Phi_i(x)$ the usual linear spline function at the point x_i [9], we seek the numerical solution in the form

$$U_h(x, t) = \sum_{i=1}^{n-1} \alpha_i(t) \Phi_i(x), \tag{2.6}$$

where $\alpha_i(t)$ $(i = 1, 2, \dots, n-1)$ are unknown functions to be determined later. Substituting (2.6) into (2.5) we get the Cauchy problem for the vector $\alpha(t) = [\alpha_i(t)]_{i=1}^{n-1}$ of the form

$$M\alpha'(t) + Q\alpha(t) = 0; \quad t > 0, \tag{2.7}$$

where $\alpha(0)$ is a given vector being an appropriate nonnegative approximation of the initial function u_0.

Here we have used the following notations:

$$\alpha(t) \in R_0^{(n-1)} \quad \text{for all fixed } t \in R_0^+, \quad M \in R^{(n-1)\times(n-1)};$$

$$M = \text{tridiag}\ \frac{h}{6}\ [1, 4, 1], \quad Q_0, Q_1, Q_2, \dots, Q_m \in R^{(n-1)\times(n-1)},$$

$$Q = a_0 Q_0 + a_1 Q_1 + a_2 Q_2 + a_3 Q_3 + \dots + a_m Q_m,$$

$$Q_i = \text{tridiag}\ \frac{1}{h^2}\ [f_i, c_i, g_i], \quad i = 0, 1, 2, 3, \dots, m,$$

where

$$f_i = \frac{h^{i+1}}{i+1}\left((k-1)^{i+1} - k^{i+1}\right); \quad k = 2,3,\dots,n-1,$$
$$c_i = \frac{h^{i+1}}{i+1}\left((1+k)^{i+1} - (k-1)^{i+1}\right); \quad k = 1,2,3,\dots,n-1,$$
$$g_i = \frac{h^{i+1}}{i+1}\left(k^{i+1} - (k+1)^{i+1}\right); \quad k = 1,2,3,\dots,n-2.$$

Using the one-step method to the discretization of (2.7) we get the following system of linear algebraic equations

$$X_1\alpha^{j+1} = X_2\alpha^j, \quad j = 0,1,2,\dots, \tag{2.8}$$

$$\alpha^0 = \alpha(0), \tag{2.9}$$

where $X_1 = M + \tau\gamma Q$, $X_2 = M - \tau(1-\gamma)Q$, α^j is the approximation of $\alpha(t)$ at time-level $t_j = \tau j$, $\tau > 0$ is the time-step parameter and $\gamma \in [0,1]$ is a given parameter.

3. Nonnegativity of full discretization

We require the nonnegativity of the matrix

$$X = X_1^{-1}X_2.$$

The most trivial condition for nonnegativity of X are the conditions $X_1^{-1} \geq 0$ and $X_2 \geq 0$. We give some condition for the number $q = \frac{\tau}{h^2}$ which guarantees these conditions. For the matrix X_2 we can do it directly and it results the upper bound

$$q \leq \frac{2}{3(1-\gamma)\left[2a_0 + \frac{a_1(L^2-(L-2h)^2)}{2h} + \dots + \frac{a_m(L^{m+1}-(L-2h)^{m+1})}{(m+1)h}\right]}. \tag{3.1}$$

For the nonnegativity of the matrix X_1^{-1} is a sufficient condition to be X_1 an M-matrix [1]. With direct computation we get the condition

$$(3.2)\qquad \frac{1}{6\gamma\left[\frac{a_0}{1}(2-1)+\frac{a_1}{2}\ h\ (2^2-1^2)+\ldots+\frac{a_m h^m}{m+1}(2^{m+1}-1^{m+1})\right]}\leq q.$$

So, denoting by

$$I_1=\left[2a_0+\frac{a_1\left(L^2-(L-2h)^2\right)}{2h}+\ldots+\frac{a_m\left(L^{m+1}-(L-2h)^{m+1}\right)}{(m+1)h}\right],$$

$$I_2=a_0\left[4(2-1)+\frac{(L-(L-2h))}{h}\right]+\frac{a_1}{2}\left[4h(2^2-1^2)+\frac{\left(L^2-(L-2h)\right)^2}{h}\right]+$$

$$+\ldots+\frac{a_m}{m+1}\left[4h^m\left(2^{m+1}-1^{m+1}\right)+\frac{\left(L^{m+1}-(L-2h)^{m+1}\right)}{h}\right],\ r=\frac{I_1}{I_2},$$

we have the following

Theorem 1. *If $\gamma\in[r,1)$ and the conditions (3.1), (3.2) are fulfilled then the solution of the numerical scheme (2.8) remains nonnegative for any initial nonnegative vector α^0.*

Remark 1. If $\gamma=1$ the bound (3.1) tends to infinity which means that the condition of the nonnegativity of numerical scheme (2.8) depends only on the lower bound (3.2).

Remark 2. When $h\to 0$ the sequence of the lower bound in Theorem 1 is monotonely increasing to the bound $\frac{1}{6\gamma a_0}$.

Analogically, the sequence of the upper bound in Theorem 1 is monotonely decreasing to the bound

$$\frac{1}{3(1-\gamma)\left[a_0+a_1\ L+a_2L^2+\ldots+a_mL^m\right]}.$$

Then the conditions of Theorem 1 turn to the conditions

$$(3.3)\qquad \frac{1}{6\gamma a_0}\leq q\leq\frac{1}{3(1-\gamma)\left[a_0+a_1\ L+a_2L^2+\ldots+a_mL^m\right]},$$

and

$$r=\frac{\left[a_0+a_1L+a_2L^2+\ldots+a_mL^m\right]}{\left[3a_0+a_1L+a_2L^2+\ldots+a_0L^m\right]}.$$

So, in case $\gamma \in [r, 1)$ the conditions (3.3) guarantee the nonnegativity of numerical scheme (2.8) for any initial nonnegative vector α^0 and for all space division n. At the same time Theorem 1 gives weaker conditions by any fixed space division n.

Remark 3. Let us examine the case when $p(x)$ is a nonnegative linear function, that is $m = 1$ and $a_1 \neq 0$. Then Theorem 1 yields the conditions

$$\text{(3.4)} \qquad \frac{1}{6\gamma\left[a_0 + \frac{a_1 h}{2} 3\right]} \leq q \leq \frac{1}{3(1-\gamma)\left[a_0 + a_1(L-h)\right]},$$

$$\frac{a_0 + a_1(L-h)}{(3a_0 + a_1(2h+L))} \leq \gamma < 1.$$

In case $h \to 0$ (3.4) reduces to the conditions

$$\text{(3.5)} \qquad \frac{1}{6\gamma a_0} \leq q \leq \frac{1}{3(1-\gamma)[a_0 + a_1 L]}, \qquad \frac{a_0 + a_1 L}{3a_0 + a_1 L} \leq \gamma < 1.$$

Obviously the upper bound (3.1) is a sufficient condition for the nonnegativity of numerical scheme (2.8). To get a greater upper bound let us apply the process, given in [8]. Denoting by

$$T = \begin{bmatrix} I & O \\ -X_2 & X_1 \end{bmatrix},$$

where I is the identity matrix of dimension $(n-1) \times (n-1)$. The problem (2.8) can be rewritten in the form

$$\text{(3.6)} \qquad T \begin{bmatrix} \alpha^j \\ \alpha^{j+1} \end{bmatrix} = \begin{bmatrix} \alpha^j \\ 0 \end{bmatrix}.$$

So, if under some conditions (3.6) conserves the nonnegativity then under the same conditions (2.8) also conserves it. Therefore we examine the condition to have T, an inverse nonnegative (monotone) matrix. For this aim, we are going to apply Theorem 3 in [6]. We partition T into the diagonal part T_d, the positive off-diagonal part T^+ and two negative off-diagonal parts T^Z and T^S. Denoting by

$$T_d = \begin{bmatrix} I & O \\ O & t_d I \end{bmatrix}, \quad T^+ = \begin{bmatrix} O & O \\ -X_2^- & O \end{bmatrix}, \quad T^Z = \begin{bmatrix} O & O \\ O & X_1^- \end{bmatrix},$$

$$T^S = \begin{bmatrix} O & O \\ -X_2^+ & O \end{bmatrix},$$

one can check that in case

(3.7) $$p \le \frac{st^-}{t_d},$$

all conditions of the above theorem are fulfilled, where we have used the notations

$$p = -\left[\frac{4}{6} - q(1-\gamma)\left[a_0\frac{(L-(L-2h))}{h} + a_1\frac{(L^2-(L-2h)^2)}{2h} + \right.\right.$$
$$\left.\left. + a_2\frac{(L^3-(L-2h)^3)}{3h} + \ldots + a_m\frac{(L^{m+1}-(L-2h)^{m+1})}{(m+1)h}\right]\right],$$

$$s = -\left[\frac{1}{6} - q(1-\gamma)\left[a_0\frac{((L-2h)-(L-h))}{h} + a_1\frac{((L-2h)^2-(L-h)^2)}{2h}\right.\right.$$
$$\left.\left. + a_2\frac{((L-2h)^3-(L-h)^3)}{3h} + \ldots + a_m\frac{((L-2h)^{m+1}-(L-h)^{m+1})}{(m+1)h}\right]\right],$$

$$t_d = \left[\frac{2}{3} + \gamma q\left[a_0\frac{(L-(L-2h))}{h} + a_1\frac{(L^2-(L-2h)^2)}{2h} + \right.\right.$$
$$\left.\left. + a_2\frac{(L^3-(L-2h)^3)}{3h} + \ldots + a_m\frac{(L^{m+1}-(L-2h)^{m+1})}{(m+1)h}\right]\right],$$

$$t^- = \left[\frac{1}{6} + q\gamma\left[a_0\frac{((L-2h)-(L-h))}{h} + a_1\frac{((L-2h)^2-(L-h)^2)}{2h} + \right.\right.$$
$$\left.\left. + a_2\frac{((L-2h)^3-(L-h)^3)}{3h} + \ldots + a_m\frac{((L-2h)^{m+1}-(L-h)^{m+1})}{(m+1)h}\right]\right].$$

Of course, in condition (3.7) we consider only the positive value of q. At the same time, for any fixed m and n, this bound is greater than the bound resulted from (3.1). For the sake of simplicity let us denote by q_{pos} the positive bound resulted from (3.7)

Theorem 2. *If the conditions*

$$\frac{1}{6\gamma\left[\frac{a_0}{1}(2-1) + \frac{a_1}{2}h(2^2-1^2) + \ldots + \frac{a_m}{m+1}h^m\left(2^{m+1}-1^{m+1}\right)\right]} \le q \le q_{pos}$$

with $\gamma \in [r, 1)$ are fulfilled, then in case $n \geq 3$ the numerical scheme (2.8) conserves the nonnegativity for any nonnegative vector α^0.

Remark 4. Considering Theorem 2 in case $m = 0$ and $a_0 = 1$ gives the bounds

$$\frac{1}{6\gamma} \leq q \leq \frac{\left[3(2\gamma - 1) + \sqrt{9 - 16\gamma(1-\gamma)}\right]}{12\gamma(1-\gamma)}, \quad \frac{1}{3} \leq \gamma < 1. \tag{3.8}$$

This bound was obtained earlier in [3].

Remark 5. Considering another case of Theorem 2 if $a_0 = a_1 = 1$, $m = = 1$, $h \to 0$, we get

$$\frac{1}{6\gamma} \leq q \leq \frac{3(2\gamma - 1) + \sqrt{9 - 16\gamma(1-\gamma)}}{12\gamma(L+1)(1-\gamma)}, \quad \frac{1+L}{3+L} \leq \gamma < 1. \tag{3.9}$$

Remark 6. In case $a_0 = a_1 = 1$ the upper bound in (3.9) is greater than (3.5) and preserves the nonnegativity of numerical scheme (2.8). So, by using (3.7), we can always get an upper bound greater than (3.1).

Under the condition of Theorem 2 T is monotone and since $T\begin{bmatrix} e \\ e \end{bmatrix} \geq 0$, where $e = [1, 1, 1, \ldots]^T \in R^{(n-1)}$, the maximum principle holds [2]. So, using the nonnegativity, we have

$$\max_{1 \leq i \leq (n-1)} \alpha_i^{j+1} \leq \max_{1 \leq i \leq (n-1)} \alpha_i^j,$$

which results the maximum norm-inequality

$$\left\|\alpha^{j+1}\right\|_c \leq \left\|\alpha^j\right\|_c, \tag{3.10}$$

where

$$\left\|\alpha^j\right\|_c = \max_{1 \leq i \leq (n-1)} \left|\alpha_i^j\right|.$$

Thus, the conditions of the nonnegativity result the monotone convergence of the numerical scheme (2.8) in maximum norm, too.

4. Generalization to the strictly positive function for heat-diffusion coefficients

In this part we generalize our result to the problem (2.1)-(2.3) by considering any arbitrary strictly positive function $p(x)$. We suppose that $p(x)$ belongs to $C^2(\Omega)$ and satisfies the strong nonnegativity condition $0 < p_0 \leq \leq p(x) < \bar{C}$, where $\bar{C}$ is some constant. First of all, we examine the possibility of approximation $p(x)$ and u_0 by some other sufficiently smooth functions, which are close to them. Our goal is to determine how far the solution of the new problem is from the original one when we approximate $p(x)$ by $\bar{p}(x)$ and u_0 by $\bar{u}_0$, we remark that $\bar{p}(x)$ satisfies the conditions $0 < p_0 \leq \bar{p}(x) < \bar{C}$, $\bar{p}(x)$ is a continuous piecewise linear function (spline). Let us consider the problem (2.5) with $p(x) \in C^2(\Omega)$, we remark that $\bar{u}_0$ is a good approximation to u_0 and the corresponding problem

$$\int_0^L \left(\frac{\partial \bar{u}}{\partial t} v + \bar{p}(x) \frac{\partial \bar{u}}{\partial x} \frac{\partial v}{\partial x} \right) dx = 0; \quad \forall v \in H_0^1(\Omega), \tag{4.1}$$

with the approximation $\bar{p}(x)$ instead of $p(x)$.

Then $\bar{u}$ means the solution of the approximated equation and satisfies

$$\bar{u}(0,t) = \bar{u}(L,t) = 0, \tag{4.2}$$

$$\bar{u}(x,0) = \bar{u}_0(x). \tag{4.3}$$

Denoting the solution of the original equation (with $p(x) \in C^2(\Omega)$) by $u(x,t)$ we want to check whether $\bar{u}$ is a good approximation of u or not.

Let $w(x,t) = u(x,t) - \bar{u}(x,t)$. Then

$$\bar{u}(x,t) = u(x,t) - w(x,t), \tag{4.4}$$

$$w(x,0) = u_0(x) - \bar{u}_0(x). \tag{4.5}$$

Subtracting (4.1) from (2.5) we consider $v = w$, integrating over the interval $(0,t)$ with respect to time we get

$$\int_0^t \int_0^L \left(\frac{\partial w}{\partial t} w + p(x) \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} - \bar{p}(x) \frac{\partial \bar{u}}{\partial x} \frac{\partial w}{\partial x} \right) dx dt = 0; \quad \forall w \in H_0^1(\Omega). \tag{4.6}$$

Substituting (4.4) into (4.6) we get

$$\frac{1}{2}\int_0^L w^2(x,t)\,dx - \frac{1}{2}\int_0^L w^2(x,0)dx + \int_0^t\int_0^L \bar{p}(x)\left(\frac{\partial w}{\partial x}\right)^2 dx\,dt =$$

$$= \int_0^t\int_0^L (\bar{p}(x) - p(x))\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}dx\,dt. \tag{4.7}$$

Using (4.7), (4.5) and Young's inequality on the right hand side for

$$\sqrt{\delta}\frac{\partial w}{\partial x}, \quad \frac{1}{\sqrt{\delta}}(\bar{p}(x) - p(x))\frac{\partial u}{\partial x},$$

where δ is arbitrary nonnegative number, one obtains

$$\frac{1}{2}\int_0^L w^2(x,t)dx + \int_0^t\int_0^L \left(\bar{p}(x) - \frac{\delta}{2}\right)\left(\frac{\partial w}{\partial x}\right)^2 dxdt \le \frac{1}{2}\int_0^L (u_0(x) - \bar{u}_0(x))^2 dx +$$

$$+\frac{1}{2\delta}\int_0^t\int_0^L (\bar{p}(x) - p(x))^2\left(\frac{\partial u}{\partial x}\right)^2 dxdt. \tag{4.8}$$

If $p_0 - \frac{\delta}{2} > \frac{\delta}{4}$ then

$$\frac{1}{2}\int_0^L w^2(x,t)dx + \frac{\delta}{4}\int_0^t\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dxdt \le \frac{1}{2}\int_0^L (u_0(x) - \bar{u}_0(x))^2 dx +$$

$$+\frac{1}{2\delta}\int_0^t\int_0^L (\bar{p}(x) - p(x))^2\left(\frac{\partial u}{\partial x}\right)^2 dxdt. \tag{4.9}$$

Denoting $\nu = \min\left\{\frac{1}{2}, \frac{\delta}{4}\right\}$

$$\int_0^L w^2(x,t)dx + \int_0^t\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dxdt \le \frac{1}{\nu}\left[\frac{1}{2}\int_0^L (u_0(x) - \bar{u}_0(x))^2 dx +\right.$$

$$+\frac{1}{2\delta}\int_0^t\int_0^L(\bar{p}(x)-p(x))^2\left(\frac{\partial u}{\partial x}\right)^2 dxdt\Bigg]. \tag{4.10}$$

Since

$$\|w\|^2_{V_2^1(\Omega\times R_0^+)}=\sup_{t\in[0,T]}\int_0^t |u(x,t)|^2dx+\int_0^t\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dxdt, \tag{4.11}$$

where $V_2^1(\Omega\times R_0^+)$ is the Banach space of all the functions from $W_2^{1,0}(\Omega\times R_0^+)$. Then (4.10) reduces to

(4.12)

$$\|w\|^2_{V_2^1(\Omega\times R_0^+)}\le c_1\int_0^L(u_0(x)-\bar{u}_0(x))^2dx+c_2\int_0^t\int_0^L(\bar{p}(x)-p(x))^2\left(\frac{\partial u}{\partial x}\right)^2 dxdt.$$

So

(4.13)

$$\|w\|^2_{V_2^1(\Omega\times R_0^+)}\le c_1\int_0^L(u_0(x)-\bar{u}_0(x))^2dx+c_2\,\|\bar{p}(x)-p(x)\|_c^2\int_0^t\int_0^L\left(\frac{\partial u}{\partial x}\right)^2 dxdt.$$

This means that $\bar{u}(x,t)$ is a good approximation of $u(x,t)$.

5. Nonnegativity of the solution of the problem in case of strictly positive heat-diffusion coefficient function

Our goal is to find some sufficient condition of the nonnegativity of numerical scheme when we replace $p(x)$ by $\bar{p}(x)$, where $\bar{p}(x)$ is piecewise linear approximation having the form

$$\bar{p}(x)=\sum_{k=0}^{n}p(x_k)\Phi_k(x). \tag{5.1}$$

The inequalitiy (4.13) and the special approximation of $\bar{p}(x)$ given in (5.1) guarantees that the order of the error of the new problem (4.1)-(4.3) is the same as for the original numerical problem. By substituting (5.1) into (4.1) and repeating the same calculation, which was given in Chapter 2, we have

$$\int_0^L \frac{\partial \bar{u}}{\partial t} v dx + \int_0^L \sum_{k=0}^{n} p(x_k)\Phi_k(x)\frac{\partial \bar{u}}{\partial x}\frac{\partial v}{\partial x} dx = 0; \quad \forall v \in H_0^1(\Omega). \tag{5.2}$$

We seek the numerical solution in the form

$$\bar{U}_h(x,t) = \sum_{i=1}^{n-1} \alpha_i(t)\Phi_i(x). \tag{5.3}$$

By substituting (5.3) into (5.2) we get a Cauchy problem of the form

$$M\alpha'(t) + Q\alpha(t) = 0; \quad t > 0, \tag{5.4}$$

where $\alpha(0)$ is a given vector being an appropriate nonnegative approximation of the initial function $\bar{u}_0$.

The M matrix, which appears in (5.4), is the same which was defined in (2.7) and

$$Q = \frac{1}{2h}\text{tridiag}[-p(x_{i-1}) - p(x_i), \quad p(x_{i-1})+$$

$$+2p(x_i)+p(x_{i+1}), \; -p(x_i)-p(x_{i+1})], \quad i = 1, 2, \ldots, n-1, \quad Q \in R^{(n-1)\times(n-1)}.$$

Using the one-step method to the discretization of (5.4) we get the following system of linear algebraic equations

$$X_1\alpha^{j+1} = X_2\alpha^j \quad j = 0, 1, 2, \ldots, \tag{5.5}$$

$$\alpha^0 = \alpha(0), \tag{5.6}$$

where X_1, X_2 and α^j are as we defined earlier. By repeating the calculation, which was given in Chapter 3 instead of (3.1), we get

$$q \le \frac{4}{3(1-\gamma)c^{**}}, \tag{5.7}$$

where $c^{**} = \max_i(p(x_{i-1}) + 2p(x_i) + p(x_{i+1}))$, $i = 1, 2, \ldots, n-1$, and instead of (3.2) we get

$$\frac{1}{3\gamma c^*} \le q, \tag{5.8}$$

where $c^* = \min_i(p(x_i) + p(x_{i+1}))$, $i = 1, 2, \ldots, n-2$, or $c^* = \min_i(p(x_{i-1}) + +p(x_i))$, $i = 2, \ldots, n-1$. We have the following

Theorem 3. *If the conditions*

$$\frac{1}{3\gamma c^*} \leq q \leq \frac{4}{3(1-\gamma)c^{**}}, \quad \frac{c^{**}}{4c^* + c^{**}} \leq \gamma < 1 \tag{5.9}$$

are fulfilled then the numerical scheme (5.5) preserves the nonnegativity for any initial nonnegative vector α^0.

Remark 7. We can estimate (5.9) by choosing $c^{**} = 4\bar{C}$ and $c^* = p_0$, then we get

$$\frac{1}{3\gamma p_0} \leq q \leq \frac{1}{3(1-\gamma)\bar{C}}, \quad \frac{\bar{C}}{\bar{C} + p_0} \leq \gamma < 1, \tag{5.10}$$

where $\bar{C} = \max_{[0,L]} p(x)$, $p_0 = \min_{[0,L]} p(x)$.

6. Numerical results

In this section we give some numerical examples. First of all, we consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{6.1}$$

$$u(0,t) = u(1,t) = 0, \tag{6.2}$$

$$u(x,0) = \sin(\pi x), \quad x \in (0,1). \tag{6.3}$$

The exact solution of (6.1)-(6.3) is $u(x,t) = \exp(-(\pi^2 t))\sin(\pi x)$. We investigate the numerical process for the different values of τ taking on the interval determined by (3.8) for the case $x = 0.5$; $\gamma = 0.5$. We notice that with increasing the number of space division the approximation will be better.

Table 1

n	h	τ	error
9	$\frac{1}{9}$	0.008	1.482046E-2
9	$\frac{1}{9}$	0.8	3.31485E-5
15	$\frac{1}{15}$	0.008	5.334079E-3
15	$\frac{1}{15}$	0.8	1.038727E-5
25	$\frac{1}{25}$	0.008	1.920938E-3
25	$\frac{1}{25}$	0.8	2.299581E-6

where the error is defined by $|u_{app}(0.5,\tau) - u(0.5,\tau)|$. The second experiment we have done for the nonhomogeneous problem

(6.4)
$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left((x+1)\frac{\partial u}{\partial x}\right) = \exp(-(x+t))(x^3 - 4x^2 + x + 3) \quad x \in (0,1),\ t > 0;$$

(6.5)
$$u(0,t) = u(1,t) = 0, \quad t > 0,$$

(6.6)
$$u(x,0) = x(1-x)\exp(-x) \quad x \in [0,1].$$

The exact solution of (6.4)-(6.6) is $u(x,t) = x(1-x)\exp(-(x+t))$. By using the value of τ from interval (3.9) we will check our approximation at $x = 0.5$ with $\gamma = 0.9$ by the scheme

(6.7)
$$X_1\alpha^{j+1} = X_2\alpha^j + \tau(\gamma F^{j+1} + (1-\gamma)F^j),$$

(6.8)
$$\alpha^0 = \alpha(0),$$

Table 2

n	h	τ	error
9	$\frac{1}{9}$	0.002	1.646742E-3
9	$\frac{1}{9}$	0.008	1.662046E-3
9	$\frac{1}{9}$	0.2	1.69801E-3
9	$\frac{1}{9}$	0.8	9.348467E-4
15	$\frac{1}{15}$	0.002	5.920082E-4
15	$\frac{1}{15}$	0.008	5.953461E-4
15	$\frac{1}{15}$	0.2	6.068423E-4
15	$\frac{1}{15}$	0.8	3.260076E-4
25	$\frac{1}{25}$	0.002	2.12878E-4
25	$\frac{1}{25}$	0.008	2.151579E-4
25	$\frac{1}{25}$	0.2	2.137274E-4
25	$\frac{1}{25}$	0.8	1.216233E-4

where $F^j = (F_j)^j$,

$$(F_i)^j = \int_0^1 \exp(-(x+jt))(x^3 - 4x^2 + x + 3)\Phi_i(x)dx$$

$$i = 1, 2, \ldots, n-1; \quad j = 0, 1, 2, \ldots.$$

In the third example we chose the value of τ resulted from the inequality (5.9) to the problem (6.4)-(6.6) for checking the approximation (5.3). We checked if u was well approximated by $\bar{u}$ when $p(x)$ was approximated by $\bar{p}(x)$. We saw that the approximation did not change. We got the same result that was given in Table 2 for the same τ and h.

Remark 8. The result, which was given in (5.9), can be used better than all theorems that were given in Chapter 3, because (5.9) can be used for any function $p(x)$ which fulfills the condition $0 < p_0 \le p(x) < \bar{C}$. But the theorems, which were given in Chapter 3, can be used only when $p(x) > 0$ is a monotone increasing polynomial and bounded above.

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