Annales Univ. Sci. Budapest., Sect. Comp. 18 (1999) 25-33

ON ADDITIVE FUNCTIONS WITH RESPECT TO THE EXPANSION OF REAL NUMBERS INTO GENERALIZED NUMBER SYSTEMS

N.L. Bassily and S. Ishak (Cairo, Egypt)
I. Kátai (Budapest/Pécs, Hungary)

To the memory of Imre Környei

1. Introduction

Let $N \neq 0, \pm 1$ be an integer, $\mathcal{A} = \{a_0 = 0, a_1, \dots, a_{t-1}\}$, $t = |N|$ be a complete residue system mod N. Let $H(\subseteq \mathbf{R})$ be the set of those x which can be written as $x = \sum_{n=-\infty}^{-1} \varepsilon_n N^n$, where ε_n are taken from the digit set $\mathcal{A}$, $(\varepsilon_n \in \mathcal{A},\ n = 1, 2, \dots)$. It is clear that

$$H = \bigcup_{a \in \mathcal{A}} \left(\frac{a}{N} + \frac{1}{N} H \right),$$

i.e. H is the attractor of the iterated function system $f_0, f_1, \dots, f_{t-1}$, where

$$f_j(y) = \frac{a_j}{N} + \frac{1}{N} y.$$

Let $M = \bigcup_{l=0}^{\infty} (N^l H)$, i.e. the set of those $x \in \mathbf{R}$ which can be written in the form

$$x = \sum_{n=-\infty}^{k} \varepsilon_n N^n, \qquad \varepsilon_n \in \mathcal{A}. \tag{1.1}$$

The research of first author was partially supported by the Associate Program of ICTP, the research of second author by the Mathematics Section of ICTP, the research of third author the Hungarian National Foundation for Scientific Research under grant OTKA 2153.

A function F defined on M is called additive (with respect to $\mathcal{A}$ and N), if $F(0)=0$, and for each $x \in M$

$$(1.2)\qquad F(x)=\sum_{n=-\infty}^{k} F(\varepsilon_n N^n), \qquad \sum_{n=-\infty}^{k} |F(\varepsilon_n N^n)|<\infty,$$

where ε_n are taken from (1.1).

Let $\mathcal{L}$ be the linear space of additive functions. A system $(\mathcal{A}, N)$ is called a number system if each integer n can be written in finite form as $n=c_0+ +c_1N+\ldots+c_kN^k$, $c_j \in \mathcal{A}$.

The following assertions are proved in Kátai [1].

(1) H is a compact set.

(2) $(\mathcal{A}, N)$ is a number-system if and only if $M=\mathbf{R}$.

(3) For each $y \in \mathbf{R}$ there is an $n \in \mathbf{Z}$ and $x \in H$ such that $y=n+x$.

(4) Let $\Gamma_l=\{\gamma \mid \gamma=\varepsilon_0+\varepsilon_1 N+\ldots+\varepsilon_l N^l,\ \varepsilon_l \in \mathcal{A}\}$. Then $(\mathcal{A}=)\Gamma_0 \subset \Gamma_1 \subset \subset \Gamma_2 \ldots$. Let $\Gamma=\bigcup \Gamma_l$. It is clear that $M=\bigcup_{\gamma \in \Gamma}(\gamma+H)$. Let λ be the Lebesque measure. Then

$$\lambda(H+\gamma_1 \cap H+\gamma_2)=0$$

holds for each $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$. If $(\mathcal{A}, N)$ is a number system, then $\Gamma=\mathbf{Z}$ and

$$\lambda(H+n_1 \cap H+n_2)=0$$

for each $n_1, n_2 \in \mathbf{Z}$, $n_1 \neq n_2$.

(5) $(\mathcal{A}, N)$ is called a just touching covering system (JTCS), if

$$\lambda(H+n_1 \cap H+n_2)=0$$

holds for each $n_1, n_2 \in \mathbf{Z}$, $n_1 \neq n_2$.

It was proved by K.-H.Indlekofer, I.Kátai and P.Racskó [2], [3], that $(\mathcal{A}, N)$ is a JTCS if and only if $\Gamma-\Gamma=\mathbf{Z}$, i.e. if each integer n can be written as

$$n=c_0+c_1N+\ldots+c_kN^k,$$

$$c_j \in \mathcal{B}=\mathcal{A}-\mathcal{A}.$$

(6) Let $S(m)$ be the set of those integers $n(\neq m)$ for which $H+m \cap H+n \neq 0$. It is obvious that $S(m)=m+S(0)$. $S(0)$ is nonempty, since in the opposite

case $\{H+n \mid n \in \mathbf{Z}\}$ would be the union of mutually disjoint compact sets, which contradicts to (3). Let $S := S(0)$.

(7) Let $\gamma \in S$, $B_\gamma = H \cap H + \gamma$, $B = \bigcup_{\gamma \in S} B_\gamma$. If $z_1 \in B_\gamma$, then $z_1 - \gamma =: \; =: z_2 \in H$, consequently $z_1 = \sum_{v=1}^{\infty} \varepsilon_v N^{-v}$, $z_2 = \sum_{v=1}^{\infty} \varepsilon'_v N^{-v}$ with suitable digits $\varepsilon_v, \varepsilon'_v \in \mathcal{A}$. Thus for $\delta_v = \varepsilon_v - \varepsilon'_v (\in \mathcal{B})$ we have

$$\gamma = \delta_1 \cdot N^{-1} + \delta_2 N^{-2} + \dots \tag{1.3}$$

On the other hand, if γ has an expansion of form (1.3), and $\delta_v = \varepsilon_v - \; -\varepsilon'_v$, $\varepsilon_v, \varepsilon'_v \in \mathcal{A}$, then $z_1 := \sum_{v=1}^{\infty} \varepsilon_v N^{-v} \in B_\gamma$. Consequently the elements of B_γ can be determined by giving all the expansion of γ in the form (1.3) and solving the equations $\delta_v = \varepsilon_v - \varepsilon'_v$, $\varepsilon_v, \varepsilon'_v \in \mathcal{A}$.

Let $\gamma \in S$ and λ be such an integer for which $\lambda N = \gamma + \delta$ holds with some $\delta \in \mathcal{B}$. Then either $\lambda = 0$ (it occurs only if $-\gamma = b \in \mathcal{B}$) or $\lambda \in S$. Indeed, if γ has the expansion (1.3), then

$$\lambda = \frac{\delta}{N} + \frac{\delta_1}{N^2} + \frac{\delta_2}{N^3} + \dots .$$

Let the directed graph $G(S)$ be defined as follows: for each $\gamma \in S$ and for each $\lambda \neq 0$ such that $\lambda N = \gamma + \delta$, $\delta \in \mathcal{B}$ let us direct an edge from λ to γ, and let us label this edge by δ.

(8) If $(\mathcal{A}, N)$ is not a JTCS, then $B = H$.

2. Characterization of additive functions. General case

We guess that for each $(\mathcal{A}, N)$ the additive functions are linear ones, i.e.

$$\mathcal{L} = \{F(x) = cx \mid c \in \mathbf{R}\}.$$

Let $F \in \mathcal{L}$. We observe that for each $l \in \mathbf{Z}$ the function

$$F_l(x) := F(xN^l)$$

belongs to $\mathcal{L}$.

Lemma 1. *Let $\gamma \in S$. Then for $\delta_1, \delta_2 \in \Gamma$ such that $\delta_1 - \delta_2 = \gamma$ the difference*

$$F(\delta_1) - F(\delta_2)$$

does depend only on γ. If $(\mathcal{A}, N)$ is a number system then

$$F(\gamma + h) - F(h) = F(\gamma) \tag{2.1}$$

holds for each $h \in \mathbf{Z}$ and $\gamma \in S$.

Proof. Let $\delta_1 - \delta_2 = \gamma$, $\delta_1^* - \delta_2^* = \gamma$, $\delta_1, \delta_1^*, \delta_2, \delta_2^* \in \Gamma$. Let $x_1, x_2 \in H$ be such numbers for which $x_1 + \delta_1 = x_2 + \delta_2$. Then $F(x_i + \delta_i) = F(x_i) + F(\delta_i)$, whence $F(\delta_1) + F(x_1) = F(\delta_2) + F(x_2)$, i.e. $F(x_2) - F(x_1) = F(\delta_1) - F(\delta_2)$. Since $x_1 + \delta_1^* = x_2 + \delta_2^*$ holds, therefore

$$F(x_2) - F(x_1) = F(\delta_1) - F(\delta_2),$$

consequently the first assertion is true.

To prove the second assertion, we should observe only that $\Gamma = \mathbf{Z}$, if $(\mathcal{A}, N)$ is a number system. The proof is complete.

Let $S^* = S \cup \{0\}$. Let us extend the graph $G(S)$ to $G(S^*)$ by drawing the edges $0 \rightarrow^{(0)} 0$, and for each $b \in S \cap \mathcal{B}$, $0 \rightarrow^{(-b)} b$. For $\gamma \in S^*$ let

$$\Delta_l(\gamma) := F_l(d_1) - F_l(d_2),$$

where $d_1, d_2 \in \Gamma$ such that $d_1 - d_2 = \gamma$. From Lemma 1 we know that the right hand side does not depend on the special choice of d_1, d_2. Let $\mathcal{T}$ denote the set of labels occurring in the set of labels of $G(S^*)$.

Lemma 2. *Let $\gamma, \eta \in S^*$ such that $\gamma \rightarrow^{(b)} \eta$. Then for each $a_u, a_v \in \mathcal{A}$ such that $a_u - a_v = b$ the difference $F_l(a_u) - F_l(a_v)$ depends only on l and b. Let*

$$F_l^*(b) := F_l(a_u) - F_l(a_v).$$

Furthermore we have

$$\Delta_l(\gamma) = F_{l-1}^*(b) + \Delta_{l-1}(\eta).$$

Proof. Let $\gamma \in S^*$ and $\delta_1, \delta_2, \dots$ be an arbitrary sequence of labels getting by walking on $G(S^*)$, starting from γ. Let $\delta_i = e_i - f_i, e_i, f_i \in \mathcal{A}, d_1, d_2 \in \Gamma$ such that $d_1 - d_2 = \gamma$. Let $x = \sum_{i=1}^{\infty} \frac{e_i}{N^i}, y = \sum_{i=1}^{\infty} \frac{f_i}{N^i}$. Then $d_1 - d_2 = x - y$, i.e. $d_1 + y = d_2 + x$, $x, y \in H$, consequently $F_l(d_1) + F_l(y) = F_l(d_2) + F_l(x)$, i.e. $\Delta_l(\gamma) = F_l(x) - F_l(y) = F_{l-1}(Nx) - F_{l-1}(Ny) = F_{l-1}(e_1 + x_1) - F_{l-1}(f_1 + y_1)$, where $x_1 = \sum_{i=1}^{\infty} \frac{e_i}{N^{i-1}}$, $y_1 = \sum_{i-2}^{\infty} \frac{f_i}{N^{i-1}}$.

The right hand side of the last equation can be rewritten as $F_{l-1}(e_1) - F_{l-1}(f_1) + F_{l-1}(x_1) - F_{l-1}(y_1)$. We observe that $F_{l-1}(x_1) - F_{l-1}(y_1)$ may depend only on $\eta = x_1 - y_1$, and that it is $\Delta_{l-1}(\eta)$. Consequently $F_{l-1}(e_1) - F_{l-1}(f_1)$ depends only on b, and so

$$\Delta_l(\gamma) = F^*_{l-1}(b) + \Delta_{l-1}(\eta).$$

This completes the proof of the lemma.

It is obvious that $\Delta_l(-\gamma) = -\Delta_l(\gamma)$.

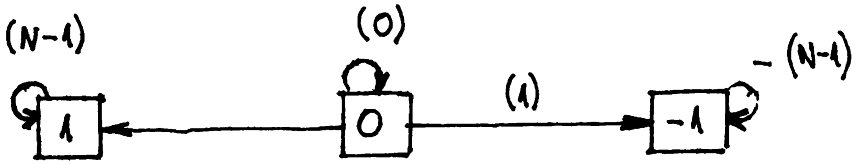
Lemma 3. *$\mathcal{T}$ is the set of those $b \in \mathcal{B}$ for which there is an $\eta \in S^*$ such that $-b \equiv \eta \ (mod\ N)$.*

Proof. Clear. If $b \in \mathcal{T}$, then there is $\gamma, \eta \in S^*$ such that $N_\gamma = b + \eta$, consequently $-b \equiv \eta \ (mod\ N)$. On the other hand if $\eta \in S$, then for $-b \equiv \eta \ (mod\ N)$, $b \in \mathcal{B}$ we have that

$$\gamma = \frac{b}{N} + \frac{\eta}{N} \in S^*,$$

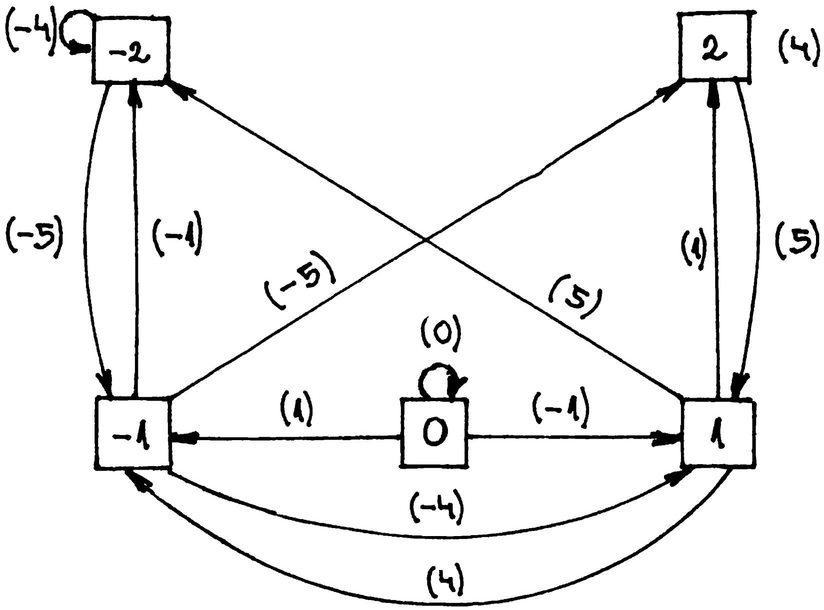
i.e. $b \in \mathcal{T}$.

Example 1. Let $\mathcal{A} = \{0, 1, \ldots, N-1\}$. Then $\mathcal{B} = \{-(N-1), \ldots, N-1\}$ and $G(S^*)$ is the following:



Hence we have $\Delta_l(0) = F^*_{l-1}(-1) + \Delta_{l-1}(1)$, $\Delta_l(1) = F^*_{l-1}(N-1) + \Delta_{l-1}(1)$. From the first equation we obtain that $\Delta_l(0) = 0$, and that $F_{l-1}(k) - F_{l-1}(k+1) + \Delta_{l-1}(1) = 0$ for $k = 0, 1, \ldots, N-2$, i.e. $F_{l-1}(k) = k\Delta_{l-1}(1)$, whence $F^*_{l-1}(N-1) = F_{l-1}(N-1) = (N-1)\Delta_{l-1}(1)$, and so $\Delta_l(1) = N\Delta_{l-1}(1)$, $(l \in \mathbf{Z})$, consequently $\Delta_l(1) = N^l\Delta_0(1)$. This immediately implies that $F(kN^l) = F_l(k) = kN^l\Delta_0(1)$, i.e. $F(x) = cx$ for each $x \in M$, where $c = \Delta_0(1)$.

Example 2. Let $N = 3$, $\mathcal{A} = \{0, 1, 5\}$. Then $\mathcal{B} = \{0, \pm 1, \pm 4, \pm 5\}$, $S^* = \{0, \pm 1, \pm 2\}$, and $G(S^*)$ is the following:



Consequently we have $0 = \Delta_l(0) = F^*_{l-1}(1) - \Delta_{l-1}(1) = F_{l-1}(1) - \Delta_{l-1}(1)$, $\Delta_l(-1) = F^*_{l-1}(-1) + \Delta_{l-1}(-2)$, i.e. $\Delta_l(1) = F_{l-1}(1) + \Delta_{l-1}(2)$, $\Delta_l(1) = F^*_{l-1}(5) + \Delta_{l-1}(-2)$, $\Delta_l(2) = F^*_{l-1}(4) + \Delta_{l-1}(2)$, $\Delta_l(1) = F^*_{l-1}(4) + \Delta_{l-1}(-1)$, $\Delta_l(2) = F^*_{l-1}(5) + \Delta_{l-1}(1)$.

Hence we obtain that $F_{l-1}(5) - F_{l-1}(1) + \Delta_{l-1}(2) = F_{l-1}(5) + \Delta_{l-1}(1)$, whence $\Delta_l(2) = 2\Delta_l(1) = 2F_l(1)$ follows. After substituting we can express all the numbers $F^*_l(b)$ in the terms of $F^*_l(1)$: $F_l(1) = F_{l-1}(1) + 2F_{l-1}(1) = 3F_{l-1}(1)$, $\Delta_l(1) = 3\Delta_{l-1}(1)$, $F_{l-1}(5) = \Delta_l(1) + \Delta_{l-1}(2) = 5F_{l-1}(1)$, whence we get that $F(a \cdot 3^l) = a \cdot 3^l F(1)$ for $a \in \mathcal{A}$ and $l \in \mathbf{Z}$. This implies immediately that $F(x) = cx$ holds for all $x \in M$.

3. Additive functions for number systems

Theorem 1. *Assume that $(\mathcal{A}, N)$ is a number system. Then $F \in \mathcal{L}$ implies that $F(x) = cx$ for $x \in \mathbf{R}$.*

The proof is based upon the following lemmas. Let $S = \{\gamma_1, \gamma_2, \ldots, \gamma_r\}$, $D = gcd(\gamma_1, \gamma_2, \ldots, \gamma_r)$, where gcd is the shorthand of the expression greatest common divisor.

Lemma 4. *We have*

$$F_l(Dk) = ckDN^l, \qquad F_l(Dk + u) = F_l(Dk) + F_l(u)$$

for each $k, l, u \in \mathbf{Z}$, where c is a suitable constant.

Proof. Let $\gamma_i \in S$. Since $(\mathcal{A}, N)$ is a number system, therefore $\gamma_i + h, h \in \Gamma$ holds for each $h \in \mathbf{Z}$, consequently by Lemma 1 we obtain that $F_l(\gamma_i + h) = F_l(\gamma_i) + F_l(h)$. Then for each $k \in \mathbf{N}$, $F_l((k+1)\gamma_i + h) = F_l(k\gamma_i + h) + F_l(\gamma_i)$, whence one can prove by induction that $F_l(k\gamma_i + h) = kF_l(\gamma_i) + F_l(h)$. It is clear that $S = -S$, i.e. $\gamma_i \in S$ implies that $-\gamma_i \in S$. Thus $F_l((-k)\gamma_i + h) = F_l(k(-\gamma_i)) + F(h) = kF_l(-\gamma_i) + F(h)$, and by $0 = F_l(\gamma_i + (-\gamma_i)) = F_l(\gamma_i) + F_l(-\gamma_i)$ we obtain that $F_l(k\gamma_i + h) = kF_l(\gamma_i) + F_l(h)$ holds for all $k \in \mathbf{Z}$.

Hence we obtain that for each $u, k_1, \ldots, k_r \in \mathbf{Z}$

$$F_l(k_1\gamma_1 + \ldots + k_r\gamma_r + u) = F(u) + k_1F_l(\gamma_1) + \ldots + k_rF_l(\gamma_r).$$

Since $D = t_1\gamma_1 + \ldots + t_r\gamma_r$ with suitable integers $t_1, t_2, \ldots, t_r$, therefore

$$F_l(Dk + u) = F_l(u) + kF_l(D)$$

holds for each $k \in \mathbf{Z}$ and $u \in \mathbf{Z}$. Applying this relation with $u = 0$, we obtain that

$$\frac{F_l(kD)}{kD} = \frac{F_l(D)}{D} = \frac{F_{l-1}(ND)}{D} = N\frac{F_{l-1}(D)}{D} = \frac{NF_{l-1}(kD)}{kD}.$$

The proof of the lemma is completed.

Proof of the theorem. If $D = 1$, then Lemma 1 implies the fulfilment of the theorem.

Assume that $D > 1$. Let $\hat{F}(x) := F(x) - cx$, where c is the constant occuring in Lemma 4. Then $\hat{F}_l(x) := \hat{F}(xN^l)$ satisfies the following relations:

$$\hat{F}_l(Dk + u) = \hat{F}_l(u) \quad \text{for} \quad u, k \in \mathbf{Z}.$$

Let D^* be the smallest positive integer for which $\hat{F}_l(D^*k+u) = \hat{F}_l(u)$ for each $u, k \in \mathbf{Z}$ holds. Then $(D^*, N) = 1$. Let us assume indirectly that $(D^*, N) = \Delta$, and that $\Delta > 1$. Let $u_0 \in \mathbf{Z}$, $u_t = u_0 + t\frac{D^*}{\Delta}$. Then

$$\hat{F}_{l+1}(u_t) = \hat{F}_l\left(Nu_0 + t\frac{N}{\Delta}D^*\right) = \hat{F}_l(Nu_0) = \hat{F}_{l+1}(u_0),$$

consequently

$$\hat{F}_l\left(\frac{D^*}{\Delta}k + u\right) = \hat{F}_l(u)$$

which contradicts to the minimality of D^*.

So we have $1 = (D^*, N)$. Then $1 = kD^* + tN$ with suitable $k, t \in \mathbf{Z}$. Furthermore $(t, D^*) = 1$. Hence $m = kmD^* + tNm$, $\hat{F}_l(m) = \hat{F}_l(kmD^*) + {} + \hat{F}_l(tNm) = \hat{F}_{l+1}(tm)$. Applying this relation $\varphi(D^*)$ - times, where φ is the Euler-function, we have

$$\hat{F}_l(m) = \hat{F}_{l+\varphi(D^*)}(t^{\varphi(D^*)}m).$$

Since $t^{\varphi(D^*)} \equiv 1 \ (mod\ D^*)$, therefore the right hand side of the last equation is

$$\hat{F}_{l+\varphi(D^*)}(m).$$

We deduced that

$$\hat{F}_l(m) = \hat{F}_{l-s\varphi(D^*)}(m)$$

holds for each $m \in \mathbf{Z}$ and $s = 1, 2, \ldots$. Consequently it holds for each $a \in \mathcal{A}$. Taking the limit for $s \to \infty$, we obtain that $\hat{F}_l(a) = 0$ for every $l \in \mathbf{Z}$ and $a \in \mathcal{A}$. Consequently $\hat{F}(x) = 0$ identically. The proof is completed.

References

[1] **Kátai I.**, Generalized number systems and fractal geometry, *Leaflets in Mathematics*, JPTE, Pécs, 1995.

[2] **Indlekofer K.-H., Kátai I. and Racskó P.**, Some remarks on generalized number systems, *Acta Sci. Math.*, **37** (1993), 543-553.

[3] **Indlekofer K.-H., Kátai I. and Racskó P.**, Number systems and fractal geometry, *Probability theory and its applications: Essays to the memory of József Mogyoródi* (eds. J.Galambos and I.Kátai), Kluwer, 1992, 319-334.

N.L. Bassily
Department of Mathematics
Faculty of Science
Ain Shams University
Cairo, Egypt

S. Ishak
Department of Mathematics
Faculty of Engineering
Ain Shams University
Cairo, Egypt

I. Kátai
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter sétány 1/D.
H-1117 Budapest, Hungary

Dept. of Applied Mathematics
Janus Pannonius University
Ifjúság u. 6.
H-7624 Pécs, Hungary