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REDUCED RESIDUE SYSTEMS AND A PROBLEM FOR MULTIPLICATIVE FUNCTIONS

Bui Minh Phong (Budapest, Hungary)

To the memory of Imre Környei
To the memory of Béla Kovács

Abstract. It is proved that if $F,\ G : \mathbb{N} \to \{0,\ 1\}$ are completely multiplicative functions such that $G(an+b) = F(An+B)$ is satisfied for some integers $a > 0,\ b,\ A > 0,\ B$ with $\Delta = Ab - aB \neq 0$ and for every positive integer n, then either $F(An+B) = G(an+b) = 0$ for all $n \in \mathbb{N}$ or $F(n) = G(m) = 1$ for all $n, m \in \mathbb{N}$, $(n, A'\Delta) = (m, a'\Delta) = 1$, where $a' = \frac{a}{(a,b)}$ and $A' = \frac{A}{(A,B)}$.

1. Introduction and results

Notations. Let $\mathbb{N}$ denote the set of all positive integers. The letters p, q, π with and without suffixes denote prime numbers. (m, n) denotes the greatest common divisor of the integers m and n. Here $m \parallel n$ denotes that m is an unitary divisor of n, i.e. that $m|n$ and $(\frac{n}{m}, m) = 1$. For each $n \in \mathbb{N}$ we denote by n^* the product of all prime divisors of n. Let $P(n)$ denote the greatest prime divisor of n. Let $\mathcal{M}$ $(\mathcal{M}^*)$ be the set of complex-valued multiplicative (completely multiplicative) functions.

P.Erdős proved in 1946 [2] that if $f : \mathbb{N} \to \mathbb{R}$ is an additive function such that $\Delta f(n) := f(n+1) - f(n) = o(1)$ as $n \to \infty$, then $f(n)$ is a constant multiple of $\log n$. This assertion has been generalized in several directions (e.g. see [1]). The characterization of multiplicative functions $f : \mathbb{N} \to \mathbb{C}$

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under suitable regularity conditions even in the simplest case $\Delta f(n) = o(1)$ is much harder. More than 15 years ago I.Kátai stated as a conjecture that $f \in \mathcal{M}$, $\Delta f(n) = o(1)$ as $n \to \infty$ imply that either $f(n) = o(1)$ or $f(n) = n^s$ $(n \in I\!N)$, $0 \leq \text{Re}\, s < 1$. This was proved by E.Wirsing in a letter to Kátai (September 3, 1984) and in a recent paper [14]. It is not hard to deduce from Wirsing's theorem that if $f,\ g \in \mathcal{M}$, $g(n+1) - f(n) = o(1)$ as $n \to \infty$, then either $f(n) = o(1)$, or $f(n) = g(n)$ $(n \in I\!N)$, and in the last case $f(n) = n^s$ $(n \in I\!N)$, $0 \leq \text{Re}\ s < 1$.

Recently, improving the above results, we proved in [9] that if $k \in I\!N$ is given and $f,\ g \in \mathcal{M}$ satisfy the condition

$$g(n+k) - f(n) = o(1) \quad \text{as} \quad n \to \infty,$$

then either $f(n) = o(1)$ as $n \to \infty$ or there are $F,\ G \in \mathcal{M}$ and a complex constant s such that

$$f(n) = n^s F(n), \quad g(n) = n^s G(n), \quad 0 \leq \text{Re}\ s < 1$$

and

$$G(n+k) = F(n)$$

are satisfied for all $n \in I\!N$. In [7]-[8], by using the result of [4], the equation $G(n+k) = F(n)$ is solved completely.

The general case concerning the characterization of those $f,\ g \in \mathcal{M}$ for which

$$g(an+b) - Ef(An+B) = o(1) \quad \text{as} \quad n \to \infty,$$

where $a > 0,\ b,\ A > 0,\ B$ are fixed integers and E is a complex constant, seems to be a hard problem. The main difficulty is that we are unable to determine all those $F,\ G \in \mathcal{M}$ for which $G(an+b) = EF(An+B)$ $(n \in I\!N)$ is satisfied, even under the assumption that the values are taken from the set $\{0,\ 1\}$. The above question was solved in [11]-[12] for $B = 0$ under the conditions $|f(n)| = |g(n)| = 1$ $(n \in I\!N)$. A similar result was obtained in [13] under the conditions $f = g$, $|g(n)| = 1$ $(n \in I\!N)$, $g(n+b) - g(n) = o(1)$ as $n \to \infty$, $(n, b) = 1$. Recently, N.L.Bassily and I.Kátai [6] showed that if $f,\ g \in \mathcal{M}$ satisfying $g(2n+1) - Df(n) = o(1)$ $(n \to \infty)$ with some constant $D \neq 0$, then either $f(n) = o(1)$ $(n \to \infty)$ and $g(m) = o(1)$ ($m \to \infty$, $(m, 2) = 1$) or $D = f(2)$, $f(n) = n^s$, $0 \leq \text{Re}s < 1$, and $f(n) = g(n)$ for odd integers n.

In order to determine those multiplicative functions $f,\ g$ which satisfy the relation $g(an+b) - Ef(An+B) = o(1)$ as $n \to \infty$, the first problem is to give all solutions of multiplicative functions F and G for which $G(an+b) = F(An+B)$ $(n \in I\!N)$ is satisfied under the assumptions that the values are taken from the set $\{0,\ 1\}$. Excluding the case $G(an+b) = F(An+B) = 0$ for all $n \in I\!N$, the solution of the last equation will use a result concerning the characterization of suitable reduced residue systems.

For fixed integers $a > 0, b, A > 0$ and B with $\Delta := Ab - aB \neq 0$, we shall denote by $\mathcal{S} = \mathcal{S}(a, b; A, B)$ the subset of positive integers which is subjected to the following properties :

(1) if $x,\ y \in \mathcal{S}$ and $(x, y) = 1$, then $xy \in \mathcal{S}$,

(2) if $x \in \mathcal{S}$ and $y \,\|\, x$, then $y \in \mathcal{S}$,

and

(3) $an + b \in \mathcal{S}$ if and only if $An + B \in \mathcal{S}$.

It is obvious that if $f \in \mathcal{M}$ and $f(an + b) = Ef(An + B)$ is satisfied for all $n \in I\!N$, then the set $\mathcal{S}_f := \{n \in I\!N \mid f(n) \neq 0\}$ satisfies the conditions (1)-(3).

Our purpose in this paper is to prove

Theorem 1. *Let $\mathcal{S} = \mathcal{S}(a, b; A, B)$ be a set subjected to the conditions (1)-(3). If there are a prime π and positive integers $w = w(\pi)$, M such that*

(4) $(\pi,\ aA) = 1,$

(5) $\{\pi^w,\ \pi^{w+1},\ \pi^{w+2}, \dots\} \subseteq \mathcal{S},$

and

(6) $AM + B \in \mathcal{S},$

then we have

$$\{\ n \mid \ (n,\ d\Delta) \ = \ 1\ \} \ \subseteq \ \mathcal{S},$$

where $d = (a, A)$.

Theorem 2. *If $F \in \mathcal{M}^*$ and $G \in \mathcal{M}^*$ such that*

$$G(an + b) = F(An + B) \quad \textit{for all} \quad n \in I\!N,$$

and the set of values of $F(An + B)$ and of $G(an + b)$ is contained in $\{0,\ 1\}$, where $a > 0,\ b,\ A > 0,\ B$ are integers with $\Delta := Ab - aB \neq 0$, then one of the following assertions holds:

(i) $F(An + B) = G(an + b) = 0$ *for all* $n \in I\!N$,

(ii) $F(n) = G(m) = 1$ *for all* $n,\ m \in I\!N,\ (n,\ A'\Delta) = (m,\ a'\Delta) = 1,$

where $a' = \frac{a}{(a,b)}$ and $A' = \frac{A}{(A,B)}$.

2. Lemmas

The proof of Theorem 1 is based on Lemmas 1-2.

Lemma 1. *If there are a prime q and a positive integer M for which $(q, aA) = 1$, $AM + B \in \mathcal{S}$ and*

$$(7) \qquad \{1, q, q^2, \ldots\} \subseteq \mathcal{S},$$

then

$$\{\, n \mid (n,\ d\Delta N) = 1 \,\} \subseteq \mathcal{S},$$

where $d = (a, A)$ and $N = N_q$ is a positive integer defined by $q^{\varphi(aA)} = aAN_q + 1$ and $\varphi(\cdot)$ denotes the Euler-function.

Proof. Assume that the set $\mathcal{S} = \mathcal{S}(a, b; A, B)$ satisfies the conditions (1)-(3), furthermore there are a prime q and a positive integer M for which $(q, aA) = 1$, $AM + B \in \mathcal{S}$ and (7) holds. First, by using (3), we can assume that $\Delta = Ab - aB > 0$. Let

$$q^{\varphi(aA)} = aAN + 1.$$

Since

$$(aAN + 1)(an + b) = a\left[(aAN + 1)n + bAN\right] + b,$$

it follows from (1)-(3) and (7) that

$$An + B \in \mathcal{S} \quad \text{if and only if} \quad A\left[(aAN + 1)n + bAN\right] + B \in \mathcal{S},$$

which implies

$$(8) \qquad An + B \in \mathcal{S} \quad \text{if and only if} \quad (aAN + 1)(An + B) + A\Delta N \in \mathcal{S}.$$

It is clear from (7) that $(aAN + 1)^{k-1} \in \mathcal{S}$ holds for all positive integers k, and so by using the fact $AM + B \in \mathcal{S}$ and (8), we have

$$(9) \qquad (aAN + 1)^k (AM + B) + A\Delta N \in \mathcal{S}$$

for all positive integers k. Let $AM + B + A\Delta N = q^C D$, where C is a non-negative integer and $(D, aAN + 1) = (D, q) = 1$. It follows from (7) that

$$(10) \qquad q^C \in \mathcal{S}.$$

On the other hand, since $(D, aAN+1)=1$ it follows from the Euler-Fermat theorem and (9) that

$$D \parallel \left((aAN+1)^{\varphi(D^2)}-1\right)(AM+B)+(AM+B+A\Delta N)$$

and

$$\left((aAN+1)^{\varphi(D^2)}-1\right)(AM+B)+(AM+B+A\Delta N)=$$

$$=(aAN+1)^{\varphi(D^2)}(AM+B)+A\Delta N \in \mathcal{S}.$$

These relations, together with (2), imply

$$D \in \mathcal{S}. \tag{11}$$

Thus, by (1), (10) and (11), we have

$$AM+B+A\Delta N \in \mathcal{S},$$

from which

$$A\Delta N m+(AM+B) \in \mathcal{S} \tag{12}$$

is satisfied for all positive integers m. Let p be a prime number which is prime to $A\Delta N$, and let α be a positive integer. Then there is a positive integer m for which the congruence

$$A\Delta N m+(AM+B) \equiv p^\alpha \pmod{p^{\alpha+1}} \tag{13}$$

holds. Thus, it follows from (1)-(3), (12) and (13) that $p^\alpha \in \mathcal{S}$, i.e

$$\{n \mid (n,\ A\Delta N)=1\} \subseteq \mathcal{S}.$$

To complete the proof of Lemma 1 it is enough to show that

$$\{n \mid (n,\ a\Delta N)=1\} \subseteq \mathcal{S}. \tag{14}$$

Let p be a prime number which is prime to $a\Delta N$, and let α be a positive integer. By (3) and the fact $AM+B \in \mathcal{S}$, we also have $aM+b \in \mathcal{S}$. Let $e=e(p,\alpha)$ be a positive integer for which

$$(aAN+1)^e(aM+b) := aM'+b \ > \ a\Delta N p^{\alpha+1}. \tag{15}$$

It is clear that $aM' + b \in \mathcal{S}$. As we proved in the proof of (9), these relations imply that

$$(aAN+1)^k(aM'+b) - a\Delta N \in \mathcal{S}$$

holds for all positive integers k. The last relation, as the proof of (12), implies that

$$aM' + b - a\Delta N m \in \mathcal{S} \tag{16}$$

holds for all positive integers m for which $aM' + b - a\Delta N m > 0$.

On the other hand, we can choose a positive integer m_0 for which

$$aM' + b - a\Delta N m_0 \equiv p^\alpha \pmod{p^{\alpha+1}}, \tag{17}$$

$$0 < m_0 \le p^{\alpha+1}$$

hold. The last relation with (15) and (16) shows that $aM' + b - a\Delta N m_0 > 0$ and

$$aM' + b - a\Delta N m_0 \in \mathcal{S}.$$

Finally, by using (2) and (17), we have $p^\alpha \in \mathcal{S}$. Thus (14) is proved.

The proof of Lemma 1 is finished.

Lemma 2. *Assume that all conditions of Theorem 1 are satisfied. Then there is a prime q such that $(q, aA) = 1$ and*

$$\{1, q, q^2, \ldots\} \subseteq \mathcal{S}. \tag{18}$$

Proof. By (4), we have $(\pi, aA) = 1$, and so

$$\pi^{w\varphi(aA)} = aAN_\pi + 1,$$

where w is the positive integer defined in (5).

As in the proof of Lemma 1, one can deduce that

$$\text{if} \quad An + B \in \mathcal{S}, \quad \text{then} \quad (aAN_\pi + 1)(An + B) + A\Delta N_\pi \in \mathcal{S},$$

which, using the following assertions

$$(aAN_\pi + 1)^{k-1}(An + B) \in \mathcal{S}, \quad \text{if} \quad An + B \in \mathcal{S},\ k \in \mathbb{N},$$

and

$$(aAN_\pi + 1)^{k-1}(An + B) \equiv B \pmod{A},$$

implies that

(19) $\quad$ if $An+B\in\mathcal{S}$, then $(aAN_\pi+1)^k(An+B)+A\Delta N_\pi\in\mathcal{S}$

holds for all positive integers k.

By (5), (19) and using the argument used in the proof of (12), we also have

(20) $\quad$ if $An+B\in\mathcal{S}$, then $\pi^{w\varphi(aA)t}\,(An+B+A\Delta N_\pi)\in\mathcal{S}$

for all $t\in\mathbb{N}$.

It is well-known from [15] that

$$P(\pi^{w\varphi(aA)t}-1)\to\infty \quad \text{as} \quad t\to\infty,$$

where $P(y)$ denotes the greatest prime divisor of y. This shows that there are a positive integer T and a prime q such that

(21) $$q \mid \pi^{w\varphi(aA)T}-1 \quad \text{and} \quad (q,\ aA\Delta N_\pi)=1.$$

Now we deduce that (18) holds for such a q.

Let q be a prime satisfying (21) and let $\Pi:=\pi^{w\varphi(aA)T}$. Let us choose $t=T$ in (20), using (4)-(6) we have

$$\text{if } AM+B\in\mathcal{S}, \text{ then } \Pi\,(AM+B+A\Delta N_\pi)\in\mathcal{S},$$

consequently

(22) $$\Pi\left[\Pi^m(AM+B)+A\Delta N_\pi\frac{\Pi^m-1}{\Pi-1}\right]\in\mathcal{S}$$

holds for all positive integers m. Then it is easily seen that there is a positive integer $m(q)$ such that

$$q\parallel G_{m(q)}:=\Pi\left[\Pi^{m(q)}(AM+B)+A\Delta N_\pi\frac{\Pi^{m(q)}-1}{\Pi-1}\right],$$

furthermore

$$q\parallel R_q:=\frac{\Pi^q-1}{\Pi-1}.$$

It is proved in [10, Theorem 4.1] that the last conditions imply that for each positive integer α there exists a positive $m(q^\alpha)$ for which

$$q^\alpha \parallel G_{m(q^\alpha)} := \Pi\left[\Pi^{m(q^\alpha)}(AM+B) + A\Delta N_\pi \frac{\Pi^{m(q^\alpha)}-1}{\Pi-1}\right].$$

This together with (22) completes the proof of (18), and so Lemma 2 is proved.

3. Proof of Theorem 1

Assume that the conditions of Theorem 1 hold. By Lemma 2, we can assume that $w = w(\pi) = 0$ in the condition (5), i.e. all conditions of Lemma 1 are satisfied with $q = \pi$. Let

$$\pi^{\varphi(aA)} = aAN_\pi + 1.$$

Thus, by Lemma 1, we have

$$\{\, n \mid (n,\ d\Delta N_\pi) = 1 \,\} \subseteq \mathcal{S}, \tag{23}$$

where $d = (a, A)$.

Let $N_\pi = N_\pi' N_\pi''$ and $\Delta = \Delta'\Delta''$, where $(N_\pi',\ N_\pi'') = (\Delta',\ \Delta'') = (aA,\ \Delta' N_\pi') = 1$ and all prime divisors of $\Delta'' N_\pi''$ are divisors of aA. Since $(aA,\ \Delta' N_\pi') = 1$, there are $N_1 \in \mathbb{N}$ and $N_2 = aAt + 1$ such that

$$aAN_1 + 1 \equiv -1 \pmod{\Delta' N_\pi'} \quad \text{and} \quad aA(aAt+1) + 1 \equiv -1 \pmod{\Delta' N_\pi'},$$

furthermore the numbers $aAN_i + 1$ $(i = 1, 2)$ are primes. It is clear from (23) that for the numbers $aAN_i + 1$ $(i = 1, 2)$ all conditions of Lemma 1 are satisfied, furthermore

$$(N_\pi,\ N_1,\ N_2) = (N_\pi' \cdot N_\pi'',\ N_1,\ N_2) \mid 2. \tag{24}$$

One can deduce from Lemma 1 that

$$\{n \mid (n,\ d\Delta N_\pi) = 1\} \cup \{n \mid (n,\ d\Delta N_1) = 1\} \cup \{n \mid (n,\ d\Delta N_2) = 1\} \subseteq \mathcal{S},$$

which with (24) implies

$$\{n \mid (n,\ 2d\Delta) = 1\} \subseteq \mathcal{S}. \tag{25}$$

Thus, the proof of Theorem 1 is completed in the case when $2|\Delta$.

Assume now that $(2, \Delta) = 1$. If aA is an even number, then

$$\left(2aA,\ 2\Delta^{'}\right) = 2 \mid (aA + 2).$$

So, we can choose a positive integer t such that

$$aA(2t+1) + 1 \equiv -1 \pmod{2\Delta^{'}} \quad \text{and} \quad aA(2t+1)+1 \ \text{is prime.}$$

Let $N_3 = 2t + 1$. We infer from Lemma 1 and (25) that

$$\{n \mid (n,\ 2d\Delta) = 1\} \cup \{n \mid (n,\ d\Delta N_3) = 1\} \subseteq \mathcal{S},$$

which gives

$$\{n \mid (n,\ d\Delta) = 1\} \subseteq \mathcal{S}.$$

Now let $2 \nmid aA\Delta$. Then we can assume that $a \equiv A \equiv B \equiv 1 \pmod 2$, $b \equiv \equiv 0 \pmod 2$. Thus, for each non-negative integer α, we can find a positive integer n_0 such that

$$an_0 + b \equiv 2^{\alpha} \pmod{2^{\alpha+1}}. \tag{26}$$

It is clear that $2|n_0$. Since $d\Delta$ is odd and n_0 is even, an application of the Chinese Remainder Theorem shows that in this case there exists a positive integer n_1 for which

$$\left(a2^{\alpha+1}n_1 + an_0 + b,\ d\Delta\right) = \left(A2^{\alpha+1}n_1 + An_0 + B,\ 2d\Delta\right) = 1. \tag{27}$$

It follows from (25) and (27) that

$$A\left[2^{\alpha+1}n_1 + n_0\right] + B \in \mathcal{S},$$

which with (3) and (26) shows that $2^{\alpha} \in \mathcal{S}$. Thus

$$\{1, 2, 2^2, \ldots\} \subseteq \mathcal{S},$$

and the proof of Theorem 1 is complete.

4. Proof of Theorem 2

Assume that $F \in \mathcal{M}^*$ and $G \in \mathcal{M}^*$ satisfy the equation

$$G(an+b) = F(An+B) \quad \text{for all} \quad n \in \mathbb{N} \tag{28}$$

and the set of values of $F(An+B)$ and of $G(an+b)$ is contained in $\{0,\ 1\}$, where $a > 0$, b, $A > 0$, B are integers with $\Delta := Ab - aB \neq 0$. Assume that (i) is not true, i.e. there is a positive integer M such that

$$G(aM+b) = F(AM+B) = 1. \tag{29}$$

It is obvious that in this case we may assume that $(a,b) = (A,B) = 1$. Let p be a prime, $p|aM+b$. Then $(p,a)=1$, and so for each $t \in \mathbb{N}$ we have

$$P_t := p^{\varphi(a)t} = aT_t + 1, \quad G(P_t) = 1. \tag{30}$$

Hence, by (28), we infer that

$$F(An+B) = G(an+b) = G(P_t)G(an+b) = G[P_t(an+b)] =$$

$$= G\left[a\left(P_t n + bT_t\right) + b\right] = F\left[A\left(P_t n + bT_t\right) + B\right] \tag{31}$$

is satisfied for all $n,\ t \in \mathbb{N}$. Let

$$\mathcal{S}_F := \{n \in \mathbb{N} \ \mid\ F(n) = 1\} \quad \text{and} \quad \mathcal{S}_G := \{n \in \mathbb{N} \ \mid\ G(n) = 1\}.$$

It follows from (29) and (31) that

$$A\left(P_t M + bT_t\right) + B \in \mathcal{S}_F \quad \text{for all} \quad t \in \mathbb{N},$$

which, using Theorem 1, implies

$$\mathbb{N}_t := \{\ n \in \mathbb{N} \ \mid\ (n,\ A\Delta T_t) = 1\ \} \ \subseteq\ \mathcal{S}_F \quad \text{for all} \quad t \in \mathbb{N}. \tag{32}$$

An application of the Chinese Remainder Theorem shows that there exists a positive integer m_0 for which

$$(am_0 + 1,\ A\Delta T_1) = 1 \quad \text{and} \quad (m_0,\ T_1) = 2.$$

Hence, by repeating the argument we used in the proof of (32), we get

$$\{\, n \in I\!N \;\mid\; (n,\; A\Delta m_0) = 1 \,\} \;\subseteq\; \mathcal{S}_F,$$

which together with (32) implies

$$\{\, n \in I\!N \;\mid\; (n,\; 2A\Delta) = 1 \,\} \;\subseteq\; \mathcal{S}_F. \tag{33}$$

The deduction of the following assertion

$$\{\, n \in I\!N \;\mid\; (n,\; 2a\Delta) = 1 \,\} \;\subseteq\; \mathcal{S}_G \tag{34}$$

is very similar to the above argument. We omit this part of the proof.

If $2 \mid aA\Delta$, then (ii) is proved. Let $2 \not| aA\Delta$, and so $2 \not| B - b$. Assume that $2 \mid B$ and $2 \not| b$. Since $G(aM+b) = F(AM+B) = 1$ and $2 \mid (aM+b)(AM+B)$, therefore either $F(2) = 1$ or $G(2) = 1$. It can be easily shown from (33)-(34) that (ii) is true in both cases. Thus, this completes the proof of (ii). Theorem 2 is proved.

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Bui Minh Phong
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter sét. 1/D.
H-1117 Budapest, Hungary
bui@compalg.inf.elte.hu