Annales Univ. Sci. Budapest., Sect. Comp. 18 (1999) 125-136

ON THE CONVEXITY OF FUZZIFIED FUNCTIONS

M. Kovács (Budapest, Hungary)

Dedicated to Imre Környei

Abstract. In this paper we will discuss the convexity concept of the fuzzy functions given by Nanda and Kar in [8]. We will apply this concept to the fuzzy functions obtained from parametrical functions by the extension principle changing the parameters by fuzzy numbers. We will show that in the most cases the fuzzified linear function will neither be convex nor concave. We introduce a new convexity concept under which the convex functions preserve the convexity property after their fuzzification.

1. Introduction

The convexity of the functions is a basic property needed in a lot of applications (e.g. in optimization) of classical analysis. It is waited that the concept of convexity of fuzzy functions will also be a useful tool for the fuzzy mathematical investigations.

The convex fuzzy sets have been discussed in several papers (e.g. [1],[6], [7]). The convexity concept of fuzzy functions was first introduced by Nanda and Kar [8]. Their concept is based on the fuzzy version of the classical Jensen inequality. In the classical analysis the convexity of a differentiable function can also be characterized as follows: f is convex if an only if it is above on every tangent plain. The first question is how this concept can be extended to the fuzzy functions. After answering for this question we can discuss the following problems, too:

This research has been partially supported by OTKA T019455 (Hungary).

i. Are the characterizations by the different convexity concept equivalent or not?

ii. When is the convexity property of a function preserved after fuzzification?

The aim of the present paper is to answer for these questions.

2. Preliminaries

In this section we collect those definitions and basic theorems which will be needed in the sequel.

Let I be the unit interval of the real line $I\!R$, $X \subset I\!R^n$ and let $I\!R_+$ be the nonnegative half line of $I\!R$. A fuzzy set on X is given by its membership function

$$\mu : X \to I.$$

The support of μ is the subset of X given by

$$\text{supp } \mu = \{x \in X : \mu(x) > 0\}.$$

The λ-cut of μ is

$$[\mu]^\lambda = \begin{cases} \{x \in X : \mu(x) \geq 0\} & \text{if } \lambda > 0, \\ \text{cl(supp } \mu) & \text{if } \lambda = 0, \end{cases}$$

where cl(supp μ) is the closure of the support set.

A fuzzy set is convex, if all λ-cuts are convex subsets of X and it is normal, if $[\mu]^1 \neq \emptyset$.

The convex, normal fuzzy sets of the real line with continuous membership function will be called fuzzy number. The set of all fuzzy numbers will be denoted by $\mathcal{F}$.

Let $g : I \to [0, \infty]$ be a continuous, strictly decreasing function with the boundary properties $g(1) = 0$ and $\lim\limits_{t \to 0} g(t) = g_0 \leq \infty$.

Let $\mathcal{F}_g$ denote the subset of fuzzy numbers with the membership function

$$\mu(a) = \begin{cases} g^{(-1)}(|a - \alpha|/d), & \text{if } d > 0, \\ \chi_{\{\alpha\}}(a), & \text{if } d = 0 \end{cases}$$

for all $\alpha \in I\!R$, $d \in I\!R_+$, where

$$g^{(-1)}(x) = \begin{cases} \{g^{-1}(x) & \text{if } x \in [0, g(0)], \\ 0 & \text{if } x \geq g(0) = g_0. \end{cases}$$

Here and in the following $\chi_A(a)$ denotes the characteristic function of the set A. The elements of $\mathcal{F}_g$ will be called *quasi-triangular fuzzy numbers* generated by g with the center α and spread d and we will recall for it by the pair (α, d).

Let $p \in [1, \infty)$ and let T_{gp} be an Archimedean t-norm given by the generator function g^p, i.e.

$$T_{gp}(a, b) := g^{(-1)}((g^p(a) + g^p(b))^{1/p}).$$

Since $\lim_{p \to \infty} T_{gp}(a, b) = \min(a, b)$, we will also use the notation T_{gp} in the case $p = \infty$ meaning min-norm for $T_{g\infty}$.

The T_{gp}-Cartesian product of n quasi-triangular fuzzy numbers generated by g will be called *$(g, p, \boldsymbol{D})$-fuzzy vector* on $\mathcal{F}_g^n$, i.e. 0

$$\boldsymbol{\mu}(\boldsymbol{a}) = (\mu_1 \times \ldots \times \mu_n)(\boldsymbol{a}) = T_{gp}(\mu_1(a_1), \ldots, \mu_n(a_n)),$$

where $\mu_i = (\alpha_i, d_i) \in \mathcal{F}_g$, $i = 1, \ldots, n$. It is obvious that $\boldsymbol{\mu} \in \mathcal{F}_g^n \subset \mathcal{F}(I\!R^n)$. It is easy to show that

$$\boldsymbol{\mu}(\boldsymbol{a}) = \boldsymbol{\mu}(a_1, \ldots, a_n) = \begin{cases} g^{(-1)}(\|\boldsymbol{D}^{\#}(\boldsymbol{a} - \boldsymbol{\alpha})\|_p) & \text{if } \exists i : d_i \neq 0, \\ \chi_{\{\alpha_1, \ldots, \alpha_n\}}(a_1, \ldots, a_n) & \text{otherwise,} \end{cases}$$

where $\boldsymbol{D} = \operatorname{diag}(d_1, \ldots, d_n)$, $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{D}^{\#}$ denotes the pseudoinverse of $\boldsymbol{D}$, i.e. $\boldsymbol{D}^{\#}$ is a diagonal matrix, the i-th element of which is $1/d_i$ if $d_i \neq 0$ and 0 if $d_i = 0$, and

$$\|\boldsymbol{a}\|_p = \begin{cases} (\sum_{i=1}^{n} |a_i|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max_{j=1,\ldots,n} a_j & \text{if } p = \infty. \end{cases}$$

We will often refer to a fuzzy vector by the notion $\boldsymbol{\mu} = (\boldsymbol{\alpha}, \boldsymbol{D}) \in \mathcal{F}_g^n$ instead of describing its membership function.

3. Fuzzy functions

Definition 1. Let $X \subset I\!R^n$. The fuzzy subset on $X \times I\!R$ with the membership function $\tilde{f}(\boldsymbol{x}, y)$ is a *fuzzy function* on X if for every fixed $x \in X$ $\widetilde{f(\boldsymbol{x})}(y) = \tilde{f}(\boldsymbol{x}, y)$ is a fuzzy number.

Definition 2. A fuzzy function $\widetilde{f(\boldsymbol{x})}$ is a (g, Ω)-*fuzzification* of a real function $f : I\!R^n \to I\!R$ if its membership function is

$$\widetilde{f(\boldsymbol{x})}(y) = \begin{cases} g^{(-1)}\left(\dfrac{|y - f(\boldsymbol{x})|}{\Omega(\boldsymbol{x})}\right) & \text{if } f(\boldsymbol{x}) - g(0)\Omega(\boldsymbol{x}) \le y \le f(\boldsymbol{x}) + g(0)\Omega(\boldsymbol{x}), \\ 0 & \text{otherwise} \end{cases}$$

with some positive function $\Omega(\boldsymbol{x})$.

Definition 3. Let $f_{\boldsymbol{a}}(\boldsymbol{x})$ be a parametrical function on $X \subset I\!R^n$ with the parameter vector $\boldsymbol{a}$, i.e. $f_{\boldsymbol{a}} : X \to I\!R$. Let $f_{\boldsymbol{a}}(x)$ be fuzzified in the parameter by a $(g, p, \boldsymbol{D})$-fuzzy vector $\boldsymbol{\mu}$ using the Zadeh extension principle [9], i.e.

$$\widetilde{f_{\boldsymbol{\mu}}(\boldsymbol{x})}(y) = \sup_{y = f_{\boldsymbol{a}}(\boldsymbol{x})} \boldsymbol{\mu}(\boldsymbol{a}).$$

Then $\widetilde{f_{\boldsymbol{\mu}}(\boldsymbol{x})}$ will be called the $(g, p, \boldsymbol{D})$-*parametrical fuzzification* of $f_{\boldsymbol{a}}(\boldsymbol{x})$.

If $\boldsymbol{a} \in I\!R^m$ and $\boldsymbol{\mu} = (\boldsymbol{\alpha}, \boldsymbol{D}) \in \mathcal{F}_g^m$, then

$$\widetilde{f_{\boldsymbol{\mu}}(\boldsymbol{x})}(y) = g^{(-1)}\left(\inf_{y = f_{\boldsymbol{a}}(\boldsymbol{x})} \|\boldsymbol{D}^{\#}(\boldsymbol{a} - \boldsymbol{\alpha})\|_p\right).$$

Definition 4. The $(g, p, \boldsymbol{D})$-fuzzification is *proper* if $d_i > 0$ for all i, and *partial* if there exists index i such that $d_i = 0$.

As it was shown in the paper [5], if the fuzzified function linearly depends on the parameters, i.e. $h_{\boldsymbol{a}}(x) = \sum_{j=1}^{m} a_j h_j(\boldsymbol{x})$, then

$$\widetilde{h_{\boldsymbol{\mu}}(\boldsymbol{x})}(y) = \begin{cases} g^{(-1)}\left(\dfrac{|y - \sum_{j=1}^{m} \alpha_j h_j(\boldsymbol{x})|}{\|\boldsymbol{D}\boldsymbol{h}(\boldsymbol{x})\|_q}\right) & \text{if } \boldsymbol{D}\boldsymbol{h}(\boldsymbol{x}) \ne \boldsymbol{0}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\boldsymbol{h}(\boldsymbol{x}) = (h_1(\boldsymbol{x}), \ldots, h_m(\boldsymbol{x}))$, $q = p/(p-1)$ if $1 \leq p < \infty$ and $q = 1$ if $p = \infty$. Particularly, if the parametrical function is linear in $\boldsymbol{x}$, too, i.e. $\ell_{\boldsymbol{a}}(\boldsymbol{x}) = \sum_{j=1}^{n} a_j x_j + a_0$ then [2]-[5]

$$\widetilde{\ell_{\mu}(\boldsymbol{x})}(y) = \begin{cases} g^{(-1)}\left(\dfrac{|y - (\sum_{j=1}^{n} \alpha_j x_j + \alpha_0)|}{\|\boldsymbol{D}(\boldsymbol{x},1)^T\|_q} \right) & \text{if } \boldsymbol{D}(\boldsymbol{x},1)^T \neq \boldsymbol{0}, \\ 0 & \text{otherwise.} \end{cases}$$

It is seen from these results, that for those functions, which linearly depend on the parameters, the $(g, p, \boldsymbol{D})$-fuzzification is equivalent with the (g, Ω)-fuzzification with $\Omega(\boldsymbol{x}) = \|\boldsymbol{D}\boldsymbol{h}(\boldsymbol{x})\|_q$.

Let $\widetilde{f(\boldsymbol{x})}$ be a fuzzy function on X. Let us introduce the *lower and upper bound functions* of its λ-cuts, namely

$$L_f^{\lambda}(\boldsymbol{x}) = \inf\{y : \widetilde{f(\boldsymbol{x})}(y) > \lambda\}$$

and

$$U_f^{\lambda}(\boldsymbol{x}) = \sup\{y : \widetilde{f(\boldsymbol{x})}(y) > \lambda\}.$$

It is obvious, if $\widetilde{f(\boldsymbol{x})}$ is a (g, Ω)-fuzzification of $f(\boldsymbol{x})$, then

$$L_f^{\lambda}(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\lambda)\Omega(\boldsymbol{x})$$

and

$$U_f^{\lambda}(\boldsymbol{x}) = f(\boldsymbol{x}) + g(\lambda)\Omega(\boldsymbol{x}).$$

4. The Nanda-Kar convexity of fuzzified functions

In the paper [8] Nanda and Kar introduced the convexity of fuzzy mappings. They based their convexity concept on the fuzzified version of the Jensen-inequality. Namely,

Definition 5. Let $X \subset I\!R^n$ be a convex set. The fuzzy function $\widetilde{f(\boldsymbol{x})}$ is convex on X if the inequality

$$[\widetilde{f(u_\alpha)}(y)]^\lambda \leq \alpha[\widetilde{f(\boldsymbol{u})}]^\lambda + (1-\alpha)[\widetilde{f(\boldsymbol{v})}]^\lambda$$

is fulfilled for all $\boldsymbol{u}, \boldsymbol{v} \in X$, $\alpha, \lambda \in I$ and $u_\alpha = \alpha\boldsymbol{u} + (1-\alpha)\boldsymbol{v}$.

The fuzzy function $\widetilde{f(\boldsymbol{x})}$ is concave on X if $-\widetilde{f(\boldsymbol{x})}$ is convex.

Since $[\widetilde{f(\boldsymbol{x})}]^\lambda = [L_f^\lambda(\boldsymbol{x}), U_f^\lambda(\boldsymbol{x})]$, the following statement is a trivial consequence of the Definition 5.

Proposition 1. *The fuzzy function $\widetilde{f(\boldsymbol{x})}$ is convex (concave) on the convex X if and only if both $L_f^\lambda(\boldsymbol{x})$ and $U_f^\lambda(\boldsymbol{x})$ are convex (concave) functions on X.*

Let us now deal with the fuzzification of linear functions.

Proposition 2. *The (g,Ω)-fuzzification of a linear function $\ell_{\boldsymbol{a}}(\boldsymbol{x})$ is convex (concave) on $X \subset I\!R^n$ if and only if one of the following properties is fulfilled:*

i. $\Omega(\boldsymbol{x})$ is a positive constant function on X;

ii. $\Omega(\boldsymbol{x})$ is a positive linear function on X.

Proof. From the convexity (concavity) of $L_f^\lambda(\boldsymbol{x})$ follows the concavity (convexity) of $g(\lambda)\Omega(\boldsymbol{x})$ and from the convexity (concavity) of $U_f^\lambda(\boldsymbol{x})$ follows the convexity (concavity) of $g(\lambda)\Omega(\boldsymbol{x})$, what is possible only in the cases i. and ii., and in these cases the convexity is really satisfied.

Corollary 2.1. *Let $\ell_{\boldsymbol{a}}(\boldsymbol{x})$ be a linear function. It has neither convex nor concave proper parametrical fuzzification on the whole $I\!R^n$.*

Proposition 3. *Let $\boldsymbol{z} \in I\!R^n$ be a vector with nonzero coordinates, and let $N_{\boldsymbol{z}} \subset I\!R^n$ be a convex neighborhood of $\boldsymbol{z}$ such that any coordinate of its points does not change sign on it. Then*

i. any $(g,\infty,\boldsymbol{D})$- or $(g,1,\boldsymbol{D})$-fuzzification of a linear function $\ell_{\boldsymbol{a}}(\boldsymbol{x})$ is locally convex on $N_{\boldsymbol{z}}$;

ii. there is neither locally convex nor locally concave $(g,p,\boldsymbol{D})$-fuzzification of a linear function $\ell_{\boldsymbol{a}}(\boldsymbol{x})$ if $1 < p < \infty$.

Proof. *i.* Since $||\boldsymbol{Dx}||_\infty$ and $||\boldsymbol{Dx}||_1$ are piecewise linear or constant on $N_{\boldsymbol{z}}$, the condition of convexity is fulfilled.

ii. For every $1 < p < \infty$ the function $||\boldsymbol{Dx}||_p$ is convex, therefore $L_f^\lambda(\boldsymbol{x})$ is concave, and $U_f^\lambda(\boldsymbol{x})$ is convex on $N_{\boldsymbol{z}}$.

Proposition 4. *Let $\Omega(\boldsymbol{x})$ be a strongly convex function on $I\!R^n$ with modulus of convexity $\kappa > 0$. Let $\widetilde{f(\boldsymbol{x})}$ be the (g, Ω)-fuzzification of the convex function f. A necessary condition for convexity of $\widetilde{f(\boldsymbol{x})}$ is the strong convexity of f with the modulus of convexity at least $g(0)\kappa$.*

Proof. A function $\varphi(\boldsymbol{x})$ is strongly convex with the modulus of convexity κ if and only if $\varphi(\boldsymbol{x}) - \kappa||\boldsymbol{x}||^2$ is convex.

We have for all $\lambda \in [0, 1]$

$$L_f^\lambda(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\lambda)\Omega(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\lambda)\kappa||\boldsymbol{x}||^2 - g(\lambda)(\Omega(\boldsymbol{x}) - \kappa||\boldsymbol{x}||^2)$$

and hence with $\lambda = 0$

$$f(\boldsymbol{x}) - g(0)\kappa||\boldsymbol{x}||^2 = L_f^\lambda(\boldsymbol{x}) + g(0)(\Omega(\boldsymbol{x}) - \kappa||\boldsymbol{x}||^2).$$

From the last equality follows the strong convexity of $f(\boldsymbol{x})$ with the modulus of convexity at least $g(0)\kappa$ because of the convexity of $L_f^\lambda(\boldsymbol{x})$ and $\Omega(\boldsymbol{x}) - \kappa||\boldsymbol{x}||^2$.

For illustration the subfigures a)-c) of Figure 1. show the λ-cuts of the parametrical fuzzifications of the linear function $y = x + 2$ by the fuzzy vector $\mu = ((1, 0.5), (2, 0.5))$ in the cases $p = 1, 2, \infty$.

5. A new convexity concept for fuzzy functions

In the classical analysis the convexity of a differentiable function can be characterized as follows: it is convex on $X \subset I\!R^n$ if and only if its epigraph is a subset of any its support halfspace or in other words the epigraph of the function is a subset of the epigraph of its support plane. To generalize this idea to the fuzzy functions we have to define the fuzzy support plane of fuzzy functions and their epigraph.

Let $\widetilde{f(\boldsymbol{x})}$ be a fuzzy function on $I\!R^n$.

Definition 6. The fuzzified epigraph $\widetilde{\text{epi}}\, f$ is a fuzzy set on $I\!R^n \times I\!R$ obtained by T_{gp}-fuzzification of the inequality $y \geq f(x)$ with the extension principle, i.e.

$$\widetilde{\text{epi}}\, f(\boldsymbol{x}, y) = \sup_{u \geq v} T_{gp}(\chi_{\{y\}}(u), \widetilde{f(\boldsymbol{x})}(v)).$$

Proposition 5. The membership function of the fuzzified epigraph is

$$\widetilde{\text{epi}}\, f(\boldsymbol{x}, y) = \begin{cases} 1 & \text{if } y \geq L_f^1(\boldsymbol{x}), \\ f(\boldsymbol{x}, y) & \text{if } L_f^0(\boldsymbol{x}) \leq y < L_f^1(\boldsymbol{x}), \\ 0 & \text{if } y < L_f^0(\boldsymbol{x}). \end{cases}$$

Proof. The statement of the proposition follows immediately from the inequality

$$\widetilde{\text{epi}}\, f(\boldsymbol{x}, y) = \sup_{u \geq v} T_{gp}(\chi_{\{y\}}(u), \widetilde{f(\boldsymbol{x})}(v)) = \\ = \sup_{y \geq v} \widetilde{f(\boldsymbol{x})}(v).$$

In the paper [4] the computation of valued inequalities has been proved, from there one can obtain the following propositions:

Proposition 6. Let $\tilde{h}_{\boldsymbol{\mu}}(\boldsymbol{x})(y)$ be the $(g, p, \boldsymbol{D})$-fuzzification of $h_{\boldsymbol{a}}(\boldsymbol{x})$. Then

$$(\widetilde{\text{epi}}\, h)(\boldsymbol{x}, y) = \sup_{\boldsymbol{a}: y \geq h_{\boldsymbol{a}}(\boldsymbol{x})} \boldsymbol{\mu}(\boldsymbol{a}) = g^{(-1)} \left(\frac{\max(0, y - \sum_{j=1}^{m} \alpha_j h_j(\boldsymbol{x}))}{||\boldsymbol{D}\boldsymbol{h}(\boldsymbol{x})||_q} \right).$$

Corollary 6.1. *Let* $\widetilde{\ell_{\boldsymbol{\mu}}}(\boldsymbol{x}, y)$ *be the fuzzification of the linear function* $\ell_{\boldsymbol{a}}(\boldsymbol{x}) = \sum_{j=1}^{n} a_j x_j + a_0$. *Then*

$$(\widetilde{\text{epi}}\, \ell)(\boldsymbol{x})(y) = g^{(-1)} \left(\frac{\max(0, y - (\sum_{j=1}^{n} \alpha_j x_j + \alpha_0))}{||\boldsymbol{D}(\boldsymbol{x}, 1)^T||_q} \right).$$

Let now $\widetilde{f(\boldsymbol{x})}$ be obtained by (g, Ω)-fuzzification from the function $f : \mathbb{R}^n \to \mathbb{R}$. Let us assume that $f(\boldsymbol{x})$ is convex on $\mathbb{R}^n$.

A support hyperplane of $f(\boldsymbol{x})$ in the point $\boldsymbol{z} \in I\!R^n$ is given by the equality

$$s_{\boldsymbol{z}}(\boldsymbol{x}) = f(\boldsymbol{z}) + c^T(\boldsymbol{x} - \boldsymbol{z})$$

where c is a subderivative of f.

Definition 7. $\widetilde{s_{\boldsymbol{z}}(\boldsymbol{x})}(y)$ is a fuzzy support plane of $f(\boldsymbol{x})(y)$ at $\boldsymbol{z}$ if it is the (g, Ω)-fuzzification of the support plane $s_{\boldsymbol{z}}(\boldsymbol{x})$.

Definition 8. The (g, Ω)-fuzzified function $\widetilde{f(\boldsymbol{x})}$ is Ω-convex at the point $\boldsymbol{z}$ if

$$\widetilde{\text{epi}}\, f \subset \widetilde{\text{epi}}\, s_z.$$

It is Ω-convex on $I\!R^n$ if it is Ω-convex for all $\boldsymbol{z} \in I\!R^n$.

It is obvious that any (g, Ω)-fuzzified linear function is Ω-convex since its support hyperplane at every point $\boldsymbol{z}$ coincides with the given function, so the inclusion for their fuzzified epigraph is trivially fulfilled.

Proposition 7. *Let $f : I\!R^n \to I\!R$ be a convex function. Then any (g, Ω)-fuzzification preserves the convexity in the sense of Ω-convexity.*

Proof. $\widetilde{\text{epi}}\, f \subset \widetilde{\text{epi}}\, s_z$ fulfills if $L_f^\lambda(x) \geq L_{s_z}^\lambda(x)$ for all $\lambda \in [0, 1]$ and $x \in I\!R^n$. From the convexity of f follows

$$\begin{aligned} L_f^\lambda(\boldsymbol{x}) &= f(\boldsymbol{x}) - g(\lambda)\Omega(\boldsymbol{x}) \geq \\ &\geq f(\boldsymbol{z}) + c^T(\boldsymbol{x} - \boldsymbol{z}) - g(\lambda)\Omega(\boldsymbol{x}) = \\ &= s_{\boldsymbol{z}}(\boldsymbol{x}) - g(\lambda)\Omega(\boldsymbol{x}) = L_{s_z}^\lambda(\boldsymbol{x}) \end{aligned}$$

for all $\lambda \in [0, 1]$.

For illustration the Figure 2 shows the fuzzified support functions for the parametrically fuzzified quadratical function $y = x^2 - 3x + 2$. The fuzzy parameters are $(1, 0.5)$, $(-3, 0.5)$, $(2, 0.5)$. So, $\Omega(x) = 0.5x^2 + 0.5|x| + 0.5$.

References

[1] **Katsaras A.K. and Liu D.B.**, Fuzzy vector spaces and fuzzy topological vector spaces, *J. Math. Anal. Appl.*, **58** (1977), 135-146.

[2] **Kovács M.**, On the g-fuzzy linear systems, *BUSEFAL*, **48** (1988), 69-77.

[3] **Ковач М.** О решении нечётко расширенных линейных систем. В *О кооперируемых работах НИВЦ МГУ и БУВЦ*, под ред. В.М. Репина и В.А. Морозова, изд. МГУ, Москва, 1990, 13-23. (Kovács M. On the solution of fuzzified linear systems. V.M. Repin and V.A. Morozov (eds), *Cooperative works of SRCC of MU and CC of BU.*, Moscow State University, 1990, 12-23.)

[4] **Kovács M.**, A stable embedding of ill-posed linear systems into fuzzy systems, *Fuzzy Sets and Systems*, **45** (1992), 305-312.

[5] **Kovács M.**, A concept of optimality for fuzzified mathematical programming problems. *Interactive Fuzzy Optimization*, M. Fedrizzi, J. Kacprzyk and M. Roubens (eds.) Springer, 1991, 36-44.

[6] **Lowen R.**, Convex fuzzy sets, *Fuzzy Sets and Systems*, **3** (1980), 291-310.

[7] **Nanda S.**, Fuzzy linear spaces over valued fields, *Fuzzy Sets and Systems*, **42** (1991), 351-354.

[8] **Nanda S. and Kar K.**, Convex fuzzy mappings. *Fuzzy Sets and Systems* **48**, (1992) 129-132.

[9] **Zadeh L.A.**, The concept of linguistic variable and its application to approximate reasoning, Part 1, *Inform.Sci.*, **8** (1975), 199-249.

M. Kovács
Department of Operations Research
Eötvös Loránd University
Rákóczi út 5.
H-1088 Budapest, Hungary

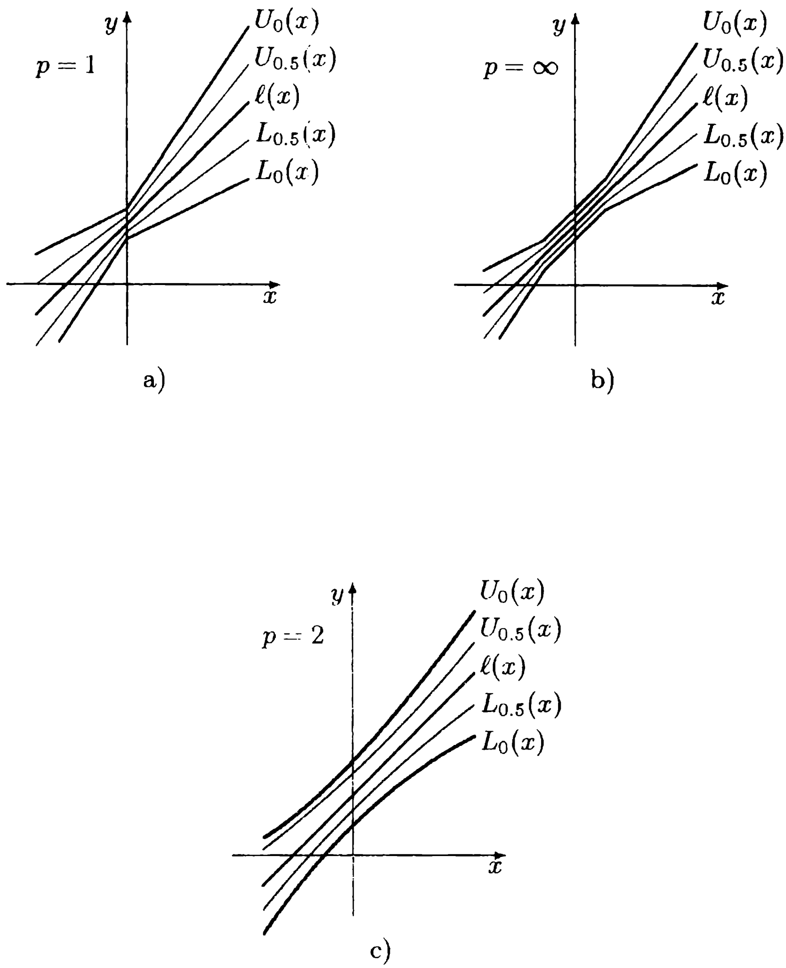
y
p = 1
U0(x)
U0.5(x)
ℓ(x)
L0.5(x)
L0(x)
x
a)
y
p = ∞
U0(x)
U0.5(x)
ℓ(x)
L0.5(x)
L0(x)
x
b)
y
p = 2
U0(x)
U0.5(x)
ℓ(x)
L0.5(x)
L0(x)
x
c)

Figure 1.

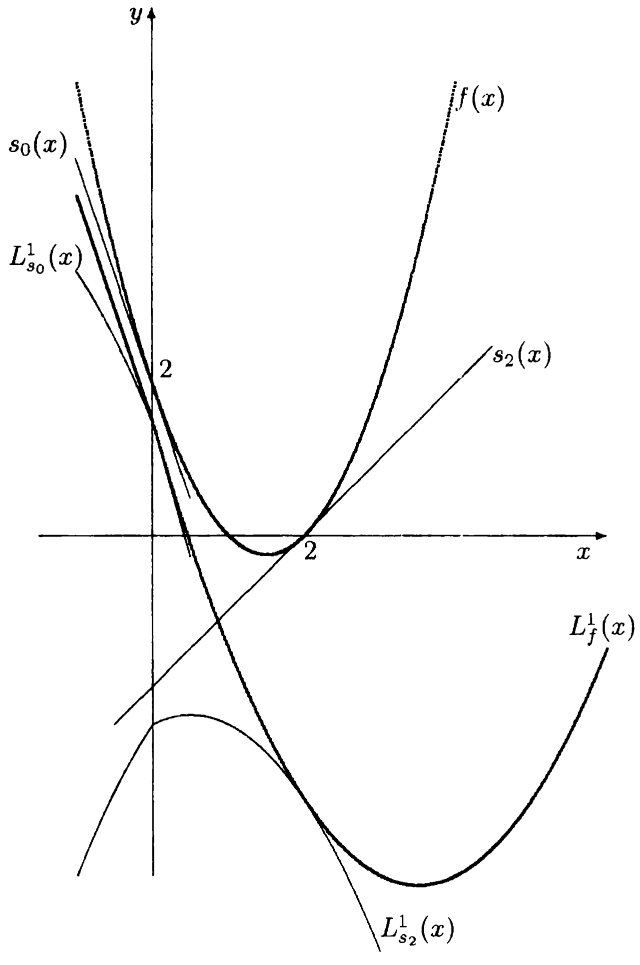

Figure 2.