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SPLINE APPROXIMATIONS TO SOLUTIONS OF INITIAL VALUE PROBLEMS

A. Sövegjártó (Budapest, Hungary)

Dedicated to Professor J.Balázs on his 75th birthday

Abstract. A family of spline methods is investigated for the numerical solutions of n-th order initial value problems for ordinary differential equations $y^{(n)}(x) = f(x, y', \dots, y^{(n-1)})$, $y^{(i)}(0) = y_0^{(i)}$, $i = 0, \dots, n-1$. This is a modification of the method that was proposed in [18], [19]. The spline S which approximates the exact solution y is of degree $n+k$, $(k = 1, 2, 3)$ and class C^{n-1}. The convergence of the method is proved. The stability is discussed for the first and second order linear test equations. Numerical examples are given. This method can be considered as a modification and extension of the methods given by Loscalzo and Talbot [13,14], Callender [4], Micula [15] and Fawzy and Soliman [7]. Furthermore, it can be considered as a modified Nordsieck method or a slight modification of Taylor expansion of order $n+k$. The proposed method has the advantage over the discrete method that it gives a global approximation of the solution and also permits the study of the behaviour of the derivatives of the approximate solution.

1. Introduction

A spline method was proposed in [18,19] for the solutions of the initial value problems of n-th order ordinary differential equations. The leading idea of the construction was to create approximate solution by splines of order $n+1$ which belongs to the class C^n on the whole integration interval and which approximates not only the exact solution of the initial value problems but its

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derivatives, too. Now the continuity conditions will be relaxed for the sake of better stability property. The constructed piecewise polynomial spline is of degree $n+k$ ($k=1,2,3$) and of class C^{n-1}.

2. Description of the method

2.1. The numerical process

Let us consider the following initial value problem of order n:

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \tag{2.1}$$

$$y^{(i)}(x_0) = y_0^{(i)}, \qquad i = 0, \dots, n-1, \tag{2.2}$$

where $y_0^i, i = 0, \dots, n-1$ are preassigned values. The smoothness condition for the function f depends on the degree of the spline: if the approximate spline is of degree $n+k$ then let $f \in C^{k+2}(D), k = 1,2,3$ and $D := \{(x, y, \dots, y^{(n-1)}) \mid x_0 \le x \le b\}$. Let assume that f is a Lipschitz continuous function, i.e.

$$|f(x, y_1, y_1', \dots, y_1^{(n-1)}) - f(x, y_2, y_2', \dots, y_2^{(n-1)})| \le L \sum_{j=0}^{n-1} |y_1^{(j)} - y_2^{(j)}|, \tag{2.3}$$

where L is the uniform Lipschitz constant.

It is known ([10], Theorem 4.1), that there exists a unique $y(x)$ solution of the problem (2.1), (2.2). We construct a polynomial spline function $S(x)$ of degree $n+k$ ($k=1,2,3$), approximating y and its derivatives. For this purpose let h be the stepsize, $h := (b - x_0)/N, N \in \mathbb{N}$, and we define in each subinterval $[x_i, x_{i+1}]$, $i = 0, \dots, N-1$ the components of S by

$$p_{i+1}(x) = a_i^{n+k}(x - x_i)^{n+k} + a_i^{n+k-1}(x - x_i)^{n+k-1} + \dots + a_i^1(x - x_i) + p_i, \tag{2.4}$$

where the coefficients a_i^j, $i = 0, \dots, N-1; j = 1, \dots, n+k$ are yet to be determined. For the sake of simplicity we consider equidistant mesh only, but the method is applicable with variable stepsizes, too.

From the initial values (2.2) and from (2.1) we get

(2.5) $$p_0 = y_0^{(0)}, \quad a_0^j = y_0^{(j)}/j!, \quad j = 1, ..., n-1,$$

(2.6) $$a_0^{n+j-1} = f^{(j-1)}(x_0, y_0^{(0)}, y_0^{(1)}, ..., y_0^{(n-1)})/(n+j-1)!, \quad j = 1, ..., k+1.$$

I. Case $n = 1$. For the first order equation let us define the recursion formulas by

(2.7) $$a_{i+1}^j = f^{(j-1)}(x_{i+1}, p_{i+2}(x_{i+1}))/j!, \quad j = 1, ..., k, \quad i = 0, ..., N-1,$$

furthermore

$$a_{i+1}^{k+1} = \frac{a_i^{k+1}}{4} + \frac{6}{4(k+1)!h^2} \times$$

(2.8) $$\times \int_{x_{i+1}}^{x_{i+2}} [f^{(k-1)}(x, p_{i+2}(x)) - k!a_{i+1}^k]\, dx, \quad k = 1, 2, 3,$$

$i = 0, ..., N-2$. The last equation is an implicit equation for a_{i+1}^{k+1}, the others are explicit ones. The formulas (2.5) – (2.8) define single-step spline method for (2.1), (2.2), when $n = 1, \quad k = 1, 2, 3$.

II. Case $n \geq 2$. The starting step of the algorithm is the same as before (formulas (2.5)-(2.6)). Furthermore, the coefficients $a_i^j, j < n$ can be determined from the continuity condition $S \in C^{n-1}$, which means

(2.9) $$p_i^{(j)}(x_i) = p_{i+1}^{(j)}(x_i), \quad i = 1, ..., N-1, \quad j = 0, 1, ..., n-1.$$

These equalities give recursion formulae for a_i^j as follows (let $m := n + k$)

(2.10) $$\begin{cases} a_{i+1}^1 = m a_i^m h^{m-1} + (m-1) a_i^{m-1} h^{m-2} + \ldots + a_i^1 \\ 2a_{i+1}^2 = m(m-1) a_i^m h^{m-2} + \ldots + 2a_i^2 \\ 3!a_{i+1}^3 = m(m-1)(m-2) a_i^m h^{m-3} + \ldots + 3!a_i^3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (n-1)!a_{i+1}^{n-1} = m(m-1)\ldots(m-n+2) a_i^m h^{m-n+1} + \ldots \\ \qquad \ldots + (n-1)!a_i^{n-1} \end{cases}$$

where $i = 0, \dots, N-1$. The other coefficients from a_{i+1}^{n} to a_{i+1}^{m-1} can be determined similarly as (2.7) by

$$(2.11)\qquad a_{i+1}^{n+j} = f^{(j)}(x_{i+1}, p_{i+2}(x_{i+1}), \dots, p_{i+2}^{(n-1)}(x_{i+1}))/(n+j)!,$$

$$j = 0, \dots, k-1, \qquad i = 0, \dots, N-1.$$

These are explicit formulae for the unknowns a_{i+1}^{n+j}.

Let us compute the last coefficients as follows

$$(2.12)\qquad a_{i+1}^{n+k} = \frac{a_i^{n+k}}{4} + \frac{6}{4(n+k)!h^2} \times$$

$$\times \int_{x_{i+1}}^{x_{i+3}} [f^{(k-1)}(x, p_{i+2}(x), p_{i+2}'(x), \dots, p_{i+2}^{(n-1)}(x)) - (n+k-1)!a_{i+1}^{n+k-1}]\, dx,$$

$$i = 0, \dots, N-2, \quad k = 1, 2, 3.$$

The algorithm now is complete.

It can be proved by the help of contractive mapping theorem, that the spline constructed above exists and is unique ([18,19], Theorem 1).

Remark 1. It is clear, that this algorithm works for $k > 3$, too, but the calculation of the higher derivatives of function f soon becomes very complicated. Dahlquist and Björk in [5] Sec. 12.3. write that the popularity of the Taylor series method has risen again. If $f(x, y)$ is composed from elementary functions, it is easy to write subroutines, which recursively compute the derivatives of f.

As we see later, formulas (2.8), (2.12) give second-order approximation for the corresponding Taylor coefficient, not only approximation of order one, as it is in the paper of Loscalzo and Talbot [12,13,14], Micula [15] and Sövegjártó [18,19]. Furthermore, this modification has a great effect for the stability property of the method (see §3.).

Remark 2. The integral term in (2.8) and (2.12) need only calculated numerically if $k = 1$. The Simpson-formula can be applied. For $k = 2, 3$ the formulas (2.8), (2.12) can be integrated. Furthermore, the implicit equations (2.8), (2.12) can be solved by simple iteration or by Newton method. This causes only small perturbation for the eigenvalues of the matrix of method (see §3), if we rewrite the above spline method into modified Nordsieck-method. This has a small effect only on the stability property of the method.

2.2 Approximate property and boundedness of the main coefficients

Lemma 1. *Let $f \in C^{k+2}(D)$ with bounded partial derivatives and let* $\max_{(x,y)\in D} |f^{(k+2)}| = K_k$, $k = 1, 2, 3$, *then*

$$(2.13) \qquad \left| a_{i+1}^{n+k} - \frac{f^{(k)}(x_{i+1}, y(x_{i+1}), \dots, y^{(n-1)}(x_{i+1}))}{(n+k)!} \right| = K_k h^2,$$

$i = 0, 1, \dots, N-1$ *and the constant K_k is independent of h and x_i.*

Proof. Let $k = 1$ for the simplicity and n be fixed. We prove by induction on i. Case $i = 0$ is trivial on the base of (2.6). Suppose, that the statement holds for a_i^{n+1}. From (2.12) considering (2.11) follows

$$a_{i+1}^{n+1} = -\frac{a_{i+1}^{n+1} - a_i^{n+1}}{3} +$$

$$+\frac{1}{(n+1)!h^2} \int_{x_{i+1}}^{x_{i+2}} [f(x, p_{i+2}(x), \dots) - f(x_{i+1}, p_{i+2}(x_{i+1}), \dots)]dx.$$

By the Taylor's formula and by the assumption we get

$$a_{i+1}^{n+1} \le -\frac{a_{i+1}^{n+1}}{3} + \frac{f'(x_i)}{3(n+1)!} + \frac{K_1}{3}h^2 + \frac{f'(x_{i+1})}{(n+1)!} + \frac{f''(x_{i+1})}{3(n+1)!}h + \frac{f'''(\xi_{i+1})}{12(n+1)!}h^2.$$

Adding to the r.h.s. $\pm f'(x_{i+1})/(3(n+1)!)$ and arranging

$$a_{i+1}^{n+1} - \frac{f'(x_{i+1})}{(n+1)!} \le -\frac{1}{3}\left[a_{i+1}^{n+1} - \frac{f'(x_{i+1})}{(n+1)!}\right] -$$

$$-\frac{f'(x_{i+1})}{3(n+1)!} + \frac{f'(x_i)}{3(n+1)!} + \frac{f''(x_{i+1})h}{3(n+1)!} + \frac{K_1}{3}h^2 + \frac{K_1}{12(n+1)!}h^2,$$

that proves the statement for $i+1$. In the cases $k = 2, 3$, the functions f' or f'' appear in the integrand, but similar arguments as above prove the statement.

To prove the boundedness of the main coefficients we need

Lemma 2. (special case of [5],Th.12.3.3.) *For the scalar difference equation*

$$\psi_{j+1} = \alpha\psi_j + \beta, \quad \psi_0 = |u_0|,$$

the following estimate holds

$$\psi = \begin{cases} \beta\dfrac{\alpha^n - 1}{\alpha - 1} + \alpha^n|u_0| & \text{if} \quad \alpha \neq 1, \\ |u_0| + \beta n & \text{if} \quad \alpha = 1. \end{cases}$$

If $\alpha < 1$, *then* $\psi_n \leq \max\left\{|u_0|, \ \dfrac{\beta}{1-\alpha}\right\}, \quad n \geq 0.$

Now we are in a position to prove

Lemma 3. *Let* $k = 1$ *and* $|f_x| \leq M$ *and* $|f_y| \leq L$. *(For* $k = 2, 3$, *instead of* f_x *and* f_y *one has to consider* f'_x, f'_y *or* f''_x, f''_y.*) If* $f_y h < 3$ *(or* $f'_y h < 3$ *and* $f''_y h < 3$*), then there exist universal constants* C_k, $k = 1, 2, 3$ *for which*

$$|a_{i+1}^{n+k}| \leq C_k, \quad i = 0, 1, ..., N-1.$$

Proof. Let $k = 1$, $n = 1$. Applying the Taylor's theorem for the function f and by integration the formula (2.12) gives

$$a_{i+1}^{n+1} = \frac{a_i^{n+1}}{4} + \frac{3}{4}\left[\frac{f_x(\xi,\eta)}{2} + \frac{f_y(\xi,\eta)a_{i+1}^{n+1}h}{3} + \frac{f_y(\xi,\eta)a_{i+1}^{n}}{2}\right],$$

and so

$$a_{i+1}^{n+1} = \frac{a_i^{n+1}}{4 - f_y(\xi,\eta)h} + \frac{3}{2[4 - f_y(\xi,\eta)h]}[f_x(\xi,\eta) + f_y(\xi,\eta)a_{i+1}^n]. \tag{2.14}$$

From (2.7) follows that $a_i^n (i = 1, 2, ...)$ are bounded if f is bounded function on the region of D. Denote A the bound of a_i^n-s, and let $B_1 := 1.5[M + LA]$, independent of i. If $f_y h < 3$ in D then $\alpha := 1/(4 - f_y h) < 1$. From (2.14) also follows immediately by Lemma 2, that

$$|a_N^{n+1}| \leq \max\left\{|a_0^{n+1}|, \frac{B_1}{1-\alpha}\right\} =: C_1.$$

If $k = 2, 3$, then instead of f we have to deal with functions f' or f''. In this cases M and L denote the bound of $|f'_x|$ and $|f'_y|$ or $|f''_x|$ and $|f''_y|$.

For the higher order equations the proofs are similar, therefore we omit.

Remark 3. From Lemma 1 and Lemma 3 can be easily derived that the other coefficients of the spline remain bounded, too. So it follows that the spline approximation is contained in a bounded domain of D, whenever the exact solution of the problem (2.1), (2.2) is bounded.

2.3. Local truncation error

In the method (2.4) the local truncation error d_{i+1} is the measure of the accuracy when: 1) the numerical solution is replaced by the exact solution which goes through the point $(x_i, y(x_i))$, and 2) we take the difference of both sides of (2.4).

Theorem. *Let $f \in C^{(k+2)}$ in (2.1). Then the local truncation error for the spline method given in Section 2.1 has the form*

$$d_{i+1} = Ch^{n+k+1} + O(h^{n+k+2}),$$

where the constant C is independent of x_i.

Proof. We proceed the proof when $n = k = 1$. For the other cases the proofs are similar. The local truncation error now is

$$d_{i+1} = y(x_{i+1}) - y(x_i) - a_i^1 h - a_i^2(y)h^2,$$

where

$$a_i^1 = f(x_i, y(x_i)), \quad a_i^2(y) = \frac{a_{i-1}^2}{4} + \frac{3}{4h^2} \int_{x_i}^{x_{i+1}} [f(x, y(x)) - f(x_i, y(x_i))]dx.$$

Applying the Taylor formula for $y(x_{i+1})$ we get

$$d_{i+1} = h^2 \left\{ \left[\frac{f'(x_i, y(x_i))}{2} - a_i^2 \right] + \right.$$

$$\left. + \left[a_i^2 - \frac{a_{i-1}^2}{4} - \frac{3}{4h^2} \int_{x_i}^{x_{i+1}} [f(x, y(x)) - f(x_i, y(x_i))]dx \right] \right\} + \frac{f''(\xi, y(\xi))}{6} h^3 =$$

$$= h^2\{I_1 + I_2\} + \frac{f''(\xi, y(\xi))}{6} h^3, \quad \xi \in (x_i, x_{i+1}).$$

On the base of Lemma 1 and formula (2.13) follows

$$|I_1| \le K_1 h^2.$$

Considering (2.8) and the Lipschitz continuity of f we get

$$|I_2| \le \left| \frac{3}{4h^2} \int_{x_i}^{x_{i+1}} [f(x, p_{i+1}(x)) - f(x, y(x))] dx \right| \le \frac{3L}{4h^2} \int_{x_i}^{x_{i+1}} |p_{i+1}(x) - y(x)| dx.$$

Applying (2.4) and the Taylor formula once more, by Lemma 1 holds

$$|I_2| \le K_0 \frac{3L}{4} h^2 + K_1 \frac{3L}{4} h^3,$$

where $K_0 := \max_{(x,y)\in D} |f''|$. Finally we get

$$|d_{i+1}| \le \frac{K_0}{6} h^3 + \left(K_0 \frac{3L}{4} + K_1 \right) h^4 + K_1 \frac{3L}{4} h^5,$$

which proves our statement.

2.4. Convergence of the method

Main theorem. *If $f \in C^{k+2}(D)$, if $h = x/N$ and if the sequence $p_{i+1}, i = 0, 1, \dots, N-1$ is defined by (2.4), and if $p_0 \to y(0)$, then $p_N(x) \to y(x)$ as $N \to \infty$ uniformly in x with the order $n+k$. Here $y(x)$ is the solution of (2.1) with initial value $y(0)$.*

Proof. For the sake of simplicity we deal with the first order differential equation $n = 1$. By the definition of the spline, if $k = 1$ let

$$p(x_{i+1}) = p_i + b_i h + a_i h^2, \tag{2.16}$$

where the coefficients a_i, b_i are defined by formulae (2.7) and (2.8). On the basis of the Taylor's theorem

$$y(x_{i+1}) = y(x_i) + y'(x_i)h + \frac{f'(x_i, y(x_i))}{2} h^2 + K_0 h^3, \tag{2.17}$$

$K_0 := \max_{(x,y)\in D} |f''(x,y)|$. Here the constant K_0 independent of h and x_i and it depends only on the upper bound of the second order partial derivatives of the function f. Subtracting (2.17) from (2.16), we get

$$|e_{i+1}| \le |e_i| + |b_i - y'(x_i)| h + \left| a_i - \frac{f'(x_i, y(x_i))}{2} \right| h^2 + K_0 h^3. \tag{2.18}$$

By the Lipschitz continuity

$$|b_i - f(x_i, y(x_i))| = |f(x_i, p_i(x_i)) - f(x_i, y(x_i))| \leq L|p_i(x_i) - y(x_i)| = L|e_i|$$

and by Lemma 1 we get from (2.18) that

$$|e_{i+1}| \leq (1 + Lh)|e_i| + \tilde{K}h^3.$$

The constant $\tilde{K}$ is independent of x and h. Let $\alpha := 1 + Lh$, $\beta := \tilde{K}h^3$ and applying Lemma 2, follows

$$|e_N| \leq \frac{\tilde{K}h^3}{Lh}[e^{Lb} - 1] + e^{Lb}|e_0|.$$

This proves the convergence of the spline method and shows that the order of the method is two. For the case $k = 2, 3$ the proof is similar, only we have to consider the Lipschitz continuity of the derivatives of function f.

Remark 4. For first-order differential equation ($n = 1$), the spline method is of order $2, 3, 4$, for second-order equation is of order $3, 4, 5$ respectively, whenewer $k = 1, 2, 3$. These results generalize the results of Loscalzo and Talbot [12-14], Micula [15], Fawzy and Soliman [7] for the first, second and n-th order differential equations respectively.

2.5. The spline method as modified Nordsieck method

Let introduce the following so-called "Nordsieck vector" [9, p.360].

$$z_i := (y_i, ha_i^1, h^2a_i^2, \ldots, h^{n+k}a_i^{n+k})^T \tag{2.19}$$

The a_i^j are meant to be approximations to $y^{(j)}(x_i)/j!$ (see [12,13]), where $y(x)$ is the exact solution of the differential equation (2.1). In order to define the integration procedure by the spline method we have to give the rule for determining z_{i+1} when z_i and the differential equation (2.1) are given. For $n \geq 2, k = 1$ such a rule is, considering (2.4), (2.10)-(2.12)

$$
(2.20)\quad
\begin{cases}
y_{i+1} = y_i + ha_i^1 + h^2a_i^2 + \ldots + h^{n+1}a_i^{n+1}, \\
ha_{i+1}^1 = ha_i^1 + 2h^2a_i^2 + \ldots + nh^na_i^n + (n+1)h^{n+1}a_i^{n+1}, \\
h^2a_{i+1}^2 = h^2a_i^2 + 3h^3a_i^3 + \ldots + \frac{n(n-1)}{2}h^na_i^n + \frac{(n+1)n}{2}h^{n+1}a_i^{n+1}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
h^{n-1}a_{i+1}^{n-1} = h^{n-1}a_i^{n-1} + nh^na_i^n + \frac{(n+1)n\ldots 3}{(n-1)!}h^{n+1}a_i^{n+1}, \\
h^na_{i+1}^n = h^nf(x_{i+1}, p_{i+2}(x_{i+1}), \ldots, p_{i+2}^{(n-1)}(x_{i+2}))/n!, \\
h^{n+1}a_{i+1}^{n+1} = \frac{h^{n+1}a_i^{n+1}}{4} + \frac{6h^{n+1}}{4(n+1)!h^2} \times \\
\qquad \times \int\limits_{x_{i+1}}^{x_{i+2}} [f(x, p_{i+2}(x), p'_{i+2}(x), \ldots, p_{i+2}^{(n-1)}(x)) - n!a_{i+1}^n]\, dx.
\end{cases}
$$

The last equation constitutes an implicit formula for a_{i+1}^{n+1}, the remaining ones are explicit. If we put $f(x, y, \ldots, y^{(n-1)}) = 0$ in (2.20), the method becomes the linear transformation

$$
(2.21)\qquad z_{i+1} = \mathbf{M}z_i,
$$

where

$$
\mathbf{M} = \begin{bmatrix}
1 & 1 & 1 & 1 & & 1 & 1 \\
 & 1 & 2 & 3 & & n & n+1 \\
 & & 1 & 3 & & \frac{n(n-1)}{2} & \frac{n(n+1)}{2} \\
 & & & & & \vdots & \vdots \\
 & & & & & 0 & 0 \\
 & & & & & 0 & \frac{1}{4}
\end{bmatrix}
$$

When the function f does not vanish, then (2.20) can be written in the form

$$
(2.22)\qquad z_{i+1} = \mathbf{M}z_i + r_i(z_{i+1}),
$$

where $r_i^T = (0, \ldots, r_{i,n+1}, r_{i,n+2})$, and here the last two components of the vector are as follows

$$
r_{i,n+1} := h^nf(x_{i+1}, p_{i+2}(x_{i+1}), \ldots, p_{i+2}^{(n-1)}(x_{i+1}))/n!,
$$

$$r_{i,n+2} := h^{n-1}\frac{6}{4(n+1)!}\int_{x_{i+1}}^{x_{i+2}}[f(x,p_{i+2}(x),\dots,p_{i+2}^{(n-1)}(x)) - n!a_{i+1}^{n}]\,dx.$$

This form of the method is more convenient for the stability considerations.

3. Linear stability analysis

3.1. First-order linear test equation

The stability of the method is discussed for the next first-order linear test differential equation

$$y' = -\lambda y, \quad \lambda > 0. \tag{3.1}$$

In case of linear equation the method is in the matrix form (2.21). If $k = 1$, the matrix of the spline method for equation (3.1) is as follows:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ -H & -H & -H \\ \dfrac{3H^2}{8+2H} & \dfrac{3H^2}{8+2H} & \dfrac{3H^2+2}{8+2H} \end{bmatrix}$$

where $H := \lambda h$. For the stability of the method the eigenvalues of $\mathbf{M}$ must lie inside the unit circle in the complex plane.

Cases k=1, 2, 3. If we apply second, third and fourth order spline, the method is stable whenever $H \leq 6$, 2.65 and 3.2. Let $H = 0$, then the eigenvalues are $\{0, 1/4, 1\}$, $\{0, 0, 1/4, 1\}$ and $\{0, 0, 0, 1/4, 1\}$ respectively.

It is proved in [13-14] that the spline method of Loscalzo and Talbot for the first order equation is divergent, whenever the degree of the spline is greater than three.

Remark 5. (On the stability property of the method, given in [18,19].) In case of first order equation (3.1) the eigenvalues of the second order spline method are 0 and $(-2H \pm (3H^2+9)^{1/2})/(3+H)$. Hence we applied the modification $\theta a_{i-1}^{n+1} + (1-\theta)a_i^{n+1}, 0 \leq \theta \leq 1$, where θ is a free parameter which can be optimized. The method is stable if $1/2 \leq \theta \leq 8/9$, for sufficiently small h. If $\theta = 8/9$, the method is stable for $\lambda h \leq 2$. The eigenvalues of

M are in this case 1 and $\pm 1/3$, if $h \to 0$. If $\theta = 1/2$, the method is stable for $\lambda h < 1$. The eigenvalues now are 1 and $(1/4) \pm (\sqrt{15}/4)i$ if $h \to 0$. For $\theta = 8/9$ the numerical results are better with order one in magnitude than for the parameter $\theta = 1/2$.

3.2. Second-order linear test equation

Consider the linear second-order differential equation

$$y'' = -\lambda^2 y, \quad \lambda > 0. \tag{3.2}$$

Dahlquist [6] proved that if the linear multistep methods for solution of the equation $y'' = f(x, y)$ are applied, the order of accuracy cannot exceed two for an unconditionally stable method. Our one-step method is conditionally stable.

Cases k=1, 2, 3. Similar analysis shows as before that for the splines of degree three, four and five, the method is stable if $\lambda^2 h^2 < 0.1$, 1.3 and 4.0. Let $\lambda h = 0$, then the eigenvalues are $\{0, 1/4, 1, 1\}$, $\{0, 0, 1/4, 1, 1\}$ and $\{0, 0, 0, 1/4, 1, 1\}$ respectively.

We have calculated the eigenvalues of the matrix **M** with the help of the software-package Mathematica.

Micula [15] extended the method of Loscalzo and Talbot for the second-order differential equations, but the derivatives on the right-hand side are absent. He proves that the method is divergent if the degree of spline is greater than four.

Thus we must stress that the method presented here works in the cases when the method of Loscalzo and Talbot and its extension by Micula does not work.

Remark 6. The stability criteria of the method given in [18,19] for the second order equation (3.2) is $\lambda^2 h^2 \leq 2.4$. The degree of the spline is three in this case.

λ	1		10		50		100		1000	
$h = 10^{-1}$	abs	rel	abs	rel	abs	rel	abs	rel	abs	rel
y	3.7-7	1.1-6	8.6-3	9.6-1	-	-	-	-	-	-
y'	"	"	" -2	"	-	-	-	-	-	-
y''	"	"	" -1	"	-	-	-	-	-	-
y'''	"	"	" +0	"	-	-	-	-	-	-
$y^{(IV)}$	2.0-3	1.9-3	8.7+2	2.4-1	-	-	-	-	-	-
$h = 10^{-2}$										
y	3.1-11	8.6-11	3.7-7	1.0-5	3.9-4	8.1-2	8.9-3	1.8+3	-	-
y'	"	"	" -6	"	1.9-2	"	" -1	"	-	-
y''	"	"	" -5	"	9.7-1	"	" +1	"	-	-
y'''	"	"	" -4	"	4.8+1	"	" +3	"	-	-
$y^{(IV)}$	2.4-5	1.9-5	2.0+1	1.9-3	2.0+5	5.3-2	9.3+6	2.7+1	-	-
$h = 10^{-3}$										
y	3.1-15	8.4-15	3.1-11	8.5-10	2.1-8	2.9-6	3.7-7	1.0-4	8.9-3	5.6+32
y'	"	"	" -10	"	1.1-6	"	" -5	"	" +0	"
y''	"	"	" -9	"	5.3-5	"	" -3	"	" +3	"
y'''	"	"	" -8	"	2.6-3	"	" -1	"	" +6	"
$y^{(IV)}$	2.5-7	1.9-7	2.4-1	1.9-5	3.4+3	4.7-4	2.0+5	1.9-3	9.3+10	8.5+30
$h = 10^{-4}$										
y	1.9-17	5.1-17	3.1-15	8.3-14	1.9-12	2.6-10	3.1-11	8.5-9	3.7-7	1.0-3
y'	"	"	" -14	"	9.7-11	"	" -9	"	" -4	"
y''	"	"	" -13	"	4.9-9	"	" -7	"	" -1	"
y'''	"	"	" -12	"	2.4-7	"	" -5	"	" +2	"
$y^{(IV)}$	2.5-9	1.9-9	2.5-3	1.9-7	3.8+1	4.7-6	2.4+3	1.9-5	2.0+9	1.9-3

Table 1.

Maximum absolute and relative errors of approximate solution and its derivatives. (We apply the notation $3.7 - 7$ for 3.7×10^{-7}.)

4. Numerical results

Numerical results have been compared with the exact results and results obtained by other methods. It can be observed that the spline method compares with other methods and can be effectively applied for the solution of nonstiff or moderately stiff problems.

Examples was performed on an IBM 486 compatible computer with program language PASCAL and with extended variables.

Example 1. (Henrici [10], pp.240-241.) (see Table 1.)

$$y' = -\lambda y, \quad 0 \le x \le 1, \quad y(0) = 1, \quad \lambda = 1, 10, 30, 50, 100, 1000.$$

The exact solution is: $y(x) = \exp(-\lambda x)$. The problem was solved with fourth order splines.

The problem was solved on the interval $x \in [0, 100]$, too. The absolute errors in $y(x)$ and its derivatives at the end point of the interval (at $x = 100$), if $\lambda = 10^2$ and $h = 10^{-3}$, are

$$y : 1.1 - 4345, \quad y' : 1.1 - 4343, \quad y'' : 1.1 - 4341,$$
$$y''' : 1.1 - 4339, \quad y^{(IV)} : 1.3 - 4337.$$

The relative errors at the end of the interval are all $1.0 - 2$, except the last derivative ($y^{(IV)}$), for which the relative error is $1.3 - 4$. The maxiumum errors are the same as in the Table 1.

Example 2. (G. Dahlquist, Å. Björk [5], Example 12.7.1.)

$$y' = 100(\sin x - y), \quad y(0) = 0, \quad 0 \le x \le 3.$$

The exact solution is

$$y(x) = \frac{1}{1.0001}\left[\sin(x) - \frac{\cos(x)}{100} + \frac{\exp(-100x)}{100}\right].$$

The absolute errors of the classical fourth order Runge-Kutta and the third order spline method with different stepsizes

h	**0.015**	**0.020**	**0.025**	**0.030**	**0.040**	**0.050**
RK4:	8.0-7	8.8-6	6.2-5	6.7+11	–	–
SP3:	7.9-6	1.6-5	2.9-5	4.8-5	1.3-4	4.6-2

Table 2.

From the results can be seen that the Runge-Kutta method for $h = 0.03$ has a frightful numerical instability, while the spline method gives an excellent result.

Example 3. (G.A. Baker, V.A. Dougalis and S.M. Serbin [2], Problem 1.)

$$y'' = -\lambda^2 y, \quad 0 \le x \le b, \quad y(0) = 1, \quad y'(0) = 0, \quad \lambda^2 = 100, 1000.$$

The exact solution is: $y(x) = \cos \lambda x$. To solve this problem, we applied the fifth-order spline ($k = 3$), with the stepsizes $h = 10^{-2}, 10^{-3}, 10^{-4}$.

λ^2=100 λ^2=1000

h	0.01		0.001		0.01		0.001		0.0001	
b=1	abs	rel	abs	rel	abs	rel	abs	rel	abs	rel
y	3.4-6	1.6-4	3.3-10	8.3-8	1.3-3	8.0-3	1.2-7	3.3-5	1.2-11	9.7-8
y'	4.2-5	1.5-4	4.0-9	8.2-8	4.3-2	9.1-3	4.2-6	3.3-5	4.2-10	2.9-8
y''	3.4-4	1.6-4	3.3-8	8.3-8	1.3+0	8.0-3	1.2-4	3.3-5	1.2-8	9.7-8
y'''	4.2-3	1.5-4	4.0-7	8.2-8	4.3+1	9.1-3	4.2-3	3.3-5	4.2-7	2.9-8
$y^{(IV)}$	3.4-2	1.6-4	3.3-6	8.3-8	1.3+3	8.0-3	1.2-1	3.3-5	1.2-5	9.7-8
$y^{(V)}$	2.5+2	7.2-3	2.5+0	1.0-4	7.8+5	1.5-1	7.9+3	3.5-3	7.9+1	1.2-5
b=10										
y	4.1-5	1.0-2	4.1-9	2.5-5	1.3-2	6.5+0	1.3-6	1.4-2	1.3-10	1.4-4
y'	4.1-4	1.7-2	4.1-8	1.8-5	4.1-1	2.1+0	4.1-5	2.0-3	4.1-9	1.4-4
y''	4.1-3	1.0-2	4.1-7	2.5-5	1.3+1	6.5+0	1.3-3	1.4-2	1.3-7	1.4-4
y'''	4.1-2	1.7-2	4.1-6	1.8-5	4.1+2	2.1+0	4.1-2	2.0-3	4.1-6	1.4-4
$y^{(IV)}$	4.1-1	1.0-2	4.1-5	2.5-5	1.3+4	6.5+0	1.3+0	1.4-2	1.3-4	1.4-4
$y^{(V)}$	2.5+2	1.2-1	2.5+0	6.2-4	1.0+6	1.1+0	7.9+3	1.6-2	7.9+1	4.6-3
b=100										
y	4.2-4	4.8+0	4.2-8	8.9-4	1.3-1	7.6+2	1.3-5	1.4+0	1.3-9	1.4-4
y'	4.2-3	4.8+0	4.2-7	2.1-3	4.0+0	2.1+2	4.2-4	1.4+0	4.2-8	1.4-4
y''	4.2-2	4.8+0	4.2-6	8.9-4	1.3+2	7.6+2	1.3-2	1.4+0	1.3-6	1.4-4
y'''	4.2-1	4.8+0	4.2-5	2.1-3	4.0+3	2.1+2	4.2-1	1.4+0	4.2-5	1.4-4
$y^{(IV)}$	4.2+0	4.8+0	4.2-4	8.9-4	1.3+5	7.6+2	1.3+1	1.4+0	1.3-3	1.4-4
$y^{(V)}$	2.6+2	2.8+0	2.5+0	5.3-3	4.3+6	2.0+2	7.9+3	8.8-1	7.9+1	4.6-3

Table 3.

Maximum absolute and relative errors of approximate solution and its derivatives

In [2], it is proposed a class of approximation schemes for linear second-order evolution equations, which are effected via a specially constructed family of rational approximations to $\cos\tau$ for $\tau \geq 0$. Their result for this problem with different observed order of accuracy at $b = 10$ and for $\lambda^2 = 100$ is as follows

$h = 0.01$ abs.error: $3.35 - 4$, the order of accuracy:3.97
$h = 0.02$ abs.error: $2.76 - 4$, the order of accuracy:5.82.

Our results are: if $h = 0.01$, then $4.1 - 5$ and if $h = 0.02$ then $6.6 - 4$. Here we applied fifth-order spline. The absolute error with eighth-order of accuracy in [2] is $2.84 - 8$ if $h = 0.02$, and $\lambda^2 = 100$.

Example 4. (R.K. Jain, R. Goel [11], Problem 1.)

$$y'' = -\lambda y', \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad \lambda = 1, 10, 30, 50, 100.$$

The exact solution is: $y(x) = (1 - e^{-\lambda x})/\lambda$.

The absolute errors of the fourth order method of Jain and Goel and the spline method of order four and five with different λ:

λ	1	10	30	50	100
JG4: y	–	4.89-7	1.38-5	6.32-5	–
JG4: y'	–	4.89-6	4.15-4	3.12-3	–
SP4: y	1.1-8	4.8-6	5.5-5	1.8-4	1.1-3
SP4: y'	1.6-8	1.8-5	6.5-4	3.6-3	4.3-2
SP5: y	2.3-11	9.8-8	3.4-6	1.9-5	2.2-4
SP5: y'	3.1-11	3.7-7	4.1-5	3.8-4	8.6-3

Table 4.

The stepsize for the JG4 method is $h = 1/32$ and for the SP4, SP5 methods is $h = 0.01$.

Conclusions. We presented a family of spline methods for the initial value problems of ordinary differential equations of order n. The order of the method depends on the degree of spline to be applied. The method gives global approximation on the whole integration interval, and approximates not only the exact solution but some of its derivatives, too. This algorithm works in the cases when the method of Loscalzo and Talbot and its extension by Micula does not work. It is effectively applicable for the solutions of nonstiff or moderately stiff problems. Burrage [3] writes the following: *"In reality, linear multistep and Runge-Kutta methods are just two small islands in a vast and* *unexplored sea of numerical methods. Before efficient methods can be found, this sea has to be charted...*". We think, the spline methods represent another island on this chart.

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A. Sövegjártó
Department of Numerical Analysis
Eötvös Loránd University
VIII. Múzeum krt. 6-8.
H-1088 Budapest, Hungary