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A CONNECTION BETWEEN A CLASS OF ITERATIVE RECURRENCE RELATIONS AND SOME WORD SEQUENCES

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To the memory of Imre Környei
To the memory of Béla Kovács

Abstract. Let $k \geq 2$ fixed integer and define a sequence G_k by $G_k(0) = 0$ and $G_k(n) = n - G_k(G_k(\dots(n-1))\dots)$, $n \geq 1$ with k iterations of G_k on the right-hand side. In this paper we show connections between properties of this sequence and generalized Zeckendorf representation of natural integers.

Let $k \geq 2$ be integer. Define the sequences $G_k(n)$ by $G_k(0) = 0$ and

$$G_k(n) = n - G_k(G_k(\dots(G_k(n-1))\dots), \quad n \geq 1, \tag{1}$$

with k iterations of G_k on the right-hand side. This class of iterativ recurrence relations was investigated by various authors, see for instance in [1], [7], [8] and [9].

In particular, it was independently shown in [2] and [3] that $G_2(n)$ can be given expicitly as

$$G_2(n) = [(n+1)\alpha] \tag{2}$$

where α is the unique positive root of the equation $x^2 + x - 1 = 0$, i.e. $\alpha = (\sqrt{5}-1)/2$ and $[x]$ denotes the greatest integer less than or equal to x.

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D. S. Meek and G. H. J. Van Rees [9] gave a solution of (1) in terms of generalized Fibonacci base reprezentations of n. For a fixed integer $k \geq 2$ they defined the sequence $F_1^{(k)}, F_2^{(k)}, \ldots$ by

$$F_n^{(k)} = n \quad \text{for} \quad n = 1, 2, \ldots, k \tag{3}$$

and

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-k}^{(k)} \quad \text{for} \quad n \geq k+1. \tag{4}$$

This sequence can be used to represent uniquely any positiv integer n as follows.

Find the largest $F_{j_1}^{(k)}$ that is less than or equal to n. Then find the largest $F_{j_2}^{(k)}$ less than or equal to $n - F_{j_1}^{(k)}$, and continue this way.

So n can be written as

$$n = \sum_{i=1}^{j_t} a_i F_i^{(k)} \tag{5}$$

or shorter

$$n = a_{j_t} a_{j_t - 1} \ldots a_2 a_1, \tag{6}$$

where $a_{j_1} = a_{j_2} = \ldots = a_{j_t} = 1$ and the other a_i's equal zeros. We shall call $n = a_{j_t} \ldots a_1$ the Zeckendorf reprezentation of n in a "generalized Fibonacci base". It is clear from this definition that

$$j_{i+1} - j_i \geq k \quad \text{for all} \quad 1 \leq i < i+1 \leq t, \tag{7}$$

i.e. in (6) between two "1"s are at least $k-1$ "0"s. The "truncation" T_k of a representation is defined in [9] as follows. Let n be represented as in (5) (or (6)).

Then

$$T_k(n) = \sum_{i=2}^{j_t} a_i F_{i-1}^{(k)} = a_{j_t - 1} a_{j_t - 2} \ldots a_2. \tag{8}$$

D.S. Meek and G.H.J. Van Rees [9] proved the following result:

$$G_k(n) = T_k(n-1) + 1 \tag{9}$$

for $n \geq 1$, where $T_k(0) = 0$.

Define $D_k(n)$ by

$$D_k(n) = G_k(n+1) - G_k(n), \quad n = 0, 1, \ldots. \tag{10}$$

From (9) we see, that we can write

$$D_k(n) = T_k(n) - T_k(n-1), \quad n = 1, 2, \ldots. \tag{11}$$

It is easy to prove [see 10] that $D_k(n) = 0$ or 1 for any non-negative integer n and $k \geq 2$.

Let $D^{(k)}$ denote the infinite string

$$D^{(k)} = D_k(0)D_k(1)\ldots, \tag{12}$$

whose $n-1$-th element is $D_k(n)$. Using the digits "0" and "1" we can form words even word sequences by "juxtaposing digits".

Let $C^{(0)}$ be an infinite string defined by

$$C_0^{(2)} = 0, \quad C_1^{(2)} = 1, \tag{13}$$

$$C_n^{(2)} = C_{n-2}^{(2)}C_{n-1}^{(2)}, \quad n \geq 2 \tag{14}$$

and

$$C^{(2)} = C_1^{(2)}C_2^{(2)}\ldots C_n^{(2)}\ldots, \tag{15}$$

where the operation of the right hand sides of (14) and (15) is the concatenation of words.

It is well known (see [12]), that

$$D^{(2)} = C^{(2)} \tag{16}$$

and the word sequence defined by (13) and (14) has been studied extensively (e.g. in [10], [11], [5] and [4]).

In this paper we extend the above result like (16) for arbitrary $k \geq 2$.

If n is a positive integer, then we denote the Zeckendorf representation of n in the basis $F_n^{(k)}$ (defined in (3) and (4)) by

$$n = A_{n,p}^{(k)}\ldots A_{n,1}^{(k)} = \sum_{s=1}^{r} A_{n,s}^{(k)}F_s^{(k)}, \tag{17}$$

and

$$A_{n,s}^{(k)} = 0 \tag{18}$$

if $s > r$ or $n = 0$. Comparing (5) to (17) we can note that $r = j_t$ and $A_{n,s}^{(k)} = a_s$ for $1 \leq s \leq r$.

We show a connection between $D_k(n)$ (defined by (10)) and $A_{n,1}^{(k)}$.

Theorem 1. *For all integers $k \geq 2,\ n \geq 0$ we have*

$$A_{n,1}^{(k)} = 1 - D_k(n). \tag{19}$$

Proof. Since $G_k(0) = 0,\ G_k(1) = G_k(2) = 1$ and so $D_k(0) = 1,\ D_k(1) = 0$, furthermore $A_{0,1} = 0$ and $A_{1,1} = 1$, (19) holds for $n = 0$ and 1. Assuming that $n \geq 2$ it follows from (11) that (19) is equvivalent to

$$T_k(n) - T_k(n-1) = 1 - A_{n,1}^{(k)}, \quad n \geq 2. \tag{20}$$

If $A_{n,1}^{(k)} = 1$, i.e. $n = A_{n,r}^{(k)} \dots A_{n,2}^{(k)}1$, then $n - 1 = A_{n,r}^{(k)} \dots A_{n,2}^{(k)}0$ and so $T_k(n) = A_{n,r}^{(k)} \dots A_{n,2}^{(k)} = T_k(n-1)$, i.e. (20) is satisfied.

If $A_{n,1}^{(k)} = 0$ then denoting by $A_{n,q}^{(k)}$ the last "1" in $A_{n,r}^{(k)} \dots A_{n,1}^{(k)}$ (i.e. $q = \min\{s \mid A_{n,s}^{(k)} = 1, 1 \leq s \leq r\}$), $n = A_{n,r}^{(k)} \dots A_{n,q+1}^{(k)}10\dots0$ (here $q \leq r$), and $n - 1 = A_{n,r}^{(k)} \dots A_{n,q+1}^{(k)}01b_{q-2}\dots b_1$, where $b_i = 1$ if and only if $q - i$ is divisible by k, furthermore if $q = r$ then $n = 10\dots0$ and $n - 1 = 1b_{q-2}\dots b_1$.

In this case

$$T_k(n) = A_{n,r}^{(k)} \dots A_{n,q+1}^{(k)}10\dots0 \neq A_{n,r}^{(k)} \dots A_{n,q+1}^{(k)}01b_{q-2}\dots b_2 = T_k(n-1),$$

from which we have $T(n) - T(n-1) = 1$, which was to be proved.

To formulate our second theorem we have to define some binary word sequences.

Let $s \geq 1$ and $k \geq 2$ be arbitray fixed integers. Let the initial words be defined by

$$Q_{m,s}^{(k)} = b_0 b_1 \dots B_{F_m^{(k)}-1} \quad \text{for all} \quad s \leq m \leq s + k - 1, \tag{21}$$

where $b_n = 1$ if and only if $F_s^{(k)} \leq n < F_{s+1}^{(k)}$ and $b_n = 0$ (in the other case), furthermore let

$$Q_{m,s}^{(k)} = Q_{m-1}^{(k)} Q_{m-k,s}^{(k)}, \quad \text{for all} \quad m \geq s + k, \tag{22}$$

where the operation of the right-hand side is the concatenation of words.

Theorem 2. *If $0 \le n < F_m^{(k)}$ and $m \ge s \ge 1$ then the $n+1$-th character of $Q_{m,s}^{(k)}$ is $A_{n,s}^{(k)}$ (defined by (17) and (18)), i.e.*

$$Q_{m,s}^{(k)} = A_{0,s}^{(k)} A_{1,s}^{(k)} \dots A_{F_m^{(k)}-1,s}^{(k)}, \quad \textit{for all} \quad m \ge s. \tag{23}$$

Proof. From the definition of $Q_{m,s}^{(k)}$ and $F_n^{(k)}$ it immediately follows that $Q_{m,s}^{(k)}$ consist of $F_m^{(k)}$ characters.

By (17) $n = A_{n,r}^{(k)} \dots A_{n,1}^{(k)}$ is the Zeckendorf representation of n in the basis $F_n^{(k)}$.

If $0 \le n < F_s^{(K)}$ then $A_{n,s}^{(k)} = 0$ (by (18)); if $F_s^{(k)} \le n < F_{s+1}^{(k)}$ then $A_{n,s}^{(k)} = A_{n,r}^{(k)} = 1$; if $F_{s+1}^{(k)} \le n < F_{s+k-1}^{(k)}$ then $A_{n,s}^{(k)} = 0$ (using (7) and the inequality $r - s < k$). Regarding these facts and (21) we have verified (23) for all m where $s \le m \le s + k - 1$.

Suppose that $m \ge s + k$

$$Q_{m-1,s}^{(k)} = A_{0,s}^{(k)} \dots A_{F_{m-1}^{(k)}-1,s}^{(k)}$$

and

$$Q_{m-k,s}^{(k)} = A_{0,s}^{(k)} \dots A_{F_{m-k}^{(k)}-1,s}^{(k)}.$$

Then from (22) we obtain

$$Q_{m,s}^{(k)} = A_{0,s}^{(k)} \dots A_{F_{m-k}^{(k)}-1,s} A_{0,s}^{(k)} \dots A_{F_{m-k}^{(k)}-1,s}^{(k)}. \tag{24}$$

If $F_{m-1}^{(k)} \le n < F_{m-1}^{(k)} + F_{m-k}^{(k)} = F_m^{(k)}$, then

$$n = A_{n,m-1}^{(k)} \dots A_{n,m-k}^{(k)} \dots A_{n,s}^{(k)} \dots A_{n,1}^{(k)}, \tag{25}$$

where $1 \le s \le m - k$, $A_{n,m-1}^{(k)} = 1$ and $A_{n,i}^{(k)} = 0$ for all i $(m - k < i \le m - 1)$.

If $q = \max\{i \mid 1 \le i < m - 1,\ A_{n,i}^{(k)} = 1,\ F_{m-1}^{(k)} < n < F_m^{(k)}\}$, then from (25) we get

$$n - F_{m-1}^{(k)} = A_{n,q}^{(k)} \dots A_{n,1}^{(k)} = A_{n-F_{m-1}^{(k)}-1}^{(k)}, \dots A_{n-F_{m-1}^{(k)},q}^{(k)}. \tag{26}$$

From (25) and (26) we have

$$A^{(k)}_{n,s} = A^{(k)}_{n-F^{(k)}_{m-1},s} \quad \text{for} \quad 1 \le s \le q. \tag{27}$$

It is clear that $A^{(k)}_{F^{(k)}_{m-1},s} = 0 = A^{(k)}_{0,s}$ $(1 \le s \le m-k)$, and by (18) and the definition of q

$$A^{(k)}_{n,s} = 0 = A^{(k)}_{n-F^{(k)}_{m-1}-s} \tag{28}$$

follows for $q \le s \le m-k$. Using (27) and (28) we can obtain (23) from (24). This completes the proof of the theorem.

Now we define the complement of the word sequences given by (21) and (22). Let $\overline{Q}^{(k)}_{m,s} = d_0 \dots d_{n-F^{(k)}_{m-1}}$ for all $s \le m \le s+k-1$, where $d_n = 0$ if and only if $F^{(k)}_s \le n \le F^{(k)}_{s+1}$ and $d_n = 1$ else. Furthermore let $\overline{Q}^{(k)}_{m,s} = \overline{Q}^{(k)}_{m-1,s}\overline{Q}_{m-k,s}$, for all $m \ge s+k$. Denote by $\overline{Q}^{(k)}$ an infinite string whose n-th character is the n-th character of $\overline{Q}^{(k)}_{m,1}$, where $n < F^{(k)}_m$.

We can simply formulate our last theorem.

Theorem 3. *For all $k \ge 2$ integers*

$$D^{(k)} = \overline{Q}^{(k)}, \tag{26}$$

where $D^{(k)}$ denotes an infinite string, too, defined by (12).

Proof. It immediately follows from Theorems 1 and 2.

Finally we note that using (26) and (16) we can obtain the equalities

$$D^{(2)} = C^{(2)} = \overline{Q}^{(2)}.$$

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