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THE MEAN VALUES OF MULTIPLICATIVE FUNCTIONS I.

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Dedicated to the memory of Professor I. Környei

1. Results

Let $g_i : \mathbb{N} \to \mathbb{C}$, $i = 1,2$, be multiplicative functions $\big(g_i(mn) = g_i(m)g_i(n)$ for $(m,n) = 1$, $g_i(1) = 1\big)$. Throughout the paper p and q denote primes; m, n, and k are natural numbers; $c_1, c_2, \ldots$ are positive constants.

In this paper we shall be concerned with the mean values of multiplicative functions

$$M_x(g_1, g_2) = \frac{1}{x} \sum_{n \le x} g_1(n+1)\, g_2(n). \tag{1}$$

Particular cases of this kind have already been studied in [3,4,5,2,8]. Estimates of (1) can be used to obtain the information on the behaviour of the distribution of the sum

$$f_1(n+1) + f_2(n)$$

where f_1 and f_2 are real-valued additive functions (see [2,8]). In the proofs we shall follow ideas and methods of A. Rényi [6], A. Hildebrand [2] and R. Warlimont [7].

Let us put

$$S(r,x) = \sum_{r<p\le x} \frac{|g_1(p)-1|^2 + |g_2(p)-1|^2}{p}$$

and

$$P(x) = \prod_{p \le x} \left(1 - \frac{2}{p} + \left(1 - \frac{1}{p}\right) \sum_{m=1}^{\infty} \frac{g_1(p^m) + g_2(p^m)}{p^m}\right). \tag{2}$$

The main result of this article is the following

Theorem. *Let the moduli of multiplicative functions g_1 and g_2 do not exceed* 1. *Then there exists a positive absolute constant c that for $x \geq r \geq 2$*

$$(3)\quad M_x(g_1, g_2) = P(x) + O\Big(x^{-1/3}\exp\left(c\, r^{2/3}\right) + (r\,\log r)^{-1/2} + \big(S(r, x)\big)^{1/2}\Big)$$

where the constant in the symbol O is absolute, too.

Let us note that the powers of x and r in the first summand of the remainder term of (3) can be changed by another ones (see (23)).

As an application we shall formulate a few corollaries.

Let φ mean the Euler function, $\sigma(n)$ be the sum of the positive divisors of n, and

$$A_k = \{n \mid p^m \| n \quad \text{implies} \quad m < k\}$$

denote the set of k-free natural numbers.

Corollary 1. *For $x \geq 2$*

$$\frac{1}{x}\sum_{n\leq x}\frac{\varphi(n+1)\,\varphi(n)}{(n+1)n} = \prod_p\left(1-\frac{2}{p^2}\right) + O\Big(\frac{1}{(\log x)^{\kappa}}\Big),$$

$$\frac{1}{x}\sum_{n\leq x}\frac{\varphi(n+1)\,\varphi(n)}{\sigma(n+1)\sigma(n)} = \prod_p\left(1-\frac{2}{p}+2\Big(1-\frac{1}{p}\Big)^2\sum_{m=1}^{\infty}\frac{1}{1+p+\ldots+p^m}\right) + O\Big(\frac{1}{(\log x)^{\kappa}}\Big),$$

$$\frac{1}{x}\sum_{\substack{n\leq x\\ n,n+1\in A_k}} 1 = \prod_p\left(1-\frac{2}{p^k}\right) + O\Big(\frac{1}{(\log x)^{\kappa}}\Big),$$

$$\frac{1}{x}\sum_{\substack{n\leq x\\ n+1\in A_k}}\frac{\varphi(n)}{n} = \prod_p\left(1-\frac{1}{p^2}-\frac{1}{p^k}\right) + O\Big(\frac{1}{(\log x)^{\kappa}}\Big)$$

where κ, $0<\kappa<1$, is arbitrary. The constants in the symbols O may depend on κ only.

Let us designate

$$\sum_{|f_i(p)|\le 1} \frac{f_i^2(p)}{p}, \quad i=1,2, \tag{4}$$

$$\sum_{|f_i(p)|>1} \frac{1}{p}, \quad i=1,2, \tag{5}$$

$$\sum_{\substack{|f_1(p)|\le 1\\ |f_2(p)|\le 1}} \frac{f_1(p)+f_2(p)}{p}. \tag{6}$$

Corollary 2. *Let f_1 and f_2 be real-valued additive functions and let the series* (4), (5), (6) *converge. Then the distribution functions*

$$\frac{1}{[x]}\#\left\{n\,\middle|\, n\le x,\ f_1(n+1)+f_2(n)\le z\right\} \tag{7}$$

converge weakly towards a limit distribution as $x\to\infty$, and the characteristic function of this limit distribution is equal to

$$\prod_p\left(1-\frac{2}{p}+\left(1-\frac{1}{p}\right)\sum_{m=1}^{\infty}\frac{\exp\left(itf_1(p^m)\right)+\exp\left(itf_2(p^m)\right)}{p^m}\right). \tag{8}$$

From Corollary 2 it follows immediately

Corollary 3. *The distribution functions*

$$\frac{1}{[x]}\#\left\{n\,\middle|\, n\le x,\ \frac{\varphi(n+1)\,\varphi(n)}{(n+1)n}\le e^z\right\},$$

$$\frac{1}{[x]}\#\left\{n\,\middle|\, n\le x,\ \frac{\sigma(n+1)\,\sigma(n)}{(n+1)n}\le e^z\right\},$$

$$\frac{1}{[x]}\#\left\{n\,\middle|\, n\le x,\ \frac{\varphi(n+1)\,\varphi(n)}{\sigma(n+1)\sigma(n)}\le e^z\right\}$$

converge weakly towards limit distributions as $x \to \infty$. The characteristic functions of these limit distributions equal

$$\prod_p \left(1+\frac{2}{p}\left(\left(1-\frac{1}{p}\right)^{it}-1\right)\right),$$

$$\prod_p \left(1-\frac{2}{p}+2\left(1-\frac{1}{p}\right)\sum_{m=1}^{\infty}\frac{\left(1+\frac{1}{p-1}\left(1-\frac{1}{p^m}\right)\right)^{it}}{p^m}\right),$$

$$\prod_p \left(1-\frac{2}{p}+2\left(1-\frac{1}{p}\right)\sum_{m=1}^{\infty}\frac{\left(\frac{p^{m-1}(p-1)^2}{p^{m+1}-1}\right)^{it}}{p^m}\right),$$

respectively.

2. Proofs

Proof of Theorem. Let us put

$$P(r,x)=\frac{P(x)}{P(r)}$$

for $2 \le r < x$. Define multiplicative functions g_{ir} and g^*_{ir}, $i = 1, 2$, by

$$g_{ir}(p^m)=\begin{cases} g_i(p^m) & \text{if } p \le r,\\ 1 & \text{if } p > r,\end{cases}$$

$$g^*_{ir}=\frac{g_i}{g_{ir}},$$

and multiplicative functions h_{ir}, $i = 1, 2$, by

$$h_{ir}(p^m)=\begin{cases} g_i(p^m)-g_i(p^{m-1}) & \text{if } p \le r,\\ 0 & \text{if } p > r,\end{cases}$$

so that $g_{ir} = 1 * h_{ir}$.

Now we can write

$$
(9)\quad \begin{aligned} M_x(g_1,g_2) - P(x) &= P(r,x)\Big(\frac{1}{x}\sum_{n\le x} g_{1r}(n+1)g_{2r}(n) - P(r)\Big) + \\ &+ \frac{1}{x}\sum_{n\le x} g_{1r}(n+1)g_{2r}(n)\left(g^*_{1r}(n+1)g^*_{2r}(n) - P(r,x)\right). \end{aligned}
$$

The moduli of the multiplicands under the multiplication sign of (2) do not exceed 1. Thereby it follows from (9) that

$$
(10)\quad \begin{aligned} \left|M_x(g_1,g_2) - P(x)\right| &\le \left|\frac{1}{x}\sum_{n\le x} g_{1r}(n+1)g_{2r}(n) - P(r)\right| + \\ &+ \frac{1}{x}\sum_{n\le x}\left|g^*_{1r}(n+1)g^*_{2r}(n) - P(r,x)\right| = R_1 + R_2. \end{aligned}
$$

First let us estimate R_1. It follows from the definition of the functions h_{ir} that

$$
(11)\quad \begin{aligned} \frac{1}{x}\sum_{n\le x} g_{1r}(n+1)g_{2r}(n) &= \frac{1}{x}\sum_{n\le x}\sum_{d\mid n+1} h_{1r}(d)\sum_{d'\mid n} h_{2r}(d') = \\ &= \frac{1}{x}\sum_{d\le x+1}\sum_{d'\le x} h_{1r}(d)h_{2r}(d')\sum_{\substack{n\le x\\ d\mid n+1\\ d'\mid n}} 1 = \sum_{d\le x+1}\sum_{\substack{d'\le x\\ (d,d')=1}} \frac{h_{1r}(d)h_{2r}(d')}{dd'} + \\ &+ O\Bigg(\frac{1}{x}\sum_{d\le x+1}\sum_{\substack{d'\le x\\ (d,d')=1}} \left|h_{1r}(d)h_{2r}(d')\right|\Bigg) = P_1 + R_3. \end{aligned}
$$

It is easy to see that

$$
R_3 \ll x^{2\alpha-1}\sum_{d=1}^{\infty}\frac{|h_{1r}(d)|}{d^{\alpha}}\sum_{d'=1}^{\infty}\frac{|h_{2r}(d')|}{d'^{\alpha}}
$$

where the value of α, $0<\alpha<1$, will be chosen later. Since

$$\sum_{d=1}^{\infty}\frac{|h_{ir}(d)|}{d^{\alpha}}=\prod_{p\le r}\left(1+\sum_{m=1}^{\infty}\frac{|h_{ir}(p^m)|}{p^{m\alpha}}\right)\le\prod_{p\le r}\left(1+\frac{2}{p^{\alpha}-1}\right)\le$$
$$\le\exp\left(c_1\sum_{p\le r}\frac{1}{p^{\alpha}}\right)\le\exp\left(\frac{c_2r^{1-\alpha}}{\log r}\right) \tag{12}$$

with suitable c_1 and c_2, then

$$R_3\ll x^{2\alpha-1}\exp\left(\frac{2c_2r^{1-\alpha}}{\log r}\right).$$

Return to (11). We have

$$P_1=\sum_{d=1}^{\infty}\sum_{\substack{d'=1\\(d,d')=1}}^{\infty}\frac{h_{1r}(d)\,h_{2r}(d')}{dd'}+$$
$$+O\Bigg(\sum_{d>x}\frac{|h_{1r}(d)|}{d}\sum_{d'=1}^{\infty}\frac{|h_{2r}(d')|}{d'}+\sum_{d'>x}\frac{|h_{2r}(d')|}{d'}\sum_{d=1}^{\infty}\frac{|h_{1r}(d)|}{d}\Bigg)= \tag{13}$$
$$=\prod_{p\le r}\left(1+\sum_{m=1}^{\infty}\frac{h_1(p^m)+h_2(p^m)}{p^m}\right)+R_4.$$

The product in the last equality is equal to $P(r)$.

For the estimation of R_4 we get analogously as in (12)

$$\sum_{d=1}^{\infty}\frac{|h_{ir}(d)|}{d}\le\exp\left(c_3\sum_{p\le r}\frac{1}{p}\right)\ll(\log r)^{c_4}$$

and

$$\sum_{d>x}\frac{|h_{ir}(d)|}{d}\le\frac{1}{x^{\beta}}\sum_{d=1}^{\infty}\frac{|h_{ir}(d)|}{d^{1-\beta}}\le\frac{1}{x^{\beta}}\exp\left(\frac{c_5r^{\beta}}{\log r}\right)$$

where β, $0<\beta<1$, will be chosen later as well. Thus

$$R_4\ll x^{-\beta}(\log r)^{c_4}\exp\left(\frac{c_5r^{\beta}}{\log r}\right).$$

From the obtained estimates of R_3 and R_4 and from (13) and (11) it follows now that

$$R_1 \ll x^{2\alpha-1}\exp\left(\frac{2c_2 r^{1-\alpha}}{\log r}\right) + x^{-\beta}\exp\left(\frac{c_6 r^{\beta}}{\log r}\right).$$

For the estimation of R_2 we shall repeat the way used by R. Warlimont [7]. Put

$$N_r = \left\{n \mid \exists\, p^m \| n+1, p > r, |1-g_1(p^m)| > \frac{1}{2} \text{ or } \exists\, q^k \| n, q > r, |1-g_2(q^k)| > \frac{1}{2}\right\}$$

and decompose R_2 into two sums:

$$(14)\qquad \begin{aligned} R_2 = {} & \frac{1}{x}\sum_{\substack{n\le x\\ n\in N_r}} \left|g_{1r}^*(n+1)g_{2r}^*(n) - P(r,x)\right| + \\ & + \frac{1}{x}\sum_{\substack{n\le x\\ n\notin N_r}} \left|g_{1r}^*(n+1)g_{2r}^*(n) - P(r,x)\right| = R_5 + R_6\,. \end{aligned}$$

Let us estimate the sum R_5. We have

$$(15)\qquad R_5 \le \frac{2}{x}\sum_{\substack{n\le x\\ n\in N_r}} 1 \le \frac{2}{x}\sum\nolimits^* \sum_{\substack{n\le x+1\\ p^m \| n}} 1 \ll \sum\nolimits^* \frac{1}{p^m}$$

where the sign $*$ means that the summation is extended over all prime powers $p^m \le x+1$ where $p > r$ and

$$\left|1 - g_1(p^m)\right| > \frac{1}{2} \quad \text{or} \quad \left|1 - g_2(p^m)\right| > \frac{1}{2}\,.$$

Continuing (15) we obtain

$$(16)\qquad \begin{aligned} R_5 \ll {} & \sum_{m=1}^{*} \frac{1}{p} + \sum_{p>r}\sum_{m=2}^{\infty} \frac{1}{p^m} \ll \sum_{r<p\le x+1} \frac{|g_1(p)-1|^2 + |g_2(p)-1|^2}{p} + \\ & + \sum_{p>r} \frac{1}{p^2} \ll S(r,x) + (r\log r)^{-1} + x^{-1}\,. \end{aligned}$$

For the estimation of R_6 we shall use the inequality

$$(17)\qquad \left|\, e^u - e^v \right| \le |u - v|$$

which is true with $\operatorname{Re} u \le 0$, $\operatorname{Re} v \le 0$. If $n \notin N_r$ it follows from (17)

$$\begin{aligned}
&\left|g_{1r}^*(n+1)g_{2r}^*(n) - P(r,x)\right| \le \\
&\le \left| \sum_{\substack{p^m \| n+1 \\ p>r}} \log g_1(p^m) + \sum_{\substack{p^m \| n \\ p>r}} \log g_2(p^m) - \log P(r,x) \right| \le \\
&\le \left| \sum_{\substack{p^m \| n+1 \\ p>r}} \big(g_1(p^m)-1\big) + \sum_{\substack{p^m \| n \\ p>r}} \big(g_2(p^m)-1\big) - \log P(r,x) \right| + \\
&+ O\Bigg(\sum_{\substack{p^m \| n+1 \\ p>r}} |g_1(p^m)-1|^2 + \sum_{\substack{p^m \| n \\ p>r}} |g_2(p^m)-1|^2 \Bigg),
\end{aligned}$$

and we have

$$\begin{aligned}
R_6 &\le \frac{1}{x}\sum_{n\le x}\left| \sum_{\substack{p^m \| n+1 \\ p>r}} \big(g_1(p^m)-1\big) - \sum_{\substack{p^m \le x \\ p>r}} \frac{g_1(p^m)-1}{p^m} \right| + \\
&+ \frac{1}{x}\sum_{n\le x}\left| \sum_{\substack{p^m \| n \\ p>r}} \big(g_2(p^m)-1\big) - \sum_{\substack{p^m \le x \\ p>r}} \frac{g_2(p^m)-1}{p^m} \right| + \\
&+ \frac{1}{x}\sum_{n\le x}\left| \sum_{\substack{p^m \le x \\ p>r}} \frac{g_1(p^m)+g_2(p^m)-2}{p^m} - \log P(r,x) \right| + \\
&+ O\Bigg(\frac{1}{x}\sum_{n\le x}\Bigg(\sum_{\substack{p^m \| n+1 \\ p>r}} |g_1(p^m)-1|^2 + \sum_{\substack{p^m \| n \\ p>r}} |g_2(p^m)-1|^2 \Bigg)\Bigg) = \\
&= R_7 + R_8 + R_9 + R_{10}\,.
\end{aligned} \tag{18}$$

For the estimation of R_7 and R_8 we shall use the Turán–Kubilius inequality ([1] Lemma 4.4):

$$\frac{1}{x}\sum_{n\le x}\left| \sum_{p^m \| n} \alpha(p^m) - \sum_{p^m\le x} \frac{\alpha(p^m)}{p^m} \right|^2 \ll \sum_{p^m \le x} \frac{|\alpha(p^m)|^2}{p^m}$$

where $\alpha(p^m)$ are complex numbers for all p^m and the constant in the symbol $\ll$ is absolute.

Thus, using the Cauchy inequality additionally, we have that $R_7\,(i=1)$ and $R_8\,(i=2)$

$$\ll \frac{1}{x}\Bigg(\sum_{n\le x+1}\Bigg|\sum_{\substack{p^m\| n\\ p>r}}(g_i(p^m)-1)-\sum_{\substack{p^m\le x+1\\ p>r}}\frac{g_i(p^m)-1}{p^m}\Bigg|^2\Bigg)^{1/2}\cdot\Bigg(\sum_{n\le x+1}1\Bigg)^{1/2}+$$

$$(19)\qquad +x^{-1}\ll\Bigg(\sum_{\substack{p^m\le x+1\\ p>r}}\frac{|g_i(p^m)-1|^2}{p^m}\Bigg)^{1/2}+x^{-1}\ll$$

$$\ll\Bigg(\sum_{r<p\le x}\frac{|g_i(p)-1|^2}{p}\Bigg)^{1/2}+\Bigg(\sum_{p>r}\frac{1}{p^2}\Bigg)^{1/2}+x^{-1/2}.$$

Therefore

$$(20)\qquad R_7+R_8\ll\big(S(r,x)\big)^{1/2}+(r\log r)^{-1/2}+x^{-1/2}.$$

Further

(21)

$$R_9=\Bigg|\sum_{r<p\le x}\frac{g_1(p)+g_2(p)-2}{p}+O\Big(\sum_{p>r}\frac{1}{p^2}\Big)-$$

$$-\sum_{r<p\le x}\Bigg(-\frac{2}{p}+\Big(1-\frac{1}{p}\Big)\sum_{m=1}^{\infty}\frac{g_1(p^m)+g_2(p^m)}{p^m}\Bigg)\Bigg|\ll\sum_{p>r}\frac{1}{p^2}\ll(r\log r)^{-1}$$

and

(22)

$$R_{10}\ll\sum_{\substack{p^m\le x+1\\ p>r}}\frac{|g_1(p^m)-1|^2+|g_2(p^m)-1|^2}{p^m}\ll S(r,x)+(r\log r)^{-1}+x^{-1}$$

in a similar way as in (19).

Finally collecting all needed estimates we obtain

$$(23)\qquad \begin{aligned}&\big|M_x(g_1,g_2)-P(x)\big|\ll x^{-\beta}\exp\Big(\frac{c_6r^{\beta}}{\log r}\Big)+\\ &+x^{2\alpha-1}\exp\Big(\frac{2c_2r^{1-\alpha}}{\log r}\Big)+\big(S(r,x)\big)^{1/2}+(r\log r)^{-1/2}\end{aligned}$$

and choosing $\alpha = \beta = 1/3$ finish the proof of Theorem.

The proof of Corollary 1 follows from Theorem using the estimate of remainder term (23). We have to choose for example

$$\beta = 1 - \alpha = \min\left(\frac{1}{2\kappa}, \frac{3}{4}\right)$$

and

$$r = c_{18}(\log x \log\log x)^{1/\beta}$$

with sufficiently small $c_{18} > 0$ and to make some simple calculations.

The proof of Corollary 2. This proof is well-known but we shall give it because of completeness.

It is known from the probability theory that a sequence of distribution functions $F_k(z)$ converges weakly towards a limit distribution $F(z)$ if the sequence of characteristic functions

$$\varphi_k(t) = \int_{-\infty}^{\infty} \exp(itz)\, dF_k(z)$$

has the limit

$$\varphi(t) = \lim_{k\to\infty} \varphi_k(t)$$

for every real t and $\varphi(t)$ is continuous at $t = 0$. Furthermore the function $\varphi(t)$ is the characteristic function of the distribution $F(z)$.

The characteristic functions of the distributions (7) equal

$$\frac{1}{[x]}\sum_{n\le x} \exp\Big(it\big(f_1(n+1) + f_2(n)\big)\Big). \tag{24}$$

Since

$$\sum_p \frac{\exp\big(it\, f_1(p)\big) + \exp\big(it\, f_2(p)\big) - 2}{p} = t \sum_{\substack{|f_1(p)|\le 1\\ |f_2(p)|\le 1}} \frac{f_1(p) + f_2(p)}{p} +$$

$$+ O\bigg(t^2 \sum_{\substack{|f_1(p)|\le 1\\ |f_2(p)|\le 1}} \frac{f_1^2(p) + f_2^2(p)}{p}\bigg) + O\bigg(\sum_{|f_1(p)|>1} \frac{1}{p}\bigg) + O\bigg(\sum_{|f_2(p)|>1} \frac{1}{p}\bigg)$$

then from the convergence of the series (4), (5), and (6) we deduce that the infinite product (8) converges for every t. Furthermore this product is continuous at $t = 0$ because it converges uniformly for $|t| \leq T$ where $T > 0$ is arbitrary.

Since for $i = 1, 2$

$$\sum_p \frac{|\exp(it f_i(p)) - 1|^2}{p} \ll t^2 \sum_{|f_i(p)| \leq 1} \frac{|f_i(p)|^2}{p} + \sum_{|f_i(p)| > 1} \frac{1}{p}$$

then from the convergence of (4) and (5) it follows that $S(r, x) \to 0$ when r tends to infinity together with x. Choosing for example $r = \log x$ in our Theorem we get that the remainder term in (3) disappears when $x \to \infty$.

Thus the characteristic functions (24) have the limit (8) for every real t and this limit is continuous at $t = 0$. Corollary 2 is proved.

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