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REMARKS ON VILENKIN BASES

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Dedicated to Prof. János Balázs on his 75th birthday

Abstract. The aim of this paper is to summarize the results about the Vilenkin system from the aspect of the bases. We shall investigate the most important function spaces. The first three sections have general characters: they contain a short summary of the results playing a basic role in the theory of the bases.

The concept of the basis

In this section we recall the concept of the basis and give a short summary of general observations with respect to the bases. (For more details see e.g. [7], [13].)

To this end let X be a (real or complex) linear space endowed with a norm $||.||$. For example, we consider the well known $L^p[0,1]$ or ℓ_p spaces with the usual norm $||.||_p$ for $1 \leq p \leq +\infty$. The space X is said to be *separable* if there exists a subset $Y \subseteq X$ which is dense in X and is at most countable. It is easy to see that the spaces $L^p[0,1]$, ℓ_p are separable for all $1 \leq p < +\infty$ but this is false for $p = +\infty$.

If X is separable and Y is a subset of X mentioned above, then the linear hull of Y is obviously dense in X. We get by an elementary argument that this is reversible, i.e. the space X is separable iff there exists a system of elements $x_n \in X$ $(n \in \mathbf{N} := \{0, 1, ...\})$ such that the subspace spanned by

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the x_n's is dense in X. In this case the system $(x_n, n \in \mathbf{N})$ is called *closed* in X. For example, if X denotes the space $C[0,1]$ (i.e. the linear space of continuous real functions defined on the interval $[0,1]$ with the maximum-norm), then the power functions form a closed system in X (in view of the well known theorem of Weierstrass), i.e. X is separable.

Let X be a separable Hilbert space with a scalar product $<,>$. If $(x_n, n \in \mathbf{N})$ is a linearly independent closed system in X, then by the Schmidt's ortogonalization procedure the orthonormality of $(x_n, n \in \mathbf{N})$ can be assumed, i.e. that $< x_n, x_k >= 0$ if $k \neq n$ and $< x_n, x_n >= 1$ $(n, k \in \in \mathbf{N})$. In this case a simple argument shows that for any $x \in X$ there exists a unique representation

$$x = \sum_{k=0}^{\infty} < x, x_k > x_k$$

(the *Fourier expansion* of x with respect to the system $(x_n, n \in \mathbf{N})$).

An analogous representation can exist also in other cases, i.e. if the space X is not necessarily an euclidean space. This observation leads to the definition of the basis. Hence, a system of elements $(x_n, n \in \mathbf{N})$ of the normed linear space X is called a (Schauder) *basis* if for any $x \in X$ there exists a unique sequence $(\alpha_k, k \in \mathbf{N})$ of coefficients such that

$$x = \sum_{k=0}^{\infty} \alpha_k x_k . \tag{1}$$

(This is a direct generalization of the elementary concept of coordinate system in finite dimensional spaces. From now on we will assume that X is not of finite dimension.)

It follows evidently from the definition that every space having basis is separable. The well known basis-problem due to S. Banach, i.e. the question whether every separable Banach space has a basis or not, had been open for a long time and answered in negative sense by P. Enflo [1] in 1973.

For example, if $1 \leq p < +\infty$, then the "coordinate sequences" $(\delta_{nk}, k \in \in \mathbf{N})$ ($n \in \mathbf{N}$, δ_{nk} stands for the Kronecker symbol) form a trivial basis in ℓ_p. Furthermore, a closed orthonormal system in a Hilbert space is a basis.

By (1) we introduce the following notations: for $x \in X$ let $\hat{x}_k(x) := := \alpha_k$ $(k \in \mathbf{N})$ and $S_n(x) := \sum_{k=0}^{n-1} \alpha_k x_k$ $(n \in \mathbf{N})$. As a simple consequence of the Banach theorem on the inverse function and of the Banach-Steinhaus theorem we obtain that the *coordinate functionals* $\hat{x}_k$ $(k \in \mathbf{N})$ are bounded (linear) mappings, furthermore, the *partial sum operators* S_n $(n \in \mathbf{N})$ are uniformly bounded. It is clear that $\hat{x}_k(x_j) = \delta_{kj} \quad (k, j \in \mathbf{N})$, i.e. the systems $(x_k, k \in \mathbf{N})$ and $(\hat{x}_k, k \in \mathbf{N})$ are *biorthogonal.*

To the characterization of the bases we need the concept of *minimality* of a system. This means that each element of the system in question is out of the closed subspace spanned by the rest elements of the system. By the Hahn-Banach theorem it can be proved that a system $(x_n, n \in \mathbf{N})$ in X is minimal iff there is a sequence of bounded linear functionals on X biorthogonal to $(x_n, n \in \mathbf{N})$. If $(x_n, n \in \mathbf{N})$ is closed, then this system of functionals is unique.

Now, the characterization theorem says that a system $(x_n, n \in \mathbf{N})$ of elements of X is a basis in X if and only if the following conditions hold:

i) $(x_n, n \in \mathbf{N})$ is closed and minimal

and

ii) if $(\varphi_n, n \in \mathbf{N})$ is the sequence of functionals biorthogonal to $(x_n, n \in \in \mathbf{N})$, then the operators

$$\sum_{k=0}^{n} \varphi_k(x)x_k \quad (x \in X \ , \ n \in \mathbf{N})$$

are uniformly bounded linear operators.

It is not hard to see that this is equivalent to the following statement: if $(x_n, n \in \mathbf{N})$ is closed in X, then it is a basis in X iff there is a constant $B \geq 0$ such that for every finite linear combination $\sum \alpha_k x_k$ and for every $n \in \mathbf{N}$

$$\left\| \sum_{k=0}^{n} \alpha_k x_k \right\| \leq B \cdot \left\| \sum_{k=0}^{\infty} \alpha_k x_k \right\|$$

holds.

Bases in function spaces

In the rest of the paper we are mainly interested in the cases $X := L^p := := L^p[0,1] \ (1 \leq p < +\infty)$ and $X := C[0,1]$. If $\Phi = (\varphi_n, n \in \mathbf{N})$ is a system in L^p having a biorthogonal system $\Psi = (\psi_n, n \in \mathbf{N})$, then by the representation theorem of F. Riesz it can be assumed that $\psi_n \in L^q \ (n \in \mathbf{N})$ where $1/p+ +1/q = 1$. Furthermore, if $f \in L^p$ and $\hat{f}(n) := \int_0^1 f\psi_n \ (n \in \mathbf{N})$, then the so-called *biorthogonal series* associated with f is $\sum_{n=0}^{\infty} \hat{f}(n)\varphi_n$. Denote by S_n $(n \in \mathbf{N})$ the partial sum operators of such expansions, i.e. let

$$S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k)\varphi_k \quad (n \in \mathbf{N}\ ,\ f \in L^p).$$

If Φ is closed and minimal in L^p, then from the above characterization theorem it follows that Φ forms a basis in L^p if and only if

$$\sup\{||S_n|| : n \in \mathbf{N}\} < +\infty\ .$$

Let $n \in \mathbf{N}$, $f \in L^p$ and write $S_n(f)$ in the following form:

$$S_n(f)(x) = \int_0^1 f(t)K_n(x,t)dt \quad (x \in [0,1])\ .$$

Here K_n denotes the so-called (n-th) *kernel function* of the system Φ, i.e.

$$K_n(x,t) := \sum_{k=0}^{n-1} \varphi_k(x)\psi_k(t) \quad (x,t \in [0,1])\ .$$

A simple calculation shows that if

$$K_n^{(1)}(t) := \int_0^1 |K_n(y,t)|dy\ , \quad K_n^{(2)}(x) := \int_0^1 |K_n(x,z)|dz \quad (t,x \in [0,1])$$

and $||K_n||_{(1,\infty)} := ||K_n^{(1)}||_\infty < +\infty$, $||K_n||_{(\infty,1)} := ||K_n^{(2)}||_\infty < +\infty$, then

$$||S_n|| \leq \frac{1}{p}||K_n||_{(1,\infty)} + \frac{1}{q}||K_n||_{(\infty,1)} \quad (1 < p < +\infty)$$

and

$$||S_n|| \leq ||K_n||_{(1,\infty)} \quad (p = 1)\ .$$

These estimates have the following consequences: if $1 \leq p < +\infty$ and Φ is closed and minimal in L^p , then

$$\sup\{||K_n||_{(1,\infty)}\ ,\ ||K_n||_{(\infty,1)} :\ n \in \mathbf{N}\} < +\infty \quad (p > 1)\ ,$$

resp.

$$\sup\{||K_n||_{(1,\infty)} \quad n \in \mathbf{N}\} < +\infty \quad (p = 1)$$

implies that Φ is a basis in L^p $(p > 1)$, resp. in L^1. Similarly, in the case $p = +\infty$ – assuming that Φ has a biorthogonal system in L^1 – we get that if

$$\sup\{||K_n||_{(\infty,1)} \; : \; n \in \mathbf{N}\} < +\infty$$

then Φ is a basis in the closed subspace of L^∞ spanned by Φ. Furthermore, if $1 < p < +\infty$ and Φ forms a basis in L^p, then its biorthogonal system is a basis in L^q.

Orthonormal bases

Now, let $(X, ||.||)$ be a Banach space such that $X \subseteq L^1$ and $||f|| \leq ||f||_1$ $(f \in X)$. Furthermore, let Φ be an orthonormal system in X that is $f\varphi_n \in$ $\in L^1$ and $\int_0^1 \varphi_n \varphi_k = \delta_{nk}$ hold for any $f \in X$ and $n, k \in \mathbf{N}$. It is easy to see that Φ is minimal and the biorthogonal series of an $f \in X$ is the usual *Fourier series* of f with respect to Φ. Hence, $\hat{f}(k) = \int_0^1 f\varphi_k$ (the k -th *Fourier coefficient*, $k \in \mathbf{N}$) and $||K_n||_{(\infty,1)} = ||K_n||_{(1,\infty)} = ||L_n||_\infty$ $(n \in \mathbf{N})$, where

$$L_n(x) := \int_0^1 |K_n(x,t)|dt = \int_0^1 \left| \sum_{k=0}^{n-1} \varphi_k(x)\varphi_k(t) \right| dt \quad (x \in [0,1])$$

is the so-called (n-th) *Lebesgue-function* of Φ. We shall investigate the following special cases:

i) $X := C[0,1]$, $||.|| := ||.||_\infty$;

ii) $X := L^p[0,1]$, $||.|| := ||.||_p$ $(1 \leq p < +\infty)$.

(In the last case it follows from the assumptions that $\varphi_n \in L^p \bigcap L^q$ $(n \in$ $\in \mathbf{N}, 1/p + 1/q = 1.)$ Our general theorems imply the following statements:

i) if Φ is closed in L^p $(1 \leq p < +\infty)$ and $\sup\{||L_n||_\infty \; : \; n \in \mathbf{N}\} <$ $< +\infty$, then Φ is a basis in L^p and for $p \neq 1$ in L^q, too ;

ii) (*principle of duality*) if $1 < p < +\infty$ and Φ is a basis in L^p, then it is a basis in L^r for every r satisfying $\min\{p,q\} \leq r \leq \max\{p,q\}$. (We recall that $1/p + 1/q = 1$.)

In the cases $p = 1$ or $X = C[0,1]$ an easy calculation shows that $||S_n|| = ||L_n||_\infty$ $(n \in \mathbf{N})$ which implies

iii) if Φ is closed in L^1 (resp. $C[0,1]$), then $\sup\{||L_n||_\infty : n \in \mathbf{N}\} < < +\infty$ is necessary and sufficient to conclude that Φ is a basis in L^1 (resp. $C[0,1]$).

The crucial part of the above statements is the uniform boundedness of the operators S_n $(n \in \mathbf{N})$ (implied by the condition $\sup\{||L_n||_\infty : n \in \mathbf{N}\} < < +\infty$) which can be investigated by means of the well known interpolation theorems ([21]). Indeed, if Φ is an orthonormal system in L^∞ and forms a basis in L^2, then by the Marcinkiewicz theorem (and by the principle of duality) the uniform weak type $(1,1)$ of S_n's implies the uniform boundedness of the operators $S_n : L^p \to L^p$, i.e. that Φ is a basis in all of the spaces L^p $(1 < p < +\infty)$. (We recall that the *uniform weak type* $(1,1)$ of S_n's means the existence of a constant $M > 0$ for which

$$\operatorname{mes}\{x \in [0,1] : |S_n(f)(x)| > y\} \leq M\frac{||f||_1}{y}$$

holds for all $f \in L^1$, $n \in \mathbf{N}$ and $y > 0$.)

The Vilenkin system

We will investigate the above questions with respect to the so-called Vilenkin system, i.e. our goal is to give a summary of results characterizing the Vilenkin system as basis in some function spaces. To this end we introduce some definitions and notations in this connection ([18]).

Let $m = (m_0, m_1, ..., m_k, ...)$ be a sequence of natural numbers, $m_k \geq \geq 2$ $(k \in \mathbf{N})$ and denote Z_{m_k} the m_k-th discrete cyclic group represented by the set $\{0, 1, ..., m_k - 1\}$ $(k \in \mathbf{N})$. If we define G_m as the complete direct product of Z_{m_k}'s, then G_m is a compact Abelian group with Haar measure 1. The elements of G_m are of the form $(x_0, x_1, ..., x_k, ...)$ $(x_k \in Z_{m_k}, k \in \mathbf{N})$ and the topology of G_m is completely determined by the sets

$$I_n := \{(x_0, x_1, ..., x_k, ...) \in G_m : x_j = 0 \quad (j = 0, ..., n-1)\} \quad (n \in \mathbf{N})$$

$(I_0 := G_m)$. Let us denote the cosets of I_n's by $I_n(x) := x + I_n$ $(x \in \in G_m, n \in \mathbf{N})$.

It is well known that the characters of G_m form a complete orthonormal system in $L^1(G_m)$ (the so-called *Vilenkin system* ([18])). The elements of the Vilenkin system can be obtained as follows. Define the sequence $(M_k, k \in \mathbf{N})$ as $M_0 := 0$ and $M_{k+1} := m_k M_k \quad (k \in \mathbf{N})$. Then any $n \in \mathbf{N}$ has a unique representation of the following form

$$n = \sum_{k=0}^{\infty} n_k M_k \quad (n_k = 0, ..., m_k - 1 \ , \ k \in \mathbf{N}).$$

If

$$r_n(x) := \exp \frac{2\pi i x_n}{m_n} \quad (n \in \mathbf{N} \ , \ x = (x_0, x_1, ...) \in G_m \ , \ i^2 = -1) \ ,$$

then the elements of our system are exactly the functions

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k} \quad (n \in \mathbf{N}).$$

We note that the group G_m can be transformed in the interval $[0, 1]$ by means of the following mapping

$$G_m \ni x \mapsto \sum_{j=0}^{\infty} \frac{x_j}{M_{j+1}} \in [0, 1] \ .$$

It is easy to see that this correspondence is almost one-to-one and measure-preserving.

In this case the kernels of *Dirichlet type* are of the form

$$K_n(x, t) = D_n(x - t) \quad (x, t \in G_m \ , \ n \in \mathbf{N})$$

where $D_n := \sum_{k=0}^{n-1} \psi_k$. In other words, the Lebesgue functions are constants for all $n \in \mathbf{N}$ (the so-called *Lebesgue constants* of the system). Moreover, it is true ([18]) that

$$||D_n||_1 = ||L_n||_\infty = O(\log n) \quad (n \mapsto \infty) \ , \ ||D_{M_n}||_1 = 1 \quad (n \in \mathbf{N})$$

and

$$\limsup_{n \mapsto \infty} \frac{||D_n||_1}{\log n} > 0 \ .$$

The first part of this statement can be improved ([4]), namely the estimation

$$\|\max\{|D_k| \ : \ k = 0, ..., n\}\|_1 = O(\log n) \quad (n \mapsto \infty)$$

is also true. Although the Vilenkin system is closed in $C(G_m)$ (:= the set of the continuous complex valued functions defined on G_m) and consequently also in $L^1(G_m)$, it follows from the above general theorems that our system does not form a basis in these spaces. In this connection we recall a non-trivial result of the orthogonal series which says that a uniformly bounded orthonormal system (defined e.g. on $[0, 1]$) cannot be a basis in the spaces $L^1[0, 1]$ and $C[0, 1]$ ([8]).

Function spaces

The space $C(G_m)$

Since $\sup\{\|D_n\|_1 \ : \ n \in \mathbf{N}\} = +\infty$, it follows by a standard argument ([18]) that for every $a \in G_m$ there is a function in $C(G_m)$ such that its Vilenkin-Fourier series diverges at a. Moreover, the partial sums of this function are not bounded.

An excellent classical result in the theory of the trigonometric Fourier series (due to L. Fejér [2]) is the construction of a function having divergent Fourier series at a prescribed point. This construction is based on the so-called *Fejér polynomials* the analogue of which for the Vilenkin system we will just give.

Namely, let $\delta_k := \left[\frac{m_k - 1}{2}\right]$ (the integer part of $\frac{m_k - 1}{2}$, $k \in \mathbf{N}$) and $N_n := \sum_{k=0}^{n} \delta_k M_k$ $(n \in \mathbf{N})$. For the sake of simplicity we will assume that $m_k > 2$ for all $k \in \mathbf{N}$. (If $m_k = 2$ for some $k \in \mathbf{N}$, then the next construction can be easily modified.) Let us introduce the following Vilenkin polynomials:

$$h_k := \sum_{j=1}^{\delta_k} \frac{r_k^j - r_k^{m_k - j}}{j} \ , \quad P_n := \psi_{N_n} \sum_{j=0}^{n} \frac{1}{M_j} h_j D_{M_j} \quad (k, n \in \mathbf{N}).$$

Then the P_n's have the following basic properties of Fejér type ([14]):

i) $\sup\{\|P_n\|_\infty \ : \ n \in \mathbf{N}\} < +\infty$;

ii) there is a constant $C > 0$ such that

$$|S_{N_n}(P_n)(0)| \geq C \cdot \log N_n \quad (n \in \mathbf{N}) .$$

We note that by the general inequality $||S_n(f)||_\infty \leq C_1 \cdot ||f||_\infty \cdot \log n$ $(0 < < n \in \mathbf{N},\ f \in C(G_m)$, for some constant C_1 depending only on m) the estimation ii) cannot be improved.

By the help of P_n's we can construct examples for functions in $C(G_m)$ having divergent Vilenkin-Fourier series. To this end let the sequences $\alpha_k \geq \geq 0$, $x^k \in G_m, n_k \in \mathbf{N}$ $(n_k < n_{k+1}, k \in \mathbf{N})$ be given and $\sum_{k=0}^{\infty} \alpha_k < +\infty$. It is clear that the function defined by

$$f(x) := \sum_{k=0}^{\infty} \alpha_k r_{n_k+1}(x) P_{n_k}(x + x^k) \quad (x \in G_m)$$

belongs to $C(G_m)$ and

$$S_{M_{n_k+1}+N_{n_k}}(f)(x) - S_{M_{n_k}+1}(f)(x) = \alpha_k r_{n_k+1}(x) S_{N_{n_k}}(P_{n_k})(x + x^k)$$

$$(x \in G_m\ ,\ k \in \mathbf{N}).$$

From these observations (by means of suitable choice of the parameters) we can deduce the following statement ([14]):

there is a function $f \in C(G_m)$ and a set $A \subseteq G_m$ such that

i) $\sup\{||S_n||_\infty :\ n \in \mathbf{N}\} < +\infty$;

ii) the Vilenkin-Fourier series of f diverges at the points of A ;

iii) for all $n \in \mathbf{N}$ and $y \in G_m$ the cardinality of the sets $A \bigcap I_n(y)$ is continuum.

Furthermore, on the growth of the partial sums of functions belonging to $C(G_m)$ we can say the following:

if $\lambda_n = \text{o}(\log n) \mapsto +\infty$ $(n \mapsto \infty)$, then there exists a function $f \in \in C(G_m)$ such that

$$\limsup_{n \mapsto \infty} \frac{|S_n(f)(0)|}{\lambda_n} > 0.$$

This means that the general estimation $S_n(g) = \text{o}(\log n)$ $(n \mapsto \infty\ ,\ g \in \in C(G_m))$ cannot be strengthened.

These statements in the Walsh case (i.e. if $m_k = 2$ for all $k \in \mathbf{N}$) are due to F. Schipp [11].

The space $L^p(G_m)$

Now, let us examine the spaces $L^p(G_m)$ $(1 \le p < +\infty)$. First let $p = 1$. We know that our system does not form a basis in $L^1(G_m)$, i.e. there is a function in $L^1(G_m)$ the Vilenkin-Fourier series of which diverges in L^1 -norm. It is quite easy to construct such a function. Indeed, since $\limsup_{n \mapsto \infty} \frac{\|D_n\|_1}{\log n} > 0$, there exist a constant $C > 0$ and a sequence $(n_k, k \in \mathbf{N})$ of indices such that $\|D_{n_k}\|_1 > C \cdot k^3$ for all $k \in \mathbf{N}$. If $\nu_k \in \mathbf{N}$ $(k \in \mathbf{N})$ denotes the index for which

$$M_{\nu_k} \le n_k < M_{\nu_k + 1}$$

holds (we can assume that the sequence $(\nu_k, n \in \mathbf{N})$ strictly increases), we consider the function

$$f := \sum_{k=1}^{\infty} \frac{1}{k^2} (D_{M_{\nu_k+1}} - D_{M_{\nu_k}}) \in L^1(G_m) .$$

Then

$$\|S_{n_k}(f)\|_1 \ge C \cdot k - 3 \cdot \sum_{j=1}^{\infty} \frac{1}{j^2} \mapsto +\infty \quad (k \mapsto \infty) .$$

This means that the Vilenkin-Fourier series of f cannot be convergent in L^1 -norm.

In the case $p > 1$ the situation is more complicated. Our main theorem is the uniform boundedness of the partial sum operators $S_n : L^p(G_m) \to$ $\to L^p(G_m)$ $(n \in \mathbf{N})$, i.e. that there exists a constant C_p (depending only on p, m) such that

$$\|S_n(f)\|_p \le C_p \cdot \|f\|_p$$

holds for all $f \in L^p(G_m)$ and $n \in \mathbf{N}$. (In this case we say that the sequence $(S_n, n \in \mathbf{N})$ is *(uniformly) of type* (p,p).) This statement for the Walsh (-Paley) system was proved by R.E.A.C. Paley [10], showing the so-called *Paley inequality*, from which the uniform (L^p, L^p) -boundedness of S_n's follows. His method can also be applied if

$$\sup\{m_k : k \in \mathbf{N}\} < +\infty$$

(Ch. Watari [19]). This is not valid for unbounded m, since in this case the generalized Paley inequality fails to be true ([19]). We note that for the Walsh-Paley system Ch. Watari [20] showed also the uniform weak type $(1,1)$ for S_n's. His method was generalized for bounded m by J. Gosselin [6]. Unfortunately, the assumption on the boundedness of m cannot be omitted in these proofs. This means that in the "unbounded" case we need a new idea for the investigations.

It is well known ([22]) that the so-called *conjugate function* plays an important role in the theory of trigonometric Fourier series. For instance the analogue of our main theorem for the trigonometric system can be showed by applying the conjugate function. Indeed, the (trigonometric) partial sum operators on the set of the trigonometric polynomials are compositions of conjugations and translations and since these operators are bounded from L^p to L^p $(1 < p < +\infty)$ (A. Zygmund [22]) the same statement follows for the partial sums, too. In the sequel we give a short summary of the analogous argument for the Vilenkin system.

To this end let us extend the definition of δ_k's also for $m_k = 2$, i.e. we write in this case $\delta_k = 1$. Furthermore, let

$$L_k := -\sum_{j=1}^{\delta_k} r_k^j + \sum_{j=\delta_k+1}^{m_k-1} r_k^j \quad (k \in \mathbf{N})$$

and $\tilde{D}_n := \sum_{k=0}^{n} L_k D_{M_k}$ $(n \in \mathbf{N})$. By the help of these *conjugate kernels* let us introduce the following sequence of operators

$$T_n(f) := f * \tilde{D}_n \quad (n \in \mathbf{N}\ ,\ f \in L^1(G_m)).$$

(Here the symbol $*$ denotes the usual *convolution operator*, i.e. for $g, h \in \in L^1(G_m)$

$$(g * h)(x) := \int_{G_m} g(t)h(x-t)dt \quad (x \in G_m)\ .)$$

The first theorem is the basic statement in this connection which means that the operators T_n $(n \in \mathbf{N})$ are uniformly of weak type $(1,1)$ ([15],[16]). From this it follows evidently that the sequence $(T_n(f)\ ,\ n \in \mathbf{N})$ converges in measure for all $f \in L^1(G_m)$. Let

$$T(f) := \lim_{n \to \infty} \text{mes}\, T_n(f)$$

(the *conjugate function* of f). Since T is of weak type $(1,1)$ and of type $(2,2)$ we get by interpolation that T is of type (p,p) for all $1 < p < +\infty$. We note that T is also of *exponential type*, i.e.

$$\text{mes}\{x \in G_m : |T(f)(x)| > y\} \leq C \cdot \exp\left(-C\frac{y}{||f||_\infty}\right) \quad (y > 0 \text{ , } f \in L^\infty(G_m))$$

(with some constant $C > 0$ depending only on m) which follows by standard extrapolation argument ([16]).

The analogue of the *trigonometric translation* (i.e. the multiplication by $\exp(inx)$ for some integer n) can be defined as follows. Let the natural number $n = \sum_{k=0}^{\infty} n_k M_k$ be given and introduce the next functions

$$d_k^+ := \sum_{j=1, j \neq m_k - n_k - 1}^{m_k - 1} r_k^j D_{M_k}, \quad d_k^- := \sum_{j=1, j \neq n_k + 1}^{m_k - 1} r_k^j D_{M_k} \quad (k \in \mathbf{N}).$$

Furthermore, let us define the operators $T_M^{n^\pm}$ as follows:

$$T_M^{n^+}(f) := \sum_{k=0}^{M} (f * d_k^+) r_k^{n_k+1}, \quad T_M^{n^-}(f) := \sum_{k=0}^{M} (f * d_k^-) r_k^{m_k - n_k - 1}$$

($M \in \mathbf{N}$, $f \in L^1(G_m)$). Then these operators are uniformly of weak type $(1,1)$ and denoting by T^{n^+}, resp. T^{n^-} their limits in measure we obtain that the *translation operators* $T^{n^\pm}$ are uniformly of type (p,p) for all $1 < p < +\infty$ ([15], [16]).

Now, if we consider the modified partial sums

$$S_n^*(f) := \bar{\psi}_n S_n(f \psi_n) \quad (n \in \mathbf{N} \text{ , } f \in L^1(G_m))$$

($\bar{\psi}_n$ is the complex conjugation of ψ_n), then a simple calculation gives a suitable expression for $S_n^*(P)$ by means of conjugations and translations where P is an arbitrary Vilenkin polynomial ([15], [16]). This leads directly to the estimation

$$||S_n^*(P)||_p \leq C_p \cdot ||P||_p$$

and since the Vilenkin system is closed in $L^p(G_m)$, the uniform type (p,p) for S_n^*'s and also for S_n's follows.

We note that there are other methods (different from the above "conjugate function technique") for the proof of the main theorem of this section. Wo-Sang

Young [21] showed that the operators S_n $(n \in \mathbf{N})$ are uniformly of weak type $(1,1)$ for all m. F. Schipp [12] pointed out that the Vilenkin system can be considered as a special product system and he proved a general statement on such systems as bases in the L^p spaces.

Gy. Gát [5] has introduced Vilenkin-like systems (containing the Vilenkin system as special case) and showed the uniform type (p,p) $(1 < p < +\infty)$ for the partial sum operators with respect to these systems. His proof is based on our main theorem.

The Orlicz spaces

Finally, we mention that the main theorem of 5.2 can also be extended to some Orlicz spaces. To formulate this extension we introduce the following notations and definitions.

Let Φ be the set of all even real functions which are nondecreasing on $[0,+\infty)$ and have the following properties:

i) $\varphi(0) = \varphi(+0) = 0$;

ii) $\varphi(x) > 0$ $(x > 0)$;

iii) $\varphi(2x) = O(\varphi(x))$ $(x \mapsto +\infty\ ,\ \varphi \in \Phi)$.

For every $\varphi \in \Phi$ let us define the space $\varphi(L)$ as the set of measurable and almost everywhere finite functions f defined on $[0,1]$, for which

$$||f||_\varphi := \int_0^1 \varphi(|f(x)|)\,dx < +\infty$$

holds. If the functions f, g belong to $\varphi(L)$, then let their φ -distance be defined as $||f-g||_\varphi$, which determines the φ -convergence in the usual way. As special cases we get the L^p spaces, the Orlicz spaces (if φ is convex), the space of a.e. finite functions with the convergence in measure.

The system of functions $g_n \in \varphi(L)$ $(n \in \mathbf{N})$ is called a ***system of** representation* in $\varphi(L)$ if for every $f \in \varphi(L)$ there exists a series $\sum a_k g_k$ with coefficients a_k $(k \in \mathbf{N})$ such that

$$\lim_{n \mapsto \infty} \left\| f - \sum_{k=0}^{n} a_k g_k \right\|_\varphi = 0 \, .$$

(Note that the uniqueness of such series is not assumed. If also this holds, then the system is a basis.) The following problem is due to P.L. Uljanov [17]: how can the spaces $\varphi(L)$ be characterized in which the classical systems of functions are systems of representation? The aim of this remark is to give the answer with respect to the Vilenkin system. Hence, the next statement is true ([3]):

the Vilenkin system is a system of representation in $\varphi(L)$ if and only if either $\varphi(L) \not\subset L^1$ or $\varphi(L)$ is equivalent to a reflexive Orlicz space.

In the case $\varphi(L) \subseteq L^1$ the Vilenkin system may be at most basis in $\varphi(L)$ since this system is uniformly bounded system of functions. In this connection P. Oswald [9] showed that if a complete orthonormal system of uniformly bounded functions is a basis in $\varphi(L)$, then $\varphi(L)$ is equivalent to an Orlicz space. It remains to answer only the question, in which Orlicz spaces is the Vilenkin system basis? Our last theorem contains the answer ([3]):

the Vilenkin system is a basis in a separable Orlicz space if and only if the space is reflexive.

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