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SOLUTION OF DIOPHANTINE EQUATIONS BY SECOND ORDER LINEAR RECURRENCES

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To the memory of Imre Környei
To the memory of Béla Kovács

Abstract. Solutions of many diophantine equations can be expressed by the terms of linear recurrences. A collection of these equations is presented in the paper. Furthermore it is shown that all positive integer solutions of the equation $x^2 + x(y-1) - y^2 = 0$ are $(x, y) = (F_{2h+1}^2, F_{2h+1}F_{2h+2})$, where F_n denotes the n-th Fibonacci number.

Let A, B, G_0, G_1 be fixed integers, such that $AB \neq 0$ and not both of G_0, G_1 are zero. Define a second order linear recurrence by a recursive formula

$$G_n = AG_{n-1} - BG_{n-2} \quad \text{for} \quad n > 1$$

which will be denoted by G or $G(A, B, G_0, G_1)$. A sequence $H(A, B, H_0, H_1)$ is the associated sequence of the sequence $G(A, B, G_0, G_1)$ if $H_0 = 2G_1 - AG_0$ and $H_1 = AG_1 - BG_0$. Special cases of sequence G are the Fibonacci sequence $F = F(1, -1, 0, 1)$, the Lucas sequence $L = L(1, -1, 2, 1)$ and the Pell sequence $P = P(2, -1, 0, 1)$.

The equation

$$x^2 - Dy^2 = N,$$

where $N \neq 0$, D is not a perfect square and $D > 0$, is called Pell equation.

We know many relationships between the Pell equation and second order linear recurrences. D.E. Ferguson [4] proved, that the only solutions of the

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equation $x^2 - 5y^2 = \pm 4$ are $x = \pm L_n, y = \pm F_n$, where L_n and F_n are the n-th terms of Lucas and Fibonacci sequences. E.M. Cohn, I.Adler, V.Thebault [3,1,2,13] showed that there is similar connection between the equation $x^2 - 2y^2 = \pm 1$ and the Pell sequence.

P. Kiss and F. Várnai [10] proved that all solutions of the equation $x^2 - 2y^2 = N$ can be determined by finitely many sequences $P(2, 1, P_0, P_1)$, such that $(x, y) = (\pm(P_{2n} + P_{2n+1}), \pm P_{2n+1})$. P. Kiss [11] generalized this theorem. Suppose that $a > 0$ is a fixed integer. If there exists a solution of the equation $x^2 - (a^2 + 1)y^2 = N$, then all solutions can be given by the terms of finitely many sequences $G(2a, -1, G_0, G_1)$.

In the same paper P. Kiss generalized the result of D. E. Ferguson, that is if there exists a solution of the equation $x^2 - (a^2 - 4)y^2 = 4N$, then all solutions can be given by the help of finitely many sequences $G(a, 1, G_0, G_1)$, such that $(x, y) = (\pm H_{2n}, \pm G_{2n})$, where H is the associated sequence of G, and if $N > 0$ then $0 < G_1 < \sqrt{N}$, if $N < 0$ then $0 \leq G_1 < a\sqrt{\frac{-N}{a^2-4}}$.

The above results can be generalized: the solutions of Pell equations for any D ($D > 0$, D is not a perfect square) can be given by the help of second order linear recurrences and all solutions can be determined by number pairs of terms of second order linear recurrences.

K. Liptai [12] proved the following result: Let N, D be integer numbers, $N \neq 0, D > 0$, and D is not a perfect square. Let (u_0, v_0) be the fundamental solution of the equation

$$x^2 - Dy^2 = 1.$$

If the equation

$$x^2 - Dy^2 = N$$

has a positive integer solution (x_0, y_0), then all solutions can be given by the help of finitely many sequences $G(2u_0, 1, G_0, G_1)$, $H(2u_0, 1, H_0, H_1)$ such that

$$(x, y) = (G_n, H_n)$$

and

$$0 \leq H_0 < v_0\sqrt{N} \qquad \text{for} \qquad N > 0,$$
$$0 < H_0 < \sqrt{\frac{-Nu_0^2}{D}} \qquad \text{for} \qquad N < 0.$$

Another type of results was shown by J.P. Jones. In [5, 6] he proved that the sets of all Fibonacci numbers, respectively Lucas numbers, equal the sets of positive values of polynomials

$$y\left(2 - (y^2 - yx - x^2)^2\right) \quad \text{and} \quad y\left(1 - \left((y^2 - yx - x^2)^2 - 25\right)^2\right),$$

if $x > 0, y > 0$.

P. Kiss [9] extended these results proving the following. Let $R = R(A, B, 0, 1)$ be a second order linear recurrence, where $A > 0$, $B = -1$ or $A > 3$ and $B = 1$. Suppose that $x \geq 0$, $y \geq 0$ are integers. Then $|x^2 - Axy + By^2| = 1$ if and only if x and y are consecutive terms of the sequence R. Furthermore he proved, that the set of the all terms of sequence R equals the set of non-negative values of polynomial

$$f(x, y) = y\left(2 - (x^2 - Axy + By^2)^2\right),$$

where $x > 0, y > 0$ are integers.

J. P. Jones [7] proved the following general theorem. Let $\mathbf{A}$ be the set of non-Fibonacci natural numbers. There exists a $P(z, x, y, v)$ polynomial of 4 variables such that $z \in \mathbf{A}$ if and only if there exist $x, y, v \in \mathbf{N}$ such that $P(z, x, y, v) = 0$. If $z \in \mathbf{A}$ is fixed, then x, y, v are uniquely determined.

V. E. Hoggatt Jr. [8] investigated a diophantine equation and stated the following.

Theorem. *All positive integer solutions of the equation*

$$x^2 + x(y - 1) - y^2 = 0 \tag{1}$$

are

$$(x, y) = (F_{2h+1}^2, F_{2h+1}F_{2h+2}). \tag{2}$$

However Hoggatt's proof of this theorem is not perfect. After some calculations he obtained an equation of the form

$$(y - 1)^2 + 4y^2 = z^2$$

and deduced from here that

$$y - 1 = m^2 - n^2, \quad 2y = 2mn \quad \text{and} \quad z = m^2 + n^2$$

with some integers m and n. But it is correct only if y is even, since $(y - 1, 2y) = 1$ or 2. Thus the theorem is proved only when $y = F_{2h+1}F_{2h+2}$ is even, i.e. y is of the form $y = F_{6k+3}F_{6k+4}$ or $y = F_{6k+5}F_{6k+6}$.

In the following we complete Hoggatt's proof.

Proof of the Theorem. In the case y is even, Hoggatt's proof is correct, that is all positive integer (x, y) solutions of (1) are the pairs in (2). Reducing the Fibonacci sequence modulo 2, it can be easily seen that y is even if and only if $2h+1$ has the form $6k+3$ or $6k+5$.

Now let (x,y) a solution of equation (1) such that $y=2t+1$ is odd. In this case, by (1), x is an integer if and only if

$$(y-1)^2+4y^2=z^2,$$

i.e. if

$$t^2+y^2=q^2$$

with some integer z and $q=z/2$. But $(t,y)=1$, so there are integers m,n such that

$$t=2mn, \quad y=m^2-n^2 \quad \text{and} \quad q=m^2+n^2. \tag{3}$$

Using that $y=2t+1$, by (3)

$$m^2-n^2=4mn+1$$

and

$$m=\frac{4n\pm\sqrt{16n^2+4n^2+4}}{2}=2n\pm\sqrt{5n^2+1} \tag{4}$$

follows. It implies that

$$5n^2+1=s^2 \tag{5}$$

with some integer s and so we obtained the Pell equation

$$s^2-5n^2=1. \tag{6}$$

The fundamental solution of (6) is $(s_0,n_0)=(9,4)$ and so, as it is well known, all positive solutions (s,n) of (6) are determined by

$$s+n\sqrt{5}=(9+4\sqrt{5})^k$$

and

$$s-n\sqrt{5}=(9-4\sqrt{5})^k \quad (k=1,2\ldots).$$

Thus we get

$$s=\frac{(9+4\sqrt{5})^k+(9-4\sqrt{5})^k}{2}, \quad n=\frac{(9+4\sqrt{5})^k-(9-4\sqrt{5})^k}{2\sqrt{5}}$$

and so, by (4) and (5),

$$m = \frac{(2+\sqrt{5})(9+4\sqrt{5})^k - (2-\sqrt{5})(9-4\sqrt{5})^k}{2\sqrt{5}}$$

and after some elementary calculations it follows that

$$y = m^2 - n^2 = \frac{1}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{6k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{6k+1}\right)\cdot$$

$$\cdot\left(\left(\frac{1+\sqrt{5}}{2}\right)^{6k+2} - \left(\frac{n+\sqrt{5}}{2}\right)^{6k+2}\right) = F_{6k+1}F_{6k+2},$$

since

$$F_q = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^q - \left(\frac{1-\sqrt{5}}{2}\right)^q\right)$$

for $q = 0, 1, 2\ldots$ by the Binet formula.

We can similarly show that

$$x = \frac{-y+1+q}{2} = \frac{n^2 - m^2 + 1 + 2m^2 + 2n^2}{2} = F_{6k+1}^2$$

which completes the proof.

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