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APPROXIMATION OF REAL NUMBERS BY RATIONALS VIA SERIES EXPANSIONS

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To the memory of Imre Környei
To the memory of Béla Kovács

Since both Imre Környei and Béla Kovács were interested in and contributed to different aspects of series expansions of real numbers, it is just fitting to write on series expansions in this memorial volume.

Let $a_n = a_n(j)$ and $b_n = b_n(j)$, $n \geq 1$, be two sequences of positive integer valued functions of the positive integers j such that the function

$$h_n(j) = \frac{a_n(j)}{b_n(j)}\, j(j-1), \qquad j \geq 2, \tag{1}$$

is integer valued for every $n \geq 1$. Let x be a real number from the interval $(0,1]$, and expand x by the following algorithm: we define the positive integers $d_k = d_k(x)$ and real numbers x_j by

$$x = x_1, \qquad 1/d_k < x_k \leq 1/(d_k - 1), \quad \text{and} \tag{2}$$

$$x_{k+1} = (x_k - 1/d_k)\, b_k(d_k)/a_k(d_k). \tag{3}$$

Upon setting

$$\frac{p_n}{q_n} = \frac{1}{d_1} + \frac{a_1(d_1)}{b_1(d_1)}\frac{1}{d_2} + \ldots + \frac{a_1(d_1)\ldots a_{n-1}(d_{n-1})}{b_1(d_1)\ldots b_{n-1}(d_{n-1})}\frac{1}{d_n}, \tag{4}$$

where we choose

$$q_n = b_1(d_1)b_2(d_2)\ldots b_{n-1}(d_{n-1})d_n, \tag{5}$$

and thus do not aim at making p_n and q_n relatively prime, we have from the algorithm

$$q_n\left(x-\frac{p_n}{q_n}\right)=r_n=\frac{a_1(d_1)a_2(d_2)\dots a_n(d_n)d_n}{b_n(d_n)}x_{n+1}. \tag{6}$$

Under the condition stated at (1)

$$x-p_n/q_n=r_n/q_n\to 0 \quad \text{as} \quad n\to+\infty,$$

that is the infinite series resulting from the algorithm (2) and (3) always converges to x. In other words (2) and (3) leads to an expansion of x into an infinite series, which expansion is known as an Oppenheim expansion. See Chapter 1 in Galambos (1976) for further details.

A special case obtains when we choose $a_n(j)=1$ and $b_n(j)=j(j-1)$, and thus $h_n(j)=1$ for all $n\geq 1$ and $j\geq 2$. The resulting series expansion is called Lüroth expansion. A recent result on Lüroth expansion by Barrionuevo et al. (1996) can be refined and generalized as follows. We use Lebesgue measure on the Borel sets of $(0,1]$ as the underlying probability $P(.)$.

Theorem 1. *In the special case of Lüroth expansion the value r_n of (6) has the same distribution function*

$$P(r_n\leq y)=F(y) \tag{7}$$

for each $n\geq 2$, where, with $i(z)$ signifying the integer part of z,

$$F(y)=y\sum_{k=2}^{i(1/y)+1}\frac{1}{k}+\frac{1}{i(1/y)+1}, \quad 0<y\leq 1. \tag{8}$$

Remark. Because of the special choice of q_n at (5), a direct comparison with continued fractions concerning the speed of convergence of r_n/q_n as a function of q_n is not possible, but a remarkably fast convergence is implicit in Theorem 1. As a matter of fact, one can expect large common factors of p_n and q_n in (4) when q_n is chosen by (5) because $b_n(j)=j(j-1)$ for each n, and thus each $b_n(d_n)$ is an even number. Furthermore, for every x, one has several indeces n for which d_n take the same value k, say (see below). One can also set up recursive formulas for p_n/q_n from which one can get further estimates.

Proof of Theorem 1. In the case of Lüroth expansion $a_n(j)=1$ and $b_n(j)=j(j-1)$, all $n\geq 1$ and $j\geq 2$. Thus r_n in (6) becomes

$$r_n=x_{n+1}/(d_n-1). \tag{9}$$

Now, by Theorem 6.1 of Galambos (1976), x_{n+1} and d_n are stochastically independent, x_{n+1} is uniformly distributed and $P(d_n = k) = 1/k(k-1)$ for $k \geq 2$ (for this last result see Theorem 4.14 in Galambos (1976)). Hence, by the total probability rule (see Galambos (1984), p.48)

$$P(r_n \leq y) = \sum_{k=2}^{+\infty} P(x_{n+1} \leq y(d_n - 1) \mid d_n = k) \frac{1}{k(k+1)}.$$

Now, by the independence of x_{n+1} and d_n

$$P(x_{n+1} \leq y(d_n - 1) \mid d_n = k) = P(x_{n+1} \leq y(k-1))$$

where, by (9), we have $0 < y \leq 1$. Finally, the uniform distribution for x_{n+1} entails that the right hand side above equals either $y(k-1)$ or 1, depending whether $y(k-1) \leq 1$ or $y(k-1) > 1$. Our computations thus yield, for $0 < y \leq 1$,

$$P(r_n \leq y) = \sum_{k=2}^{i(1/y)+1} \frac{y(k-1)}{k(k-1)} + \sum_{k=i(1/y)+2}^{+\infty} \frac{1}{k(k-1)}$$

which does not depend on n, and, indeed, equals $F(y)$ of (8). The proof is completed.

We can combine Theorem 1 with the specific form of q_n at (5), yielding

$$x - \frac{p_n}{q_n} = r_n e^{-\log q_n} \tag{10}$$

where r_n is distributed as $F(y)$ at (8), and the cited distributional properties of $b_j(d_j) = d_j(d_j - 1)$, combined with the strong law of large numbers, entail that

$$\log q_n = n(1/n)(\log d_1(d_1 - 1) + \ldots + \log d_{n-1}(d_{n-1} - 1) + \log d_n) =$$
$$= n(c + o(1)),$$

where

$$c = \sum_{k=2}^{+\infty} \frac{\log k(k-1)}{k(k-1)}. \tag{11}$$

We thus have the asymptotic form

$$x - \frac{p_n}{q_n} = r_n e^{-n(c+o(1))} \tag{12}$$

valid for almost all x for the case of Lüroth expansion. In this form one can of course simplify by the common factors of p_n and q_n, allowing comparison with any other expansion. It should also be pointed out that at (12) the role of r_n diminishes because of the unspecified error term $o(1)$ in the exponential factor. On the other hand, in the exact form (10) (or (6)), the property of r_n as expressed in Theorem 1 plays the dominant role. With this in mind, we compare Lüroth expansions with another special Oppenheim expansion, called Engel series.

If we choose $a_n(j) = 1$ and $b_n(j) = j$ for all $n \geq 1$ and $j \geq 2$, the algorithm (2) and (3) lead to the Engel series expansion of x. The function $h_n(j)$ of (1) becomes $h_n(j) = j - 1$, satisfying the requirement of its being integer valued. The expressions (5) and (6) become

$$q_n = d_1 d_2 \dots d_n \quad \text{and} \quad q_n \left(x - \frac{p_n}{q_n} \right) = r_n = x_{n+1}.$$

We set

$$u_{n+1} = h_n(d_n)\, x_{n+1} = (d_n - 1)x_{n+1}.$$

Then $r_n = u_{n+1}/(d_n - 1)$. Appealing to Theorem 6.1 of Galambos (1976) again, we have that u_{n+1} is uniformly distributed on $(0,1)$ and u_{n+1} and d_n are independent. We could proceed as with the case of Lüroth expansion, however, the distribution of d_n itself is no longer of simple form because the d_n are not independent among themselves; rather, they satisfy $d_{n-1} \leq d_n$. Hence, r_n strongly depends on n, which in fact converges to zero. The asymptotic formula also differs from (12): by Corollary 6.25 of Galambos (1976) one gets

$$x - \frac{p_n}{q_n} = \exp\left\{-n(n+1)(1/2 + o(1))\right\}.$$

We conclude the paper by noting that a theorem corresponding to Theorem 6.1 of Galambos (1976) remains valid for the alternating Lüroth-type expansion as defined in Kalpazidou et al. (1990). We therefore have that for such expansion, with the obvious change in the definition of p_n and q_n,

$$q_n \left| x - \frac{p_n}{q_n} \right| = r_n$$

has the same distribution for all $n \geq 1$. We have

$$P(r_n \leq z) = F_A(z) = \sum_{k=2}^{i(1/z)} \frac{z}{k-1} + \frac{1}{i(1/z)}, \qquad 0 < z \leq 1,$$

which distribution function was introduced in Barrionuevo et al. (1996) in connection with a related limit theorem.

References

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