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ON PLEASANT BUT NOT COMPLETELY PLEASANT KNAPSACK PROBLEMS

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Abstract. This paper is devoted to describe the smallest possible pleasant but not completely pleasant knapsack problems. A knapsack problem of n variables is pleasant if the greedy solution is optimal for every right-hand side. The problem is completely pleasant if the same holds for each of the subproblems involving only the first k ($k \leq n$) variables. It is shown that the minimal value of n for which there exist pleasant but not completely pleasant knapsack problems is 4. This result represents a difference between the properties of the ≤ 3 and ≥ 4 dimensional spaces.

1. Introduction

In this paper the following knapsack problems will be considered

$$\max\ (\min) \sum_{j=1}^{n} c_j x_j,$$

$$(1) \qquad \sum_{j=1}^{n} a_j x_j = b,$$

$$\mathbf{x} \in Z_+^n.$$

Thus one of them is a maximization problem (MAXKP) and the other one is a minimization problem (MINKP), but all of the constraints are formally the same. In this paper we assume that the following, so-called *regularity conditions* are satisfied. In the problems the coefficients are positive integers, i.e.

$$(2) \qquad \forall j \quad c_j, a_j \in N.$$

The order of variables is in accordance with their weights and relative weights, i.e.

$$a_1 < a_2 < \ldots < a_n \tag{3}$$

and

$$\frac{c_1}{a_1} \le \frac{c_2}{a_2} \le \ldots \le \frac{c_n}{a_n} \tag{4a}$$

in the case of MAXKP and

$$\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \ge \ldots \ge \frac{c_n}{a_n} \tag{4b}$$

in the case of the MINKP. These latter constraints will be referred as (4). As both (3) and (4) do not necessarily hold for arbitrary sequences $\{a_n\}$ and $\{c_n\}$, thus the two constraints restrict the knapsack problems to be considered. To ensure that the greedy solution is always feasible the following restriction is needed

$$a_1 = 1. \tag{5}$$

The constraints (2), (3), (4) and (5) are the regularity conditions.

Then the greedy solution is defined by

$$x_j^g(n,b) = \left\lfloor \frac{b - \sum\limits_{i=j+1}^{n} a_i x_i^g(n,b)}{a_j} \right\rfloor \qquad j = n,\ n-1, \ldots, 1. \tag{6}$$

The greedy solution will be considered as a function of the right-hand side and the number of variables, as sometimes greedy solutions of problems having only k ($\le n$) variables are needed. More precisely it will always mean the first k variables. The vector $\mathbf{x}^g(k,b)$ is considered as an n-dimensional vector with $x_j^g(k,b) = 0$, $j == k+1, \ldots, n$. A particular optimal solution of the problem will be denoted by $\mathbf{x}^*(n,b)$. The optimal value of the objective function is $f(n,b)$ and the value of the greedy solution is $g(n,b)$, i.e.

$$g(n,b) = \sum_{j=1}^{n} c_j x_j^g(n,b).$$

If it is unambiguous we are speaking about problem P_k emphasizing that the subproblem of the first k variables is considered.

Definition 1. Let u and v be values of the objective function belonging to different solutions. The value u is better or at least as good, resp., as the value v if

$$\begin{cases} u > v \text{ or } u \geq v, & \text{resp., in the case of the MAXKP,} \\ u < v \text{ or } u \leq v, & \text{reps., in the case of the MINKP.} \end{cases}$$

Definition 2. A parametric knapsack problem of type MAXIP or MINKP is called pleasant if the greedy solution is optimal for every right-hand side. A right-hand side b is called pleasant, as well, if the greedy solution of that particular right-hand side is optimal, i.e. $f(n,b) = g(n,b)$. The problem is completely pleasant if for every k $(1 \leq k \leq n)$ the subproblem containing only the first k variables is pleasant, i.e. $\forall b$ and $\forall k$ $f(k,b) = g(k,b)$. Finally a problem is called relatively pleasant if it is pleasant but not completely pleasant.

The most easiest way of testing the pleasantness of a knapsack problem is based on a theorem of [1], [3], [4] and [8]. This is Theorem 3 below. But this test gives a positive answer only if the problem is completely pleasant. This test is an $O(n^2)$ algorithm. In [7] a polynomial algorithm of $O(n^4)$ is discussed, which solves the problem in all of the cases if the regularity conditions are satisfied. From a practical point of view the difference 2 in the degree is huge. Therefore it is interesting to obtain more knowledge about the relatively pleasant problems. The aim of this paper is to describe the relatively pleasant knapsack problems with a minimal number of variables.

2. Some previous results

Until now only the following case of MINKP has been investigated

$$c_1 = c_2 = \ldots = c_n = 1. \tag{7}$$

In this case the smallest possible n is 5 according to a theorem of [9]. In [1] all of the values $a_1, \ldots, a_5$ which give rise to a pleasant but not completely pleasant knapsack problems are shown to be of the following form:

$$1, 2, a, \ a+1, \ 2a,$$

where $a \geq 4$ is a parameter. The complete description of the different kinds of pleasant MINKPs in the case $n = 6$ can be found in [9]. Here the structure is much more complicated.

The following three theorems are needed later on.

Theorem 1. *[6] Assume that the regularity conditions are satisfied. Then any problem with $n = 1$ or 2 is pleasant.*

This can be obtained very easily. An immediate consequence of this is

Proposition 1. *If $n = 3$ and the regularity conditions are satisfied and the knapsack problem is pleasant, then it is completely pleasant.*

The not completely pleasant problems have non-pleasant subproblems and these have non-pleasant right-hand sides. The smallest one is denoted by m, i.e.

$$m = \min\{b \ : \ \exists s, \ f(s,b) \neq g(s,b)\}.$$

The following two theorems have been discovered for MINKP. Theorem 2 is discussed in [2] under condition (7). Theorem 3 has been found under the same condition in [1] and independently for the general MINKP in [3]. A general version of the theorems for both MINKP and MAXKP can be found in [8].

Theorem 2. *Suppose that a non-pleasant knapsack problem satisfies the regularity conditions. Let m be the smallest right-hand side for which $f(n,b) \neq$ $\neq g(n,b)$, define the indices k and p in the following way*

$$k = \max\{j \ : \ x_j^g(n,m) > 0\}$$

and

$$p = \max\{j \ : \ \exists \mathbf{x}^*(n,m), \ x_j^*(n,m) > 0\}.$$

Then there are two indices q and r, such that

$$p \geq q > r \geq 1, \ x_r^g(p, a_k) > 0,$$

and

$$m = a_q + \sum_{j=q}^{p} a_j x_j^g(p, a_k). \tag{8}$$

Theorem 3. *Assume that the regularity conditions are satisfied and the subproblem P_{n-1} is pleasant. Let the integers s and t be defined as follows.*

$$s = \left\lceil \frac{a_n}{a_{n-1}} \right\rceil, \quad t = s a_{n-1} - a_n. \tag{9}$$

Then the following two statements are equivalent:

(i) the problem is pleasant,

(ii) $c_n + g(n,t)$ *is at least as good as* sc_{n-1}.

3. The relatively pleasant problems with $n = 4$

An immediate consequence of Proposition 1 is that if a relatively pleasant knapsack problem of 4 variables exists then only its subproblem P_3 is non-pleasant. This observation plays an important role in the proofs of this section.

Proposition 2. *Assume that the regularity conditions are satisfied in a relatively pleasant problem of 4 variables. Then the following conditions are satisfied*

(10) $$a_2 \text{ is not a divisor of } a_3,$$

(11) $$a_3 < a_4 \leq \left\lceil \frac{a_3}{a_2} \right\rceil a_2,$$

(12) $$\left(\left\lceil \frac{a_3}{a_2} \right\rceil \right) c_2 \quad \text{is better than} \quad c_3 + \left(\left\lceil \frac{a_3}{a_2} \right\rceil a_2 - a_3 \right) c_1,$$

(13) $$c_4 + \left(\left\lceil \frac{a_3}{a_2} \right\rceil a_2 - a_4 \right) c_1 \quad \text{is at least as good as} \quad \left\lceil \frac{a_3}{a_2} \right\rceil c_2,$$

where (10) and (11) are consequences of (12) and (13).

Proof. As P_3 is not pleasant it follows from Theorem 1 and 3 that condition (12) must be satisfied. Then (10) follows from (4). But P_4 is pleasant, thus $\mathbf{x}^g\left(4, \left\lceil \frac{a_3}{a_2} \right\rceil a_2\right)$ is optimal. If $a_4 < \left\lceil \frac{a_3}{a_2} \right\rceil a_2$ then $\mathbf{x}^g\left(4, \left\lceil \frac{a_3}{a_2} \right\rceil a_2\right) = \mathbf{x}^g\left(3, \left\lceil \frac{a_3}{a_2} \right\rceil a_2\right)$. As $\mathbf{x}^g\left(3, \left\lceil \frac{a_3}{a_2} \right\rceil a_2\right)$ is not optimal, neither is $\mathbf{x}^g\left(4, \left\lceil \frac{a_3}{a_2} \right\rceil a_2\right)$ which is a contradiction. Thus (11) holds. On the left-hand side of (13) is the value of $g\left(4, \left\lceil \frac{a_3}{a_2} \right\rceil a_2\right)$ and $f\left(3, \left\lceil \frac{a_3}{a_2} \right\rceil a_2\right)$ is on the right-hand side. As the first one is the optimal value, (13) follows immediately.

Theorem 4. *Assume that the regularity conditions hold for a problem of 4 variables. The problem is relatively pleasant if and only if (11) and (12) and (13) and the following conditions are satisfied*

$$c_4 + g(2, 2a_3 - a_4) \text{ is at least as good as } 2c_3, \tag{14}$$

$$c_4 + (a_3 + a_2 - a_4)c_1 \text{ is at least as good as } c_3 + c_2. \tag{15}$$

Proof. It follows from Proposition 2 that (11) and (12) and (13) are necessary for the relative pleasantness. If (14) or (15), resp., are violated then the problem is not pleasant for the right-hand side $2a_3$ or $a_2 + a_3$, resp.

Suppose that (11)-(15) hold. It follows from (12) that P_3 is not pleasant. Thus it is enough to show that P_4 is pleasant. Theorem 2 will be applied. For the distinct values of the triplets (k, p, q) there are the following cases.

k	p	q
4	3	3
4	3	2
4	2	2
3	2	2

Let w_{kpq} be the integer on the right-hand side of (8). To prove the pleasantness of the problem it is sufficient to show that the greedy solution is optimal for all w_{kpq}.

Case (i): w_{433}. It follows from (11) that $\left\lceil \frac{a_4}{a_3} \right\rceil = 2$ and thus $w_{433} = 2a_3$. For this right-hand side the following statement is obtained from the regularity conditions. Among those feasible solutions, where $x_4 = 0$, the best one is $(0,0,2,0)$. Thus (14) is necessary and sufficient to the pleasantness of this right-hand side.

Case (ii): w_{432}. The following inequality is obtained from (10) and (11).

$$\left\lfloor \frac{a_3}{a_2} \right\rfloor a_2 < a_3 < a_4 \leq \left\lceil \frac{a_3}{a_2} \right\rceil a_2. \tag{16}$$

Hence $x_2^g(3, a_4) = 0$ and thus $w_{432} = a_3 + a_2$. Again, it is easy to see from the regularity conditions that $\mathbf{x}^*(3, a_3 + a_2) = (0,1,1,0)$. Thus (15) is necessary and sufficient to the pleasantness of this right-hand side.

Case (iii): w_{422}. It follows also from (16) that $w_{422} = \left\lceil \frac{a_3}{a_2} \right\rceil a_2$. It is clear from (12) that $\mathbf{x}^*(3, \lceil \frac{a_3}{a_2} \lceil a_2) = (0, \left\lceil \frac{a_3}{a_2} \right\rceil, 0, 0)$. Thus (13) is necessary and sufficient to the pleasantness of this right-hand side.

Case (iv): w_{322}. It follows from (16) that $w_{322} = w_{422}$, thus there is no need for further condition.

Examples. It is easy to see that the conditions of Theorem 4 are satisfied in the following examples, i.e. relatively pleasant problems with 4 variables exist for both MAXKP and MINKP. Assume that for the positive integers a, c, k and m the inequalities $2k < a$ and $2 \leq m$ and $2m + 2 \leq c$ hold.

$$\begin{aligned} &\max\ 1x_1 + (a+2)x_2 + ((m-1)a + k + 2m - 1)x_3 + (am + c)x_4 \\ &\qquad 1x_1 + ax_2 + ((m-1)a + k)x_3 + amx_4 = b \\ &\qquad x_1, \dots, x_4 \in Z_+, \end{aligned}$$

and

$$\begin{aligned} &\min\ 1x_1 + (a-2)x_2 + ((m-1)a + k - 2m + 1)x_3 + (am - c)x_4 \\ &\qquad 1x_1 + ax_2 + ((m-1)a + k)x_3 + amx_4 = b \\ &\qquad x_1, \dots, x_4 \in Z_+. \end{aligned}$$

In the latter problem it is assumed that $am > c$. To see that these are not the only possible type of problems, the following one is provided

$$\begin{aligned} &\max\ 1x_1 + 7x_2 + 9x_3 + cx_4 \\ &\qquad 1x_1 + 5x_2 + 6x_3 + 9x_4 = b \\ &\qquad x_1, \dots, x_4 \in Z_+, \end{aligned}$$

where $c \leq 15$.

Finally an easy consequence of the above mentioned results is discussed. It is also a direct consequence of a more general theorem of [9].

Proposition 3. *Let us consider any MINKP of 4 variables satisfying the regularity conditions and (7). If it is pleasant then it is completely pleasant.*

Proof. (15) takes now the following form

$$1 + a_3 + a_2 - a_4 \leq 2.$$

Hence $a_3 + a_2 \leq a_4 + 1$. It follows from (16) that

$$a_4 = a_3 + a_2 - 1. \tag{17}$$

Hence the current form of (15) is

$$1 + g(2, a_3 - a_2 + 1) \leq 2.$$

Thus the value of $g(2, a_3 - a_2 + 1)$ is 0 or 1. The first one is not possible as $a_3 - a_2 + 1 > 1$. Then the only possibility for the satisfaction of the equation $g(2, a_3 - a_2 + 1) = 1$ is that

$$a_3 - a_2 + 1 = a_2.$$

Therefore $a_3 = 2a_2 - 1$. Hence

$$\left\lceil \frac{a_3}{a_2} \right\rceil = 2.$$

Thus the current form of (12) is

$$2 < 2$$

which is a contradiction.

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