Annales Univ. Sci. Budapest., Sect. Comp. 18 (1999) 137-150

ON THE $M^X/G/1$ SYSTEM

L. Lakatos (Budapest, Hungary)

To the memory of Imre Környei

1. Introduction

For the equilibrium description of the M/G/1 system one usually uses the embedded Markov chain technique leading to the Pollaczek-Khinchin formula, the generating function of ergodic distribution. From mathematical viewpoint it gives the complete solution of the problem, but in practice it is not always simple to obtain the concrete probabilities from it. The most natural way would be the differentiation, but it gives very complicated expressions. In [2] Brière and Chaudry investigate this problem from another viewpoint in case of bulk-arrival systems, the inversion of generating function is realized by comparing the coefficients at the corresponding powers of z, a recursive algorithm is obtained for different concrete service time distributions. Their work includes both sample numerical results and easily implementable algorithms. Software packages realizing these algorithms are also available on such systems [4]. [3] presents a unified approach for the numerical solutions of the M/G/1 queue. It gives an overview of existing methods and discusses the possibility of their applications. There is mentioned e.g. Neuts' matrix-analytic method which by Powell and Van Hoorn is good for mathematical treatment, but it is difficult to implement computationally at least for high values of parameters. The authors present a possible solution in case the service time distribution has a rational Laplace-Stieltjes transform, under such assumptions explicit closed-form expressions can be obtained in terms of roots of associated characteristic equation (it corresponds to the denominator of generating function).

The functioning of M/G/1 system may be characterized by the help of Kovalenko's piecewise linear processes [1], it gives possibility to compute

---

The research was supported by the Hungarian National Foundation for Scientific Research under grant OTKA T022796.

the desired probabilities on the basis of a busy period and the transition probabilities of the embedded Markov chain. In this paper we use this technique to find the ergodic probabilities when the arriving requests form groups, the arrival rate, the mean value of service time and the probability of appearence of a given number of requests for the service time are the primary information for us, the desired probabilities are obtained directly from them avoiding the generating functions, even we have not to know the concrete service time distribution. This method was used for the simple M/G/1 system in [5], for system with vacation in [6].

2. Results on piecewise-linear processes and some notations

The piecewise linear processes were introduced by Kovalenko and they are described in the third chapter of [1]. There is a useful theorem concerning how to determine the ergodic distribution at the same place in the fourth chapter. Since the functioning of the M/G/1 system may be described by means of these processes, we shortly point out the basic moments which are necessary for the fulfilment of conditions of these theorems. In case of investigation of the M/G/1 system by the help of embedded Markov chain its states are identified by the number of requests there at moments $t_n + 0$, i.e. they coincide with the number of requests after having serviced the n-th one. There is no restriction on the waiting room, so the number of states is countable. The ergodicity theorem from [1] (a detailed proof is given in [7]) requires a finite number of states, so we unite into one the cases when the number of present ones is equal to or greater than k. Furthermore, we assume that the mean value of service time of a request is finite, and if the service process of a request started, it is continued till the end without interruption. Under these conditions according to [1] the ergodic distribution exists and it can be computed on the basis of the mean value of duration of busy period and the mean value of sojourn on different levels for it (the expressions to have k requests in the system and to stay at level k we will use in the same sense). So our purpose is to find the mean value of a busy period and the mean values of time spent in different states for it.

We will use the following notations:

λ - the arrival rate;

$G(z) = \sum_{i=1}^{\infty} g_i z^i$ - the generating function of number of requests in the arriving group, $\alpha = G'(1)$ - its mean value;

$B(x)$ - the distribution function of the service time of a request, $\tau = \int_0^\infty x dB(x)$ - the mean value of this service time, $b(s) = \int_0^\infty e^{-sx} dB(x)$ - its Laplace-Stieltjes transform;

$\rho = \lambda\tau$ - the utilization factor of the system;

$H_1(x)$ - the distribution function of the duration of busy period on condition that it starts with the presence of one request;

$\Gamma_1(s)$ - the Laplace-Stieltjes transform of the busy period's distribution function on condition that it starts with the presence of one request;

$\Gamma(s)$ - the Laplace-Stieltjes transform of the busy period's distribution function;

$P_i(k)$ - the probability of event that in i groups together appear k requests;

c_i - the probability of appearence of i requests for the service of a request;

ζ - the mean value of duration of the busy period;

ζ_i - the mean value of time spent above the i-th level for a busy period;

ξ_i - the mean value of time spent on the i-th level for a busy period;

ζ_{1i} - the mean value of time spent above the i-th level on condition that the busy period begins with presence of one request;

ξ_{1i} - the mean value of time spent on the i-th level for a busy period on condition that it begins with presence of one request.

3. Results

Theorem. *In the M/G/1 system with bulk arrivals under condition $\alpha\rho < 1$ ($\rho = \lambda\tau$, $\alpha = \sum_{i=1}^{\infty} i g_i$) the equilibrium distribution exists and the ergodic probabilities are determined by*

$$p_i = \frac{\xi_i}{\zeta}, \qquad i = 0, 1, 2, \ldots,$$

where ζ is the mean value of duration of the busy period and ξ_i ($i = 0, 1, 2, \ldots$) are the mean values of times spent on the i-th level for it.

Lemma 1. *The mean value of duration of the busy period is*

$$\zeta = \frac{\alpha\tau}{1 - \alpha\rho}.$$

Lemma 2. *In the M/G/1 system with bulk arrivals*

$$\xi_0 = \tau, \quad \xi_1 = \frac{\tau}{c_0} - g_1\tau, \quad \xi_2 = (1-g_1)(\xi_{10}+\xi_{11}) + \frac{1-c_0-c_1}{c_0}(\xi_{10}+\xi_{11}) - g_2\tau,$$

and the mean values of time ξ_k ($k \geq 3$) spent in different states for a busy period satisfy the recurrence relation

$$\xi_k = \sum_{i=1}^{k-2}(1-g_1-\ldots-g_i)\xi_{1,k-i} + \sum_{i=1}^{k-2}\frac{1-c_0-\ldots-c_i}{c_0}\xi_{1,k-i} +$$

$$+(1-g_1-\ldots-g_{k-1})(\xi_{10}+\xi_{11}) + \frac{1-c_0-\ldots-c_{k-1}}{c_0}(\xi_{10}+\xi_{11}) - g_k\tau,$$

where

$$\xi_{10} = \tau, \qquad \xi_{11} = \frac{\tau}{c_0} - \tau, \qquad \xi_{12} = \frac{1-c_0-c_1}{c_0}(\xi_{10}+\xi_{11}),$$

$$\xi_{1k} = \sum_{i=1}^{k-2}\frac{1-c_0-\ldots-c_i}{c_0}\xi_{1,k-i} + \frac{1-c_0-\ldots-c_{k-1}}{c_0}(\xi_{10}+\xi_{11}), \qquad k \geq 3.$$

4. Proofs

The proof of Theorem is the direct consequence of results about piecewise-linear processes [1] and the two lemmas.

Proof of Lemma 1. First we show that the Laplace-Stieltjes transform $\Gamma_1(s)$ satisfies the functional equation

$$\Gamma_1(s) = b(s + \lambda - \lambda G(\Gamma_1(s))).$$

We determine $H_1(x)$. In this case the busy period starts with the entry of one request. During the service of this request there are i moments when groups of requests appear and they contain together k requests. We have

$$H_1(x) = \int_0^x \sum_{i=0}^{\infty} e^{-\lambda y}\frac{(\lambda y)^i}{i!}\sum_{k=i}^{\infty} P_i(k) H_1^{*k}(x-y)dB(y),$$

where $H_1^{*k}(x)$ means that the convolution of $H_1(x)$ is taken k times. Its Laplace-Stieltjes transform is

$$\Gamma_1(s) = \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{k=i}^{\infty} P_i(k) \Gamma_1^k(s) \int_0^{\infty} e^{-(s+\lambda)x} (\lambda x)^i dB(x) =$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!} G^i(\Gamma_1(s)) \int_0^{\infty} e^{-(s+\lambda)x} (\lambda x)^i dB(x) = \sum_{i=0}^{\infty} (-1)^i \frac{\lambda^i [G(\Gamma_1(s))]^i b^{(i)}(s+\lambda)}{i!} =$$

$$= b(s + \lambda - \lambda G(\Gamma_1(s))).$$

The existence of solution of this equation can be proved by using the geometrical approach applied in the case of simple M/G/1 system [1]. $G(x)$ is monotone increasing function of x on $[0, \infty)$, so its inverse function exists. Assuming $s + \lambda - \lambda G(\Gamma_1(s)) = x$ we have

$$G(\Gamma_1(s)) = \frac{s + \lambda - x}{\lambda} \qquad \Longrightarrow \qquad b(x) = \Gamma_1(s) = G^{-1}\left(\frac{s + \lambda - x}{\lambda}\right),$$

where G^{-1} is the inverse of G. We are interested in the behaviour of $G^{-1}\left(\frac{s+\lambda-x}{\lambda}\right)$. Denoting $y = G^{-1}\left(\frac{s+\lambda-x}{\lambda}\right)$, its inverse function is $y = s + \lambda - \lambda G(x)$, for which $y' = -\lambda G'(x) < 0$, $y'' = -\lambda G''(x) < 0$, $y(0) = s + \lambda$, $y(1) = s$, so it is a monotone decreasing function taking on the zero value at a point $x > 1$. From these facts $G^{-1}\left(\frac{s+\lambda-x}{\lambda}\right)$ is also monotone decreasing function taking on the zero value at $s + \lambda$, 1 at s and a value greater than 1 at 0. So it has a point of intersection with $b(x)$ between s and $s + \lambda$, it gives the solution for fixed s. From this functional equation we obtain the mean value of duration of the busy period (starting with one request)

$$\mu = \frac{\tau}{1 - \alpha\lambda\tau} = \frac{\tau}{1 - \alpha\rho},$$

and since the Laplace-Stieltjes transform of the busy period's distribution function is

$$\sum_{i=1}^{\infty} g_i b^i(s + \lambda - \lambda G(\Gamma_1(s))) = G(b(s + \lambda - \lambda G(\Gamma_1(s)))),$$

the desired mean value is

$$\zeta = \frac{\alpha\tau}{1 - \alpha\rho}.$$

The lemma is proved.

We determine the transition probabilities of the embedded Markov chain. Its states are identified by number of requests in the system at moments $t_n + 0$, i.e. by the number of ones which are left in the system after the service of the n-th one is completed. For their generating function we have

$$\int_0^\infty e^{-\lambda x} dB(x) + \int_0^\infty \sum_{i=1}^\infty \sum_{k=i}^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} P_i(k) z^k dB(x) =$$

$$= \int_0^\infty e^{-\lambda x} \sum_{i=0}^\infty \frac{(\lambda x)^i}{i!} \sum_{k=i}^\infty P_i(k) z^k dB(x) = \int_0^\infty e^{-\lambda x} \sum_{i=0}^\infty \frac{(\lambda x G(z))^i}{i!} dB(x) =$$

$$= b(\lambda(1 - G(z))) = \sum_{j=0}^\infty c_j z^j .$$

From it the mean value of number of requests appearing for the service time of a request

$$\sum_{k=1}^\infty k c_k = [b(\lambda(1 - G(z)))]'|_{z=1} = \alpha\rho.$$

During the busy period we have intervals of two types: we stay on the first level and above the first level. We remember that the states of embedded chain are identified by the number of requests left in the system after completion of a concrete one, but for us it will be more convenient to characterize the system with the number of requests at the starting moments of services and this state does not change till completion. The such defined notions of state and number of present requests must be distinguished, the difference is clear from the following reasoning. If one considers service periods of requests when at the starting moment there is no other one, they correspond to state 1 excluding two cases. The first case is the jump to a level above the first, then the service of last request from this period from the viewpoint of states corresponds to the new level (from the viewpoint of the number of present requests to the first). But the whole duration of time spent on the first level does not change because coming from the second level to the first one the inverse situation takes place. The picture is analoguous for the levels above the first, too. The second case is the service of last request in the busy period, it corresponds to the zero state, so it must be excluded from the number of requests serviced on the first level.

Proof of Lemma 2. We determine the mean value of period during which there is only one request in the system. If there is only one request this state is continued with probability c_1 and finished with probability $1 - c_1$ (no request enters or more requests enter). For such a period with probability $c_1^{k-1}(1-c_1)$ are serviced k requests, the mean value of its length is

$$\sum_{k=1}^{\infty} k c_1^{k-1}(1-c_1)\tau = \frac{\tau}{1-c_1}.$$

The mean value of a period above the first level may be computed on the following way. After the first level with probability $\frac{c_k}{1-c_0-c_1}$ we come to the k-th one. In order to return to the first level it is necessary to service $k-1$ of them with the entering during this service ones, the mean value of time is

$$\sum_{k=2}^{\infty} \frac{c_k}{1-c_0-c_1}(k-1)\frac{\tau}{1-\alpha\rho} = \frac{\tau}{(1-\alpha\rho)(1-c_0-c_1)}\left(\sum_{k=2}^{\infty} k c_k - \sum_{k=2}^{\infty} c_k\right) =$$

$$= \frac{\tau}{(1-\alpha\rho)(1-c_0-c_1)}[\alpha\rho - c_1 - (1-c_0-c_1)] = \frac{\alpha\rho - 1 + c_0}{(1-\alpha\rho)(1-c_0-c_1)}\tau,$$

where we used the equalities $\alpha\rho = \sum_{k=1}^{\infty} k c_k$ and $\sum_{k=0}^{\infty} c_k = 1$.

The busy period begins with presence of one request with probability g_1, with more than one request with probability $1-g_1$. In the first case we have $0, 1, 2, \ldots$ periods above the first level with probabilities

$$\frac{c_0}{1-c_1}, \quad \frac{1-c_0-c_1}{1-c_1}\frac{c_0}{1-c_1}, \ldots, \frac{(1-c_0-c_1)^k}{(1-c_1)^k}\frac{c_0}{1-c_1}, \ldots,$$

the mean value of time spent on and above the first level are respectively

$$\xi_{10} + \xi_{11} = \sum_{k=1}^{\infty} k \frac{(1-c_0-c_1)^{k-1}}{(1-c_1)^{k-1}} \frac{c_0}{1-c_1} \frac{\tau}{1-c_1} = \frac{\tau}{c_0},$$

$$\zeta_{11} = \sum_{k=1}^{\infty} k \frac{(1-c_0-c_1)^k}{(1-c_1)^k} \frac{c_0}{1-c_1} \frac{\alpha\rho - 1 + c_0}{(1-\alpha\rho)(1-c_0-c_1)}\tau = \frac{\alpha\rho - 1 + c_0}{c_0(1-\alpha\rho)}\tau.$$

In the second case the busy period starts with the presence of i requests with probability $\frac{g_i}{1-g_1}$, having serviced $i-1$ requests with the entering ones we come to the first level and we will be in the previous situation. The mean value of time to come to the first level is

$$\sum_{i=2}^{\infty} \frac{g_i}{1-g_1}(i-1)\frac{\tau}{1-\alpha\rho} = \frac{\alpha - 1}{(1-g_1)(1-\alpha\rho)}\tau,$$

after it we will stay above the first level on average $\frac{\alpha\rho-1+c_0}{c_0(1-\alpha\rho)}\tau$ till the end of the busy period.

Resuming our results we have

- the mean value of time spent on zero level is $\xi_0 = \xi_{10} = \tau$;

- the mean value of time spent on the first level is

$$\xi_1 = g_1\left(\frac{\tau}{c_0} - \tau\right) + (1-g_1)\left(\frac{\tau}{c_0} - \tau + \tau\right) = \frac{\tau}{c_0} - g_1\tau$$

(in the first summand we subtract the mean value of time on the zero level, in the second summand it is necessary still to add the service time of last request above the first level from the first period);

- the mean value of time spent above the first level is

$$\zeta_1 = g_1\frac{\alpha\rho-1+c_0}{c_0(1-\alpha\rho)}\tau + (1-g_1)\left[\frac{(\alpha-1)\tau}{(1-g_1)(1-\alpha\rho)} - \tau + \frac{\alpha\rho-1+c_0}{c_0(1-\alpha\rho)}\tau\right] =$$

$$= \frac{\alpha\rho-1+c_0}{c_0(1-\alpha\rho)}\tau + \frac{(\alpha-1)\tau}{1-\alpha\rho} - (1-g_1)\tau.$$

The sum of these three terms

$$\tau + \frac{\tau}{c_0} - g_1\tau + \frac{\alpha\rho-1+c_0}{c_0(1-\alpha\rho)}\tau + \frac{(\alpha-1)\tau}{1-\alpha\rho} - (1-g_1)\tau = \frac{\alpha\tau}{1-\alpha\rho}$$

gives the mean value of length of the busy period.

Now let us determine the mean value of time spent above the second level. It is necessary to consider two cases: 1) the busy period begins on the first level and 2) the busy period begins above the first level. In the first case there are two possibilities:

- from the first level we come to the second level, spending on average ζ_{11} above it we return to the first one;

- from the first level we come to a level above the second, then for

$$\sum_{k=3}^{\infty}\frac{c_k}{1-c_0-c_1-c_2}(k-2)\frac{\tau}{1-\alpha\rho} = \frac{\alpha\rho-2+2c_0+c_1}{(1-\alpha\rho)(1-c_0-c_1-c_2)}\tau = \varepsilon_2$$

we return to the second level and we are in the previous situation. These two possibilities give together

$$\frac{c_2}{1-c_0-c_1}\zeta_{11} + \frac{1-c_0-c_1-c_2}{1-c_0-c_1}(\zeta_{11}+\varepsilon_2).$$

We have i such periods with probability $\frac{(1-c_0-c_1)^i}{(1-c_1)^i}\frac{c_0}{1-c_1}$, from it

$$\zeta_{12}=\sum_{i=1}^{\infty} i\frac{(1-c_0-c_1)^i}{(1-c_1)^i}\frac{c_0}{1-c_1}(\zeta_{11}+\varepsilon_2')=\frac{1-c_0-c_1}{c_0}\zeta_{11}+\frac{1-c_0-c_1-c_2}{c_0}\varepsilon_2,$$

where

$$\varepsilon_2'=\frac{\alpha\rho-2+2c_0+c_1}{(1-\alpha\rho)(1-c_0-c_1)}\tau.$$

Now we consider the second case. We have again two possibilities:

- the busy period begins on the second level, spending ζ_{11} above it we return to the first one, after it we have the first case;

- the busy period begins above the second level, after having serviced a certain number of requests we return to the second one and from it to the first, the corresponding mean value is

$$\sum_{i=3}^{\infty}\frac{g_i}{1-g_1-g_2}(i-2)\frac{\tau}{1-\alpha\rho}-\tau+\zeta_{11}=\frac{\alpha-2+g_1}{(1-\alpha\rho)(1-g_1-g_2)}\tau-\tau+\zeta_{11}$$

(the first summand gives the mean value of time to return to the second level, the last request belongs to the second level, so τ must be subtracted, spending ζ_{11} above the second we will be on the first one). The mean value of this first period

$$\frac{g_2}{1-g_1}\zeta_{11}+\frac{1-g_1-g_2}{1-g_1}\left[\frac{\alpha-2+g_1}{(1-g_1-g_2)(1-\alpha\rho)}\tau-\tau+\zeta_{11}\right]=$$

$$=\zeta_{11}+\frac{\alpha-2+g_1}{(1-g_1)(1-\alpha\rho)}\tau-\frac{1-g_1-g_2}{1-g_1}\tau.$$

Now we have again the first possibility, we come to the first level, and the first case takes place. The probabilities of two cases are g_1 and $1-g_1$ respectively, the mean value of time spent above the second level for a busy period will be

$$\zeta_2=g_1\left[\frac{1-c_0-c_1}{c_0}\zeta_{11}+\frac{1-c_0-c_1-c_2}{c_0}\varepsilon_2\right]+(1-g_1)\times\left[\zeta_{11}+\right.$$

$$\left.+\frac{\alpha-2+g_1}{(1-g_1)(1-\alpha\rho)}\tau-\frac{1-g_1-g_2}{1-g_1}\tau+\frac{1-c_0-c_1}{c_0}\zeta_{11}+\frac{1-c_0-c_1-c_2}{c_0}\varepsilon_2\right]=$$

$$=(1-g_1)\zeta_{11}+\frac{1-c_0-c_1}{c_0}\zeta_{11}+\frac{1-c_0-c_1-c_2}{c_0}\varepsilon_2+\frac{\alpha-2+g_1}{1-\alpha\rho}\tau-(1-g_1-g_2)\tau.$$

Subtracting ζ_2 from ζ_1 we obtain the mean value of time spent on the second level for the busy period, namely

$$\xi_2 = \zeta_1 - \zeta_2 = \frac{\alpha\rho - 1 + c_0}{c_0(1-\alpha\rho)}\tau + \frac{(\alpha-1)\tau}{1-\alpha\rho} - (1-g_1)\tau - (1-g_1)\zeta_{11} -$$

$$-\frac{1-c_0-c_1}{c_0}\zeta_{11} - \frac{\alpha\rho - 2 + 2c_0 + c_1}{c_0(1-\alpha\rho)}\tau - \frac{\alpha - 2 + g_1}{1-\alpha\rho}\tau + (1-g_1-g_2)\tau =$$

$$= \frac{1-c_0-c_1}{c_0}\left(\frac{\tau}{1-\alpha\rho} - \zeta_{11}\right) + (1-g_1)\left(\frac{\tau}{1-\alpha\rho} - \zeta_{11}\right) - g_2\tau =$$

$$= (1-g_1)(\xi_{10}+\xi_{11}) + \frac{1-c_0-c_1}{c_0}(\xi_{10}+\xi_{11}) - g_2\tau.$$

We compute the mean value of time spent above the k-th level for a busy period. The busy period can start on and above the first level. At first we investigate the case of first level. After the service of a certain number of requests on the first level with probabilities $\frac{c_i}{1-c_0-c_1}$ $(i = 2, 3, \ldots, k)$ we come to the i-th level and with probability $\frac{1-c_0-\ldots-c_k}{1-c_0-c_1}$ above the k-th level. If after the first level we come to the second one, then spending on average $\zeta_{1,k-1}$ above the k-th we return to the first; if we come to the third then spending $\zeta_{1,k-2}$ above the k-th we come to the second level and after $\zeta_{1,k-1}$ to the first, i.e. the mean value of time spent above the k-th level is $\zeta_{1,k-2} + \zeta_{1,k-1}$. From similar considerations in case of the k-th level we return to the first one spending on average $\zeta_{11} + \zeta_{12} + \ldots + \zeta_{1,k-1}$ above the k-th one. If we come to a level above the k-th one then for

$$\sum_{i=k+1}^{\infty} \frac{c_i}{1-c_0-c_1-\ldots-c_k}(i-k)\frac{\tau}{1-\alpha\rho} =$$

$$= \frac{\alpha\rho - k + kc_0 + (k-1)c_1 + \ldots + c_{k-1}}{(1-\alpha\rho)(1-c_0-\ldots-c_k)}\tau$$

we return to the k-th level and spending $\zeta_{11} + \ldots + \zeta_{1,k-1}$ above the k-th we return to the first one. These possibilities give together that we spend above the k-th level

$$\frac{c_2}{1-c_0-c_1}\zeta_{1,k-1} + \frac{c_3}{1-c_0-c_1}(\zeta_{1,k-2} + \zeta_{1,k-1}) + \ldots +$$

$$+\frac{c_k}{1-c_0-c_1}(\zeta_{11} + \zeta_{12} + \ldots + \zeta_{1,k-1}) + \frac{1-c_0-\ldots-c_k}{1-c_0-c_1}\times$$

$$\times\left(\frac{\alpha\rho - k + kc_0 + (k-1)c_1 + \ldots + c_{k-1}}{(1-\alpha\rho)(1-c_0-\ldots-c_k)}\tau + \zeta_{11} + \ldots + \zeta_{1,k-1}\right) =$$

$$= \zeta_{1,k-1} + \frac{1-c_0-c_1-c_2}{1-c_0-c_1}\zeta_{1,k-2} + \ldots + \frac{1-c_0-\ldots-c_{k-1}}{1-c_0-c_1}\zeta_{11} +$$

$$+ \frac{\alpha\rho - k + kc_0 + (k-1)c_1 + \ldots + c_{k-1}}{(1-\alpha\rho)(1-c_0-c_1)}\tau.$$

We have i such periods with probability $\frac{(1-c_0-c_1)^i}{(1-c_1)^i}\frac{c_0}{1-c_1}$, consequently the mean value of time spent above the k-th level for a busy period starting with the appearence of one request at the beginning will be

$$\sum_{i=1}^{\infty} i\frac{(1-c_0-c_1)^i}{(1-c_1)^i}\frac{c_0}{1-c_1}\left[\zeta_{1,k-1} + \frac{1-c_0-c_1-c_2}{1-c_0-c_1}\zeta_{1,k-2} + \ldots + \right.$$

$$\left. + \frac{1-c_0-\ldots-c_{k-1}}{1-c_0-c_1}\zeta_{11} + \frac{\alpha\rho - k + kc_0 + \ldots + c_{k-1}}{(1-\alpha\rho)(1-c_0-c_1)}\tau\right] =$$

$$= \frac{1-c_0-c_1}{c_0}\zeta_{1,k-1} + \frac{1-c_0-c_1-c_2}{c_0}\zeta_{1,k-2} +$$

$$+ \ldots + \frac{1-c_0-\ldots-c_{k-1}}{c_0}\zeta_{11} + \frac{\alpha\rho - k + kc_0 + \ldots + c_{k-1}}{c_0(1-\alpha\rho)}\tau.$$

Now we consider the case when the busy period begins above the first level. After having serviced a certain number of requests we come to the first level and will be in the previous situation. The busy period starts with probabilities $\frac{g_2}{1-g_1}, \frac{g_3}{1-g_1}, \ldots, \frac{g_k}{1-g_1}, \frac{1-g_1-\ldots-g_k}{1-g_1}$ on the second, third, ..., k-th and above the k-th level, the desired mean values of time till to return to the first one are $\zeta_{1,k-1}, \zeta_{1,k-2} + \zeta_{1,k-1}, \ldots, \zeta_{11} + \zeta_{12} + \ldots + \zeta_{1,k-1}$ respectively, in the last case for

$$(*) \quad \sum_{i=k+1}^{\infty} \frac{g_i}{1-g_1-\ldots-g_k}(i-k)\frac{\tau}{1-\alpha\rho} = $$

$$= \frac{\alpha - k + (k-1)g_1 + (k-2)g_2 + \ldots + g_{k-1}}{(1-\alpha\rho)(1-g_1-\ldots-g_k)}\tau$$

we come to the k-th level and spending $\zeta_{11} + \zeta_{12} + \ldots + \zeta_{1,k-1}$ above the k-th return to the first one. The mean value of time spent above the k-th level while we come to the first is

$$\frac{g_2}{1-g_1}\zeta_{1,k-1} + \frac{g_3}{1-g_1}(\zeta_{1,k-2} + \zeta_{1,k-1}) + \ldots + \frac{g_k}{1-g_1}(\zeta_{11} + \ldots + \zeta_{1,k-1}) +$$

$$+\frac{1-g_1-\ldots-g_k}{1-g_1}\left[\frac{\alpha-k+(k-1)g_1+\ldots+g_{k-1}}{(1-\alpha\rho)(1-g_1-\ldots-g_k)}\tau-\tau+\zeta_{11}+\ldots+\zeta_{1,k-1}\right]$$

$$=\zeta_{1,k-1}+\frac{1-g_1-g_2}{1-g_1}\zeta_{1,k-2}+\ldots+\frac{1-g_1-\ldots-g_{k-1}}{1-g_1}\zeta_{11}+$$

$$+\frac{\alpha-k+(k-1)g_1+\ldots+g_{k-1}}{(1-\alpha\rho)(1-g_1)}\tau-\frac{1-g_1-\ldots-g_k}{1-g_1}\tau$$

((*) is modified by τ since the service of last request belongs already to the k-th level) and after staying

$$\frac{1-c_0-c_1}{c_0}\zeta_{1,k-1}+\frac{1-c_0-c_1-c_2}{c_0}\zeta_{1,k-2}+\ldots+\frac{1-c_0-\ldots-c_{k-1}}{c_0}\zeta_{11}+$$

$$+\frac{\alpha\rho-k+kc_0+\ldots+c_{k-1}}{c_0(1-\alpha\rho)}\tau$$

above the k-th level the busy period will be terminated. Since the busy period starts with probabilities g_1 and $1-g_1$ on and above the first level, the mean value of time spent above the k-th level for a busy period

$$\zeta_k=g_1\left[\frac{1-c_0-c_1}{c_0}\zeta_{1,k-1}+\ldots+\frac{1-c_0-\ldots-c_{k-1}}{c_0}\zeta_{11}+\right.$$

$$\left.+\frac{\alpha\rho-k+kc_0+\ldots+c_{k-1}}{c_0(1-\alpha\rho)}\tau\right]+$$

$$+(1-g_1)\left[\zeta_{1,k-1}+\frac{1-g_1-g_2}{1-g_1}\zeta_{1,k-2}+\ldots+\frac{1-g_1-\ldots-g_{k-1}}{1-g_1}\zeta_{11}+\right.$$

$$+\frac{\alpha-k+(k-1)g_1+\ldots+g_{k-1}}{(1-\alpha\rho)(1-g_1)}\tau-\frac{1-g_1-\ldots-g_k}{1-g_1}\tau+$$

$$\left.+\frac{1-c_0-c_1}{c_0}\zeta_{1,k-1}+\ldots+\frac{1-c_0-\ldots-c_{k-1}}{c_0}\zeta_{11}+\frac{\alpha\rho-k+kc_0+\ldots+c_{k-1}}{c_0(1-\alpha\rho)}\tau\right]$$

$$=(1-g_1)\zeta_{1,k-1}+(1-g_1-g_2)\zeta_{1,k-2}+\ldots+(1-g_1-\ldots-g_{k-1})\zeta_{11}+$$

$$+\frac{\alpha\rho-k+(k-1)g_1+\ldots+g_{k-1}}{1-\alpha\rho}\tau-(1-g_1-\ldots-g_k)\tau+$$

$$+\frac{1-c_0-c_1}{c_0}\zeta_{1,k-1}+\ldots+\frac{1-c_0-\ldots-c_{k-1}}{c_0}\zeta_{11}+\frac{\alpha\rho-k+kc_0+\ldots+c_{k-1}}{c_0(1-\alpha\rho)}\tau.$$

The similar expression for the mean value of time spent above the $k-1$st level

$$\zeta_{k-1} = (1-g_1)\zeta_{1,k-2} + (1-g_1-g_2)\zeta_{1,k-3} + \ldots + (1-g_1-\ldots-g_{k-2})\zeta_{11} +$$
$$+\frac{\alpha-(k-1)+(k-2)g_1+\ldots+g_{k-2}}{1-\alpha\rho}\tau - (1-g_1-\ldots-g_{k-1})\tau +$$
$$+\frac{1-c_0-c_1}{c_0}\zeta_{1,k-2} + \ldots + \frac{1-c_0-\ldots-c_{k-2}}{c_0}\zeta_{11} +$$
$$+\frac{\alpha\rho-(k-1)+(k-1)c_0+\ldots+c_{k-2}}{c_0(1-\alpha\rho)}\tau.$$

The difference of two expressions $\xi_k = \zeta_{k-1} - \zeta_k$ gives the mean value of time spent on the k-th level for the busy period, namely

$$\xi_k = (1-g_1)\xi_{1,k-1} +$$
$$+(1-g_1-g_2)\xi_{1,k-2} + \ldots + (1-g_1-\ldots-g_{k-2})\xi_{12} - (1-g_1-\ldots-g_{k-1})\zeta_{11} +$$
$$+\frac{\alpha-(k-1)+(k-2)g_1+\ldots+g_{k-2}}{1-\alpha\rho}\tau - \frac{\alpha-k+kg_1+\ldots+g_{k-1}}{1-\alpha\rho}\tau -$$
$$-(1-g_1-\ldots-g_{k-1})\tau + (1-g_1-\ldots-g_{k-1}-g_k)\tau +$$
$$+\frac{1-c_0-c_1}{c_0}\xi_{1,k-1} + \ldots + \frac{1-c_0-\ldots-c_{k-2}}{c_0}\xi_{12} - \frac{1-c_0-\ldots c_{k-1}}{c_0}\zeta_{11} +$$
$$+\frac{\alpha\rho-(k-1)+(k-1)c_0+\ldots+c_{k-2}}{c_0(1-\alpha\rho)}\tau - \frac{\alpha\rho-k+kc_0+\ldots+c_{k-1}}{c_0(1-\alpha\rho)}\tau =$$
$$= \sum_{i=1}^{k-2}(1-g_1-\ldots-g_i)\xi_{1,k-i} + \sum_{i=1}^{k-2}\frac{1-c_0-\ldots-c_i}{c_0}\xi_{1,k-i} +$$
$$+(1-g_1-\ldots-g_{k-1})(\xi_{10}+\xi_{11}) + \frac{1-c_0-\ldots-c_{k-1}}{c_0}(\xi_{10}+\xi_{11}) - g_k\tau.$$

This expression contains the mean values ξ_{1i} $(i \geq 0)$ of times which we spend on different levels in a busy period starting with presence of one request. The formulas given for them in the lemma can be easily derived from this proof regarding only the possibilities when at the beginning of busy period there is one request in the system, or they can be taken in ready form from [5]. The lemma is proved.

References

[1] **Гнеденко Б.В. и Коваленко И.Н.**, *Введение в теорию массового обслуживания*, Наука, Москва, 1987.

[2] **Brière G. and Chaudry M.L.**, Computational analysis of single-server bulk-arrival queues $M^X/G/1$, *Comput. Oper. Res.*, **15** (3) (1988), 283-292.

[3] **Chaudry M.L., Gupta U.C. and Agarwal M.**, Exact and approximate numerical solutions to steady-state single-server queues: M/G/1 - a unified approach, *Queueing Systems*, **10** (1992), 351-380.

[4] **Chaudry M.L. and Hasham,** Software package for the queueing system $M^X/G/1$, *Oper. Res. Letters*, **6** (1987), 195-196.

[5] **Lakatos L. and Čerić V.**, Another approach to the ergodic distribution in the M/G/1 system, *Probability theory and applications: Essays to the memory of József Mogyoródi*, eds. J.Galambos and I.Kátai, Kluwer, 1992, 289-298.

[6] **Lakatos L.**, Ergodic distribution for the M/G/1 system with vacation, *Theory of Stochastic Processes*, **3 (19)** (1997), 244-255.

[7] **Tomkó J.**, Tartózkodási időproblémák Markov láncokra, *Alk. Mat. Lapok*, **8** (1982), 91-108.

L. Lakatos
Department of Computer Algebra
Eötvös Loránd University
P.O.B. 32.
H-1518 Budapest, Hungary