ALMOST-PERIODIC MULTIPLICATIVE FUNCTIONS ON THE SET OF SHIFTED PRIMES

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To the memory of Imre Környei and Béla Kovács

Abstract. Let $f \in \mathcal{L}_2$ be multiplicative and almost-periodic with non-empty Fourier-Bohr spectrum. If in addition

$$\sum_{p \leq x} |f(p)|^2 \leq Ax \log^{-\rho} x$$

with some $\rho > 0$, then, for each $\varepsilon > 0$, there exists a periodic function $P_\varepsilon : N \to C$ such that

$$\limsup_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |f(p+1) - P_\varepsilon(p+1)| < \varepsilon.$$

Let $f : N \to C$ be a complex-valued multiplicative function, let p denote a prime and let $\pi(x)$ be the number of primes below x.

For each positive α we shall say that f belongs to the class $\mathcal{L}_\alpha$ if

$$||f||_\alpha := \limsup_{x \to \infty} \left(\frac{1}{x} \sum_{n \leq x} |f(n)|^\alpha \right)^{\frac{1}{\alpha}} < +\infty.$$

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Let $\alpha \geq 1$ and $\mathcal{B}_\alpha$ be the space of $(\mathcal{B}_\alpha)$-almost periodic functions, i.e. $f \in \mathcal{B}_\alpha$ in case $f \in \mathcal{L}_\alpha$ and, for each $\varepsilon > 0$, there exists a trigonometric polynomial $P_\varepsilon(n) = \sum a_j \exp(2\pi i \alpha_j n)$, $\alpha_j \in \mathbb{R}$, such that

$$\|f - P_\varepsilon\|_\alpha < \varepsilon.$$

For an arbitrary arithmetical function f we define the *Fourier-Bohr spectrum* $\sigma(f)$ of f by

$$\sigma(f) := \left\{ \alpha \in R/\mathbb{Z} : \limsup_{x\to\infty} \left| \frac{1}{x} \sum_{n\leq x} f(n) \exp(-2\pi i n\alpha) \right| > 0 \right\}.$$

Using the results of the previous papers [1], [2] we prove

Theorem. *Let $f \in \mathcal{B}_2$ be multiplicative and satisfy the condition*

$$\sum_{p\leq x} |f(p)|^2 \leq A_1 x \ln^{-\rho} x, \tag{1}$$

where $\rho > 0$. If $\sigma(f) \neq \emptyset$, then for each $\varepsilon > 0$ there exists a periodic function $P_\varepsilon : N \to C$ such that

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} |f(p+1) - P_\varepsilon(p+1)| < \varepsilon.$$

Remark. If $f \in \mathcal{L}_\alpha$, $\alpha > 2$, then $f \in \mathcal{L}_2$ and

$$\sum_{p\leq x} |f(p)|^2 \leq \left(\sum_{n\leq x} |f(n)|^\alpha \right)^{\frac{2}{\alpha}} \pi(x)^{1-\frac{2}{\alpha}} \ll x \ln^{\frac{2}{\alpha}-1} x.$$

Hence f satisfies condition (1) with $\rho = 1 - \frac{2}{\alpha}$.

The next result will play a key role in the proof of the Theorem.

Lemma 1 (see Theorem 2 of [1]). *Let f_i, $i = 1, \dots k$, be complex-valued multiplicative functions satisfying the conditions*

$$A(n) = \sum_{i=1}^{k} \alpha_i f_i(n) \geq 0, \qquad n = 1, 2, \dots,$$

where $\alpha_i \in \mathbb{C}$ *and where* $f_i \in \mathcal{L}_2$ *and satisfy (1),* $i = 1, \ldots, k$. *Then, for some* $\delta > 0$,

$$\frac{1}{\pi(x)} \sum_{p \leq x} A(p+1) \ll \frac{\ln y}{x} \sum_{n \leq x} A(n) + \frac{1}{y^\delta}$$

holds uniformly for $2 \leq y \leq \ln x$. *If* f_i, $i = 1, \ldots, k$, *satisfy conditions (1) and*

$$\sum_{n \leq x} |f_i(n)|^2 \leq A_2 x \tag{2}$$

then the constant implied in $\ll$ *depends only on* A_1 *and* A_2.

Lemma 2 (see Lemma 3 of [2]). *Let* f *be a complex-valued multiplicative function, assume that* $f \in \mathcal{L}_2$ *and* f *satisfies condition (1). Suppose further that*

$$\sum_{p \leq x} |1 - f(p)| \frac{\ln p}{p} \leq \varepsilon(x) \ln x, \tag{3}$$

where $\varepsilon(x) \downarrow 0$ *but* $\varepsilon(x)\sqrt{\ln x} \to \infty$ *as* $x \to \infty$. *Then we have*

$$\frac{1}{\pi(x)} \sum_{p \leq x} f(p+1) = \prod_{p \leq x} \left(1 + \sum_{r=1}^{\infty} \frac{1}{\varphi(p^r)} \left(f(p^r) - f(p^{r-1}) \right) \right) + \\ + o\left(\exp\left(\sum_{p \leq x} \frac{|f(p)| - 1}{p} \right) + 1 \right).$$

Proof of Theorem. For any arithmetical function set

$$M(f) := \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n),$$

if the limit exists. Let $f \in \mathcal{B}_2$ and $\sigma(f) \neq \emptyset$. In this case (see [3], Corollary 1) there exists a Dirichlet-character $\chi_d \pmod d$ such that the mean value $M(f_{\chi_d})$ exists and is different from zero. Using Theorem of [4] we obtain that $M(|f|)$ exists, $M(|f|) \neq 0$ and the series

$$\sum_{p} \frac{|f(p)| - 1}{p}, \qquad \sum_{|f(p)| \leq \frac{3}{2}} \frac{(|f(p)| - 1)^2}{p},$$

$$\sum_{||f(p)| - 1| \geq \frac{1}{2}} \frac{|f(p)|^2}{p}, \qquad \sum_{p} \sum_{r \geq 2} \frac{|f(p^r)|}{p^r} \tag{4}$$

converge.

For a multiplicative function f we define a multiplicative function $\sqrt{f}$ by

$$(\sqrt{f(n)})^2 = f(n) \quad \text{and} \quad \sqrt{f(p^r)} \cdot {}^{(0)}\!\sqrt{\chi_d(p^r)} = {}^{(0)}\!\sqrt{f(p^r)\chi_d(p^r)},$$

where we use the notation ${}^{(0)}\!\sqrt{z} = \sqrt{|z|} \cdot \exp(\frac{1}{2} i \arg z)$, $\arg z \in [-\pi, \pi)$.

If $p \nmid d$ we obtain

$$\left| \sqrt{f(p)} + \sqrt{\overline{\chi}_d(p)} \right| \geq 1. \tag{5}$$

Now define a multiplicative function f_1 by

$$f_1(p^r) = \begin{cases} \overline{\chi}_d(p^r) & \text{if} \quad r \geq 2,\ p^r \geq t, \\ \overline{\chi}_d(p) & \text{if} \quad r = 1,\ |f(p)| \geq \frac{3}{2},\quad p \geq t, \\ f(p^r) & \text{otherwise.} \end{cases}$$

We have

$$\sum_{n \leq x} |f_1(n)|^2 \leq \sum_{\substack{m \leq x \\ p|m \to p^2|m}} \sum_{n \leq \frac{x}{m}} |f(n)|^2 \leq A_2 x \prod_p \left(1 + \frac{1}{p^2}\right).$$

Therefore f_1 satisfies the conditions (1) and (2). Hence, using Lemma 1, we obtain

$$\frac{1}{\pi(x)} \sum_{p \leq x} \left| \sqrt{f(p+1)} - \sqrt{f_1(p+1)} \right|^2 \ll \frac{\ln y}{x} \sum_{n \leq x} |f(n) - f_1(n)|^2 + \frac{1}{y^\beta},$$

where $\beta > 0$ and $2 < y < \ln x$. We have (see (4))

$$\sum_{n \leq x} \left| \sqrt{f(n)} - \sqrt{f_1(n)} \right|^2 \ll$$

$$\ll \sum_{r \geq 2} \sum_{p^r \geq t} \frac{x}{p^r} (|f(p^r)| + 1) + x \sum_{\substack{p > t \\ |f(p)| \geq \frac{3}{2}}} \frac{|f(p)|^2}{p} \ll \varepsilon(t) \cdot x,$$

where $\varepsilon(t) \to 0$ as $t \to \infty$. Thus we proved

$$\frac{1}{\pi(x)} \sum_{p \leq x} \left| \sqrt{f(p+1)} - \sqrt{f_1(p+1)} \right|^2 \ll \varepsilon(t) \ln y + \frac{1}{y^\beta}. \tag{6}$$

Now let the multiplicative function f_2 be defined by

$$f_2(p^r) = \begin{cases} \overline{\chi}_d(p) & \text{if } r = 1,\, p > t,\ \ |f(p)| \le \frac{3}{2}, \\ f_1(p^r), & \text{otherwise.} \end{cases}$$

Using Lemma 2 we shall determine the asymptotic behaviour of the sum

$$\frac{1}{\pi(x)} \sum_{p \le x} \Big(|f_1(p+1)| - \overline{\sqrt{f_1(p+1)}} \sqrt{f_2(p+1)} -$$

$$- \sqrt{f_1(p+1)}\, \overline{\sqrt{f_2(p+1)}} + |f_2(p+1)| \Big) .$$

For a fixed number t we have proved that f_1 satisfies the conditions (1) and (2). Concerning f_2 we have $|f_2(p^r)| = |\overline{\chi}_d(p^r)|$, if $p^r \ge t$, and therefore f_2 satisfies the conditions (1) and (2). Applying Cauchy's inequality we obtain that $\overline{\sqrt{f_1}}\sqrt{f_2}$, $\sqrt{f_1}\,\overline{\sqrt{f_2}}$ satisfy these conditions, too. Next we prove that for all these functions the condition (3) holds. We show the details in the case $\overline{\sqrt{f_1}}\sqrt{f_2}$.

We have

$$\frac{1}{\ln x} \sum_{p \le x} \left| \overline{\sqrt{f_1(p)}} \sqrt{f_2(p)} - 1 \right| \frac{\ln p}{p} =$$

$$= \frac{1}{\ln x} \sum_{p \le t} \big||f(p)| - 1\big| \frac{\ln p}{p} + \frac{1}{\ln x} \sum_{\substack{p \le x \\ |f(p)| \le \frac{3}{2}}} \left| \overline{\sqrt{f(p)}} \sqrt{\overline{\chi}_d(p)} - 1 \right| \frac{\ln p}{p}.$$

Using (5) and Cauchy's inequality we conclude that the second sum is

$$\le \frac{1}{\ln x} \sum_{\substack{t < p \le x \\ |f(p)| \le \frac{3}{2}}} \big||f(p)| - 1\big| \frac{\ln p}{p} \le \left(\sum_{|f(p)| \le \frac{3}{2}} \frac{(|f(p)| - 1)^2}{p} \frac{1}{\ln^2 x} \sum_{p \le y} \frac{\ln^2 p}{p} \right)^{\frac{1}{2}} +$$

$$+ \left(\sum_{\substack{p > y \\ |f(p)| \le \frac{3}{2}}} \frac{(|f(p)| - 1)^2}{p} \frac{1}{\ln^2 x} \sum_{p \le x} \frac{\ln^2 p}{p} \right)^{\frac{1}{2}}$$

Set $y = \ln x$. By (4) we obtain that the right side tends to zero as $x \to \infty$. Hence (3) holds. Using Lemma 2 we get

$$\frac{1}{\pi(x)} \sum_{p \le x} \left| \sqrt{f_1(p+1)} - \sqrt{f_2(p+1)} \right|^2 =$$

$$= \prod_{p\le t}\left(1+\sum_{r=1}^{\infty}\frac{1}{\varphi(p^r)}(|f_1(p^r)|-|f_1(p^{r-1})|)\right)\left\{\prod_{\substack{t<p\le x\\ |f(p)|\le\frac32}}\left(1+\frac{|f(p)|-1}{p}\right)-\right.$$

$$\left.\prod_{\substack{t<p\le x\\ |f(p)|\le\frac32}}\left(1+\frac{\sqrt{f(p)}\overline{\sqrt{\chi_d(p)}}-1}{p}\right)-\prod_{\substack{t<p\le x\\ |f(p)|\le\frac32}}\left(1+\frac{\overline{\sqrt{f(p)}\sqrt{\chi_d(p)}}-1}{p}\right)+1\right\}$$

$$+o\left(\exp\left(\sum_{\substack{p\le x\\ |f(p)|\le\frac32}}\frac{1}{p}\left(\sqrt{|f(p)|}-1+|f(p)|-1\right)\right)+1\right).$$

Recall that the mean value $M(f\chi_d)$ exists and is different from zero. Hence (see Theorem of [4]) the series

$$\sum_{p}\frac{\chi_d(p)f(p)-1}{p},\qquad \sum_{|f(p)|\le\frac32}\frac{|\chi_d(p)f(p)-1|^2}{p} \tag{7}$$

converge. We have

$$\sqrt{f(p)}\overline{\sqrt{\chi_d(p)}} = {}^{(0)}\!\!\sqrt{f(p)\chi_d(p)} = 1+\frac12(f(p)\chi_d(p)-1)+O(|\chi_d(p)f(p)-1|^2).$$

Using the convergence of the series (4) and (7) we see that the series

$$\sum_{|f(p)|\ge\frac32}\frac{|f(p)\chi_d(p)-1|}{p}\le\sum_{||f(p)|-1|\ge\frac12}\frac{|f(p)|^2}{p},$$

$$\sum_{|f(p)|\le\frac32}\frac{\chi_d(p)f(p)-1}{p},\qquad \sum_{|f(p)|\le\frac32}\frac{{}^{(0)}\!\!\sqrt{f(p)\chi_d(p)}-1}{p}$$

converge. In the same way as before we obtain that the series

$$\sum_{|f(p)|\le\frac32}\frac{|f(p)|-1}{p},\quad \sum_{p}\frac{\sqrt{|f(p)|}-1}{p},\quad \sum_{|f(p)|\le\frac32}\frac{\overline{\sqrt{f(p)}}\sqrt{\chi_d(p)}-1}{p}$$

converge, too. Thus

$$\frac{1}{\pi}\sum_{p\le x}\left|\sqrt{f_1(p+1)}-\sqrt{f_2(p+1)}\right|^2\ll\exp\left(\sum_{p\le t}\frac{|f(p)|-1}{p}\right).$$

$$\cdot\left(\left|\sum_{\substack{t<p\leq x\\ |f(p)|\leq\frac{3}{2}}}\frac{|f(p)|-1}{p}\right|+\left|\sum_{\substack{t<p\leq x\\ |f(p)|\leq\frac{3}{2}}}\frac{\sqrt[(Q)]{f(p)\chi_d(p)}-1}{p}\right|\right)+o(1)\ll$$

$$\ll\varepsilon(t)+o(1), \tag{8}$$

where $\varepsilon(t)\to 0$ as $t\to\infty$. We have

$$\frac{1}{\pi(x)}\sum_{p\leq x}|f(p+1)-f_2(p+1)|\leq\frac{1}{\pi(x)}\sum_{p\leq x}(|f(p+1)-f_1(p+1)|+$$

$$+\left|\sqrt{f_1(p+1)}-\sqrt{f_2(p+1)}\right|^2+2\left|\sqrt{f_1(p+1)}-\sqrt{f_2(p+1)}\right|\left|\sqrt{f_1(p+1)}\right|\Big).$$

Therefore, by (6) and (8), using Cauchy's inequality we get

$$\frac{1}{\pi(x)}\sum_{p\leq x}|f(p+1)-f_2(p+1)|\ll$$

$$\left(\left(\varepsilon(t)\ln y+y^{-\beta}\right)^{\frac{1}{2}}+(\varepsilon(t)+o(1))^{\frac{1}{2}}\right)\left(\frac{1}{\pi(x)}\sum_{p\leq x}(|f(p+1)|+f_1(p+1)|)\right)^{\frac{1}{2}}+$$

$$+\varepsilon(t)+o(1).$$

Recall that $f_1,f_2\in\mathcal{L}_2$. Hence by Lemma 1 the second factor equals $O(1)$. Let $y=1/\varepsilon(t)$. Then we obtain

$$\frac{1}{\pi(x)}\sum_{p\leq x}|f(p+1)-f_2(p+1)|\ll\varepsilon_1(t)+o(1),$$

where $\varepsilon_1(t)\to 0$ as $t\to\infty$. On the other hand the function

$$f_2(n)=\prod_{\substack{p^r\|n,\\ p^r\leq t}}f(p^r)\prod_{\substack{p^r\|n,\\ p^r>t}}\chi_d(p^r)$$

is a periodic function with period

$$T=d2^{r_1}\dots p_k^{r_k},$$

where p_i $(i = 1, \dots, k)$ are primes less than t and

$$r_i = \left[\frac{\ln t}{\ln p_i}\right], \qquad i = 1, \dots, k.$$

Thus we have proved the theorem.

References

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