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ON PERTURBATIONS OF INITIAL–BOUNDARY VALUE PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS

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Dedicated to Professor J. Balázs on his 75-th birthday

This paper is devoted to certain nonlinear parabolic equations in unbounded domains of the space variable. Consider e.g. the problem

$$D_t u + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D_x^\alpha \left[f_\alpha(t,x,u,\dots,D_x^\beta u,\dots)\right] = g \quad \text{in} \quad Q_T = (0,T)\times\Omega,$$

$$u(0,x) = 0, \qquad x\in\overline{\Omega},$$

$$D_x^\gamma u(t,x) = 0, \quad t\in[0,T], \quad x\in\partial\Omega, \quad |\gamma|\le m-1,$$

where $\Omega\subset\mathbb{R}^n$ is an unbounded domain.

There will be formulated conditions such that the weak solution of this problem can be obtained as the limit (as $k\to\infty$) of weak solutions u_k of problems

$$D_t u_k + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D_x^\alpha \left[f_\alpha^k(t,x,u_k,..,D_x^\beta u_k,..)\right] = g_k \quad \text{in} \quad Q_T^k = (0,T)\times\Omega_k,$$

$$u_k(0,x) = 0 \qquad \text{in} \qquad \overline{\Omega_k},$$

$$D_x^\gamma u_k(t,x) = 0, \quad t\in[0,T], \quad x\in\partial\Omega_k, \quad |\gamma|\le m-1,$$

where $\Omega_k\subset\Omega$ is a bounded domain such that $B_k\cap\Omega\subset\Omega_k$, $B_k = \{x\in \in\mathbb{R}^n : |x| < k\}$.

Similar results have been proved e.g. in [4]-[8] for nonlinear elliptic equations.

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In §1 we shall prove a rather general perturbation theorem on nonlinear evolution equations with pseudo-monotone type operators. In §2 it will be formulated several applications of this theorem.

1. The general perturbation theorem

Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain and $\Omega_k \subset \Omega$ be bounded domains with the cone property (see [10]) such that $\Omega_k \supset \Omega \cap B_k$ for sufficiently large $k \in \mathbb{N}$. Let $p \geq 2$ and m a positive integer. Denote by $W_p^m(\Omega)$ the usual Sobolev space of real valued functions u whose distributional derivatives of order $\leq m$ belong to $L^p(\Omega)$. The norm on $W_p^m(\Omega)$ is defined by

$$\|u\|_{W_p^m(\Omega)} = \left\{ \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^p \right\}^{1/p},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j = \dfrac{\partial}{\partial x_j}$. The expression $W_{p,0}^m(\Omega)$ will denote the closure in $\|\cdot\|_{W_p^m(\Omega)}$ of $C_0^\infty(\Omega)$, the infinitely differentiable functions with compact support contained in Ω.

Let X be a closed linear subspace of $W_p^m(\Omega)$, by $L^p(0,T;X)$ will be denoted the Banach space of the set of measurable functions $u: (0,T) \to X$ such that $|u|^p$ is integrable. The dual space of $L^p(0,T;X)$ is $L^q(0,T;X')$ where $1/p + 1/q = 1$ and X' is the dual space of X (see e.g. [3]).

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a fixed function with the properties

$$\varphi(x) = 1 \quad \text{if} \quad |x| \leq 1/2, \qquad \varphi(x) = 0 \quad \text{if} \quad |x| \geq 1,$$

and define φ_k by

$$\varphi_k(x) = \varphi(x/k)), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.$$

Further, let X_k be a closed linear subspace of $W_p^m(\Omega_k)$ and define the restriction operator $\mathcal{M}_k$ by $\mathcal{M}_k w = w|_{\Omega_k}$, $w \in X$.

Assume that

I. For any $w \in X \quad \mathcal{M}_k(\varphi_k w) \in X_k$.

Then for any $u \in L^p(0,T;X)$ we have $M_k(\varphi_k u) \in L^p(0,T;X_k)$, where the operator M_k is defined by

$$(M_k\nu)(t,x) = [\mathcal{M}_k\nu(t,.)](x), \quad \nu \in L^p(0,T;X).$$

Further, assume that there exist linear continuous (extension) operators $\mathcal{N}_k: \ X_k \to X$ such that $\mathcal{N}_k w|_{\Omega_k} = w$ a.e. and the norms of $\mathcal{N}_k$ are bounded $(k \in N)$. Then we have linear continuous operators

$$N_k: \quad L^p(0,T;X_k) \to L^p(0,T;X)$$

such that the norms of N_k are bounded, where operators N_k are defined by

$$(N_k\nu)(t,x) = [\mathcal{N}_k\nu(t,.)](x), \qquad \nu \in L^p(0,T;X_k).$$

II. Let $A_k: \ L^p(0,T;X_k) \to L^q(0,T;X_k')$ be (nonlinear) operators such that if

$$u_k \in L^p(0,T;X_k) \quad \text{and} \quad \|u_k\|_{L^p(0,T;X_k)}$$

is bounded, then $\|A_k(u_k)\|_{L^q(0,T;X_k')}$ is bounded $(k \in \mathbb{N})$.

III. The operators A_k satisfy the following coercivity condition: $u_k \in$ $\in L^p(0,T;X_k)$ and

$$\lim_{k\to\infty} \|u_k\|_{L^p(0,T;X_k)} = \infty \qquad \text{imply} \qquad \lim_{k\to\infty} \frac{[A_k(u_k),u_k]}{\|u_k\|} = +\infty$$

($[A_k(u_k),\nu]$ denotes the value of the functional $A_k(u_k)$ at $\nu \in L^p(0,T;X_k)$).

IV. There exists an operator $A: \ L^p(0,T;X) \to L^q(0,T;X')$ such that if $u_k \in L^p(0,T,X_k)$, $(N_k u_k) \to u$ weakly in $L^p(0,T;X)$ to some $u \in L^p(0,T;X_k)$ such that for the distributional derivatives of functions $u_k \in L^p(0,T;X_k)$ we have $\frac{du_k}{dt} \in L^q(0,T;X_k')$, the norms $\left\|\frac{du_k}{dt}\right\|_{L^q(0,T;X_k')}$ are bounded and

$$\limsup[A_k(u_k), \quad u_k - M_k(\varphi_k u)] \le 0,$$

then

$$\widetilde{A}_k\,(u_k) \to A(u) \qquad \text{weakly in } L^q(0,T;X'),$$

where the "extensions" $\widetilde{A}_k\,(u_k)$ are defined by

$$\left[\widetilde{A}_k\,(u_k),\nu\right] = [A_k(u_k),\ M_k(\varphi_k\nu)], \quad \nu \in L^p(0,T;X).$$

V. The functionals $h_k \in L^q(0,T,W_p^m(\Omega_k)')$ are such that for their extensions defined by

$$\left[\hat{h}_k,\nu\right] = [h_k, M_k\nu], \qquad \nu \in L^p(0,T;X)$$

$(\hat{h}_k) \to h$ in the norm of $L^q(0,T;X')$ with some $h \in L^q(0,T;X')$.

Theorem 1. *Assume I-V. If $u_k \in L^p(0,T;X_k)$ satisfy*

$$\frac{du_k}{dt} + A_k(u_k) = h_k, \qquad \frac{du_k}{dt} \in L^q(0,T;X'_k), \tag{1.1}$$

$$u_k(0) = 0,$$

then there exist a subsequence (u_{k_l}) of (u_k) and $u \in L^p(0,T;X)$ such that $(N_{k_l}u_{k_l}) \to u$ weakly in $L^p(0,T;X)$ and u satisfies

$$\frac{du}{dt} + A(u) = h, \qquad \frac{du}{dt} \in L^q(0,T;X'), \tag{1.2}$$

$$u(0) = 0.$$

Remark 1. Since X_k is continuously and densely imbedded into $L^2(\Omega)$ thus $X_k \subset L^2(\Omega) \subset X'_k$ and so

$$u_k \in L^p(0,T;X_k), \qquad \frac{du_k}{dt} \in L^q(0,T;X'_k)$$

imply $u \in C(0,T;L^2(\Omega))$, consequently $u(0)$ is well defined (see e.g. [3]).

Remark 2. Existence theorems on problem (1.1) with monotone type operators A_k can be found e.g. in [1].

Remark 3. Clearly, if the solution of (1.2) is unique then also $(N_k u_k)$ tends weakly to u in $L^p(0,T;X)$.

The proof of Theorem 1. By III the norms $\|u_k\|_{L^p(0,T;X_k)}$ are bounded. Because for the solutions of (1.1) we have

$$\left[\frac{du_k}{dt},\ u_k\right] + [A_k(u_k),u_k] = [h_k,u_k],$$

where

$$\left[\frac{du_k}{dt},\ u_k\right] = \int_0^T \left\langle \frac{du_k}{dt}(t,.),\ u_k(t,.)\right\rangle dt = \frac{1}{2}\int_0^T \frac{d}{dt}\langle u_k(t,.),\ u_k(t,.)\rangle dt =$$

$$= \frac{1}{2}\int_0^T \frac{d}{dt}(u_k(t,.),\ u_k(t,.))_{L^2(\Omega_k)}dt = \frac{1}{2}(u_k(T,.),\ u_k(T,.))_{L^2(\Omega_k)} \geq 0$$

($\langle w,\nu\rangle$ denotes the value of the functional $w \in X'$ at $\nu \in X$, $(w,\nu)_{L^2(\Omega)}$ denotes the scalar product of functions $w,\nu \in L^2(\Omega)$, see e.g. [3]). Thus

$$\frac{[A_k(u_k),u_k]}{\|u_k\|} \leq \frac{[h_k,u_k]}{\|u_k\|} \leq \|\hat{h}_k\|_{L^q(0,T;X')},$$

where the right hand side is bounded. Consequently, III implies that $\|u_k\|_{L^p(0,T;X_k)}$ are bounded.

Therefore $(N_k u_k)$ is a bounded sequence in $L^p(0,T;X)$. By assumption II the sequence $(A_k u_k)$ is bounded in $L^q(0,T;X_k')$ and so by the definition of $\widetilde{A}_k\ (u_k)$, $(\widetilde{A}_k\ (u_k))$ is a bounded sequence in $L^q(0,T;X')$. Since $L^p(0,T;X)$ and $L^q(0,T;X')$ are reflexive Banach spaces, thus there exist a subsequence (u_{k_l}), $u \in L^p(0,T;X)$ and $a \in L^q(0,T;X')$ such that

(1.3) $$(N_{k_l}u_{k_l}) \to u \quad \text{weakly in} \quad L^p(0,T;X)$$

and

(1.4) $$(\widetilde{A}_{k_l}\ (u_{k_l})) \to a \quad \text{weakly in} \quad L^q(0,T;X').$$

First we show that by (1.3), (1.4), V we obtain from (1.1)

(1.5) $$\frac{du}{dt} + a = h, \qquad \frac{du}{dt} \in L^q(0,T;X'),$$

$$u(0) = 0.$$

Let $\nu \in L^p(0,T;X) \cap C^1(0,T;L^2(\Omega))$ be an arbitrary fixed function with $\nu(T) = = 0$. Then from (1.1) we obtain

$$\left[\frac{du_{k_l}}{dt},\ M_{k_l}(\varphi_{k_l}\nu)\right] + [A_{k_l}(u_{k_l}),\ M_{k_l}(\varphi_{k_l}\nu)] = [h_{k_l},\ M_{k_l}(\varphi_{k_l}\nu)],$$

i.e. by the definition of $\widetilde{A}_k\ (u_k)$ and by using the definition $[\widetilde{h}_k,\nu] = = [h_k, M_k(\varphi_k\nu)]$ we have

(1.6) $$\left[-u_{k_l},\ \frac{d}{dt}(M_{k_l}(\varphi_{k_l}\nu))\right] + [\widetilde{A}_{k_l}\ (u_{k_l}),\nu] = [\widetilde{h}_{k_l},\nu].$$

Clearly,

$$
(1.7)\quad \begin{aligned}\left[u_{k_l},\ \frac{d}{dt}(M_{k_l}(\varphi_{k_l}\nu))\right] = \left[u_{k_l},\ M_{k_l}\left(\varphi_{k_l}\frac{d\nu}{dt}\right)\right] = \left[N_{k_l}u_{k_l},\ \varphi_{k_l}\frac{d\nu}{dt}\right] = \\ = \int_0^T \left(N_{k_l}u_{k_l}(t),\ \varphi_{k_l}\frac{d\nu}{dt}(t)\right)_{L^2(\Omega)} dt.\end{aligned}
$$

It is easy to show that

$$
\varphi_{k_l}\frac{d\nu}{dt} \to \frac{d\nu}{dt} \quad \text{in the norm of} \quad L^2(0,T;L^2(\Omega))
$$

and by (1.3)

$$
(N_{k_l}u_{k_l}) \to u \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).
$$

Consequently,

$$
\lim_{l\to\infty}\left[N_{k_l}u_{k_l},\ \varphi_{k_l}\frac{d\nu}{dt}\right] = \int_0^T \left(u(t),\ \frac{d\nu}{dt}(t)\right)_{L^2(\Omega)} dt.
$$

Denote the last term by $\left(u,\ \frac{d\nu}{dt}\right)_{L^2(0,T;L^2(\Omega))}$. It is easy to show that

$$
\lim_{k\to\infty}\left\|\hat{h}_k - \widetilde{h}_k\right\|_{L^q(0,T;X')} = 0,
$$

thus, by V,

$$
\lim_{k\to\infty}[\widetilde{h}_k,\ \nu] = [h,\nu].
$$

Consequently, by (1.4), (1.6) one obtaines as $k\to\infty$

$$
(1.8)\quad -\left(u,\ \frac{d\nu}{dt}\right)_{L^2(0,T;L^2(\Omega))} + [a,\nu] = [h,\nu].
$$

Since the functions $\nu\in C^1(0,T;L^2(\Omega))$ with $\nu(0)=\nu(T)=0$ are dense in $L^p(0,T;X)$, thus we obtain that for the distributional derivative $\frac{du}{dt}$ of u

$$
(1.9)\quad \frac{du}{dt}\in L^q(0,T;X') \quad \text{and} \quad \frac{du}{dt}+a=h.
$$

Further, applying (1.8) to functions $\nu \in C^1(0,T;L^2(\Omega))$ with $\nu(T)=0$ we get

$$\left[\frac{du}{dt},\ \nu\right] + (u(0),\nu(0))_{L^2(\Omega)} + [a,\nu] = [h,\nu],$$

thus by (1.9) we obtain

$$(u(0),\nu(0))_{L^2(\Omega)} = 0, \qquad \text{hence} \ \ u(0)=0,$$

i.e. we have shown (1.5).

Now we prove that $a = A(u)$. By IV it is sufficient to show the inequality

$$\limsup_{l\to\infty}\,[A_{k_l}(u_{k_l}),\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)] \le 0. \tag{1.10}$$

By (1.1) we have

$$\left[\frac{du_{k_l}}{dt},\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)\right] + [A_{k_l}(u_{k_l}),\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)] =$$

$$= [h_{k_l},\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)]\,. \tag{1.11}$$

For the right hand side

$$[h_{k_l},\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)] = [h_{k_l},\ M_{k_l}(N_{k_l}u_{k_l}) - M_{k_l}(\varphi_{k_l}u)] =$$

$$= \left[\hat{h}_{k_l},\ N_{k_l}u_{k_l} - \varphi_{k_l}u\right] \to 0 \tag{1.12}$$

holds since

$$\lim_{l\to\infty}\|\hat{h}_{k_l} - h\|_{L^q(0,T;X')} = 0 \quad \text{and} \quad (N_{k_l}u_{k_l} - \varphi_{k_l}u) \to 0$$

weakly in $L^p(0,T;X)$ because of (1.3) and

$$\lim_{l\to\infty}\|\varphi_{k_l}u - u\|_{L^p(0,T;X)} = 0.$$

Further, for the first term in the left of (1.11)

$$\left[\frac{du_{k_l}}{dt},\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)\right] =$$

$$\left[\frac{du_{k_l}}{dt} - \frac{d(M_{k_l}(\varphi_{k_l}u))}{dt},\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)\right] + \left[\frac{dM_{k_l}(\varphi_{k_l}u)}{dt},\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)\right]$$

$$= \frac{1}{2}\int_0^T \frac{d}{dt}(u_{k_l}(t) - M_{k_l}(\varphi_{k_l}u)(t),\ u_{k_l}(t) - M_{k_l}(\varphi_{k_l}u)(t))_{L^2(\Omega_{k_l})}dt +$$

$$+ \left[M_{k_l}\left(\varphi_{k_l}\frac{du}{dt}\right),\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)\right] =$$

$$= \frac{1}{2}(u_{k_l}(T) - M_{k_l}(\varphi_{k_l}u)(T),\ u_{k_l}(T) - M_{k_l}(\varphi_{k_l}u)(T)) +$$

$$+ \left[\varphi_{k_l}\frac{du}{dt},\ N_{k_l}u_{k_l} - \varphi_{k_l}u\right] \geq \left[\varphi_{k_l}\frac{du}{dt},\ N_{k_l}u_{k_l} - \varphi_{k_l}u\right],$$

where the last term tends to 0 since

$$\varphi_{k_l}\frac{du}{dt} \to \frac{du}{dt} \quad \text{in the norm of } L^q(0,T;X') \text{ and}$$
$$N_{k_l}u_{k_l} - \varphi_{k_l}u \to 0 \quad \text{weakly in } L^p(0,T;X).$$

Hence

$$\liminf_{l\to\infty}\left[\frac{du_{k_l}}{dt},\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)\right] \geq 0,$$

thus (1.11), (1.12) imply (1.10). So we have shown that $a = A(u)$, thus by (1.5) the proof of Theorem 1 is complete.

From the above proof it easily follows a modification of Theorem 1:

Theorem 2. *Assume I, IV, V. If $u_k \in L^p(0,T;X_k)$ satisfy (1.1), further, $(N_k u_k) \to u$ weakly in $L^p(0,T;X)$ and $(\widetilde{A}_k\,(u_k)) \to z$ weakly in $L^q(0,T;X')$ with some $z \in L^q(0,T;X')$, then u satisfies (1.2).*

2. Applications

It will be formulated several special cases when the conditions of Theorem 1 are satisfied.

Clearly, the assumption I is satisfied, e.g. if

a) $X = W^m_{p,0}(\Omega), \quad X_k = W^m_{p,0}(\Omega_k);$

b) $\partial\Omega$ is bounded, $\Omega_k = \Omega \cap B_k$, $X = W_p^m(\Omega)$ and $X_k = W_p^m(\Omega_k)$ or $X_k = \{\nu \in W_p^m(\Omega_k) : D^\gamma \nu|_{S_k} = 0 \text{ for } |\gamma| \le m-1\}$, where $D^\gamma\nu|_{S_k}$ denotes the trace of $D^\gamma \nu$ on the sphere $S_k = \{x \in \mathbb{R}^n : |x| = k\}$.

c) $\partial\Omega \in C^m$ is bounded, $\Omega_k = \Omega \cap B_k$, $X = W_{p,0}^m(\Omega)$, $X_k = \{\nu \in W_p^m(\Omega_k) : D^\gamma \nu|_{\partial\Omega} = 0 \text{ for } |\gamma| \le m-1\}$.

The following special operators A_k satisfy assumptions II-IV.

A) Let N and M be the number of multiindices β satisfying $|\beta| \le m$ resp. $|\beta| \le m-1$. The vectors $\xi \in \mathbb{R}^n$ will also be written in the form $\xi = (\eta, \zeta)$, where $\eta \in \mathbb{R}^M$ consists of those coordinates ξ_β for which $|\beta| \le m-1$ and ζ consists of coordinates ξ_β with $|\beta| = m$.

Assume that

(2.1) The functions $f_\alpha^k : Q_T^k \times \mathbb{R}^N \to \mathbb{R}$, $f_\alpha : Q_T \times \mathbb{R}^N \to \mathbb{R}$ satisfy the Carathéodory conditions, i.e. they are measurable in (t,x) for each fixed $\xi \in \mathbb{R}^N$ and continuous in ξ for almost all $(t,x) \in Q_T^k$ resp. Q_T.

(2.2) $|f_\alpha^k(t,x,\xi)| \le c_1|\xi|^{p-1} + k_1(t,x)$ for a.e. $(t,x) \in Q_T^k$, all $\xi \in \mathbb{R}^N$, $k \in N$ with some $c_1 > 0$, $k_1 \in L^q(Q_T)$.

(2.3)
$$\sum_{|\alpha|=m} \left[f_\alpha^k(t,x,\eta,\varsigma) - f_\alpha^k(t,x,\eta,\varsigma')\right](\xi_\alpha - \xi'_\alpha) > 0,$$
$$\text{if } \ \zeta \ne \zeta' \ \text{ for a.e. } \ (t,x) \in Q_T^k, \ \text{all } \ (\eta,\zeta), \ \ (\eta,\zeta') \in \mathbb{R}^N.$$

(2.4)
$$\sum_{|\alpha|\le m} f_\alpha^k(t,x,\xi)\xi_\alpha \ge c_2|\xi|^p - k_2(t,x)$$

for a.e. $(t,x) \in Q_T^k$, all $\xi \in \mathbb{R}^N$, $k \in N$ with some $c_2 > 0$, $k_2 \in L^1(Q_T)$.

(2.5) $f_\alpha^k(t,x,\xi) \to f_\alpha(t,x,\xi)$ (as $k \to \infty$) uniformly in $\xi \in G$ for any bounded $G \in \mathbb{R}^N$ and a.e. $(t,x) \in Q_T$.

Let

$$[B_k(u),\nu] = \sum_{|\alpha|\le m} \int_0^T \left[\int_{\Omega_k} f_\alpha^k((t,x,u,\dots,D_x^\beta u,\dots)D_x^\alpha \nu\, dx\right] dt,$$

$$u,\nu \in L^p(0,T;X_k),$$

$$[B(u),\nu] = \sum_{|\alpha|\le m} \int_0^T \left[\int_\Omega f_\alpha(t,x,u,\dots,D_x^\beta u,\dots) D_x^\alpha \nu dx \right] dt,$$

$$u,\nu \in L^p(0,T;X).$$

Theorem 3. *Assume (2.1)-(2.5). Then operators* $A_k = B_k$, $A = B$ *satisfy II-IV.*

Proof. Conditions II, III directly follow from (2.1), (2.2), (2.4).

In order to prove IV assume that $u_k \in L^p(0,T;X_k)$,

$$(2.6)\qquad \begin{gathered} (N_k u_k) \to u \text{ weakly in } L^p(0,T;X), \quad \frac{du_k}{dt} \in L^q(0,T;X_k'), \\ \text{the norms } \left\| \frac{du_k}{dt} \right\|_{L^q(0,T;X_k')} \text{ are bounded} \end{gathered}$$

and

$$(2.7)\qquad \limsup [B_k(u_k),\ u_k - M_k(\varphi_k u)] \le 0.$$

Since for arbitrary fixed k_0 the sequence (u_k) is bounded in $L^p(0,T;W_p^m(\Omega_{k_0}))$, $\left(\frac{du_k}{dt}\right)$ is bounded in $L^q(0,T;W_p^m(\Omega_{k_0})')$, and $\Omega_{k_0} \subset \mathbb{R}^n$ is a bounded domain, thus there is a subsequence of (u_k) which is convergent in $L^p(0,T;W_p^{m-1}(\Omega_{k_0}))$ (see e.g. [3]), so we can choose a subsequence (u_k) for which

$$(2.8)\qquad D_x^\gamma(M_{k_l} u_{k_l}) \to D_x^\gamma u \quad \text{a.e. in } Q_T \text{ if } |\gamma| \le m-1.$$

Since

$$\lim_{k\to\infty} \|M_k(\varphi_k u) - u\|_{L^p(0,T;W_p^k(\Omega_k))} = 0$$

and $\|B_k(u_k)\|_{L^q(0,T;W_p^k(\Omega_k)')}$ is bounded, thus from (2.7) follows

$$(2.9)\qquad \limsup \sum_{|\alpha|\le m} \int_0^T \left[\int_{\Omega_k} f_\alpha^k(t,x,u_k,\dots,D_x^\beta u_k,\dots)(D_x^\alpha u_k - D_x^\alpha u) \right] \le 0.$$

Define functions p_k by

$$p_k = \begin{cases} \sum_{|\alpha|\le m} \left[f_\alpha^k(t,x,u_k,..,D_x^\beta u_k) - f_\alpha^k(t,x,u,..,D_x^\beta u,..) \right] (D_x^\alpha u_k - D_x^\alpha u), \\ \qquad\qquad\qquad\qquad (t,x) \in Q_T^k, \\ 0, \qquad\qquad\qquad\qquad (t,x) \in Q_T \setminus Q_T^k. \end{cases}$$

Then (2.9), (2.2) and (2.6) imply

$$\limsup \int\limits_{Q_T} p_k \leq 0.$$

By using arguments of Lemma 9 of [6], based on the work [2] of F.E.Browder, we obtain that there exist subsequences (u_{k_l}) and (p_{k_l}) of (u_k) resp. (p_k) such that

$$\lim(p_{k_l}) = 0 \qquad \text{a.e. in} \quad Q_T \tag{2.10}$$

and for $|\delta| = m$

$$\sup_l \left| D^\delta u_{k_l}(t,x) \right| < +\infty \quad \text{for a.e.} \quad (t,x) \in Q_T. \tag{2.11}$$

From (2.5), (2.8), (2.10), (2.11) it follows

$$\lim_{l\to\infty} \sum_{|\alpha|=m} \left[f_\alpha^{k_l}(t,x,u_{k_l},..,D_x^\beta u_{k_l},..) - f_\alpha^{k_l}(t,x,u,..,D_x^\gamma u,\dots,D_x^\delta u_{k_l},..) \right] \times$$

$$\times (D_x^\alpha u_{k_l} - D_x^\alpha u) = 0 \tag{2.12}$$

a.e. in Q_T, where $|\gamma| \leq m-1$, $|\delta| = m$ (see e.g. [2], [6]).

Finally, by (2.3), (2.11), (2.12) one obtains

$$D_x^\delta(N_{k_l} u_{k_l}) \to D_x^\delta u \qquad \text{a.e. in} \quad Q_T.$$

Thus (2.5), (2.8) and Vitali's theorem imply that

$$\widetilde{B}_{k_l}(u_{k_l}) \to B(u) \quad \text{weakly in} \quad L^q(0,T;X').$$

By virtue of II $\widetilde{B}_k(u_k)$ is bounded in $L^q(0,T;X')$, thus from the above argument it follows that

$$\widetilde{B}_k(u_k) \to B(u) \quad \text{weakly in} \quad L^q(0,T;X'),$$

i.e. we have shown IV.

B) Assume that operators $C_k : L^p(0,T;X_k) \to L^q(0,T;X_k')$ satisfy II, i.e.

(2.13) If $\|u_k\|_{L^p(0,T;X_k)}$ is bounded then $\|C_k(u_k)\|_{L^q(0,T;X_k')}$ is bounded $(k \in \mathbb{N})$.

There is a number ρ with $1 < \rho < p$ such that

$$(2.14) \qquad |[C_k(\nu),\ \nu]| \le c_3 \|\nu\|^{\rho}_{L^p(0,T;X_k)} + \tilde{c}_3, \qquad \nu \in L^p(0,T;X_k), \quad k \in \mathbb{N}$$

with some constants $c_3,\ \tilde{c}_3$.

There exist positive numbers $\delta,\ r$ such that

(2.15) if $\|u_k\|_{L^p(0,T;X_k)} \le c_4$ then $|[C_k(u_k),\ \nu]| \le \tilde{c}_4\ \|\nu\|_{L^p(0,T;W^{m-\delta}_{\rho}(\Omega_r))}$ with some constant $\tilde{c}_4$ (depending on c_4).

Finally, there exists $C:\ L^p(0,T;X) \to L^q(0,T;X')$ such that

(2.16) if $(N_k u_k) \to u$ weakly in $L^p(0,T;X)$, $\dfrac{du_k}{dt}$ is bounded in $L^q(0,T;X'_k)$ then

$$(\tilde{C}_k\ (u_k)) \to C(u) \quad \text{weakly in} \quad L^q(0,T;X').$$

Theorem 4. *Let operators $B_k,\ B$ be defined in A) and assume (2.13)-(2.16). Then operators $A_k = B_k + C_k$, $A = B + C$ satisfy II-IV.*

Proof. Conditions II, III easily follow from (2.1), (2.2), (2.4) and (2.13), (2.14). Further, assume that $(N_k u_k) \to u$ weakly in $L^p(0,T;X)$, $\left\|\dfrac{du_k}{dt}\right\|_{L^q(0,T;X'_k)}$ is bounded and

$$(2.17) \qquad \limsup [A_k(u_k),\ u_k - M_k(\varphi_k u)] \le 0.$$

Then

$$\|u_{k_l} - M_{k_l}(\varphi_{k_l} u)\|_{L^p(0,T;W^{m-\delta}_{\rho}(\Omega_r))} \to 0$$

for a subsequence (see e.g. [3]), hence by (2.15)

$$(2.18) \qquad \lim_{l\to\infty} [C_{k_l}(u_{k_l}),\ u_{k_l} - M_{k_l}(\varphi_{k_l} u)] = 0$$

and by (2.16)

$$(2.19) \qquad (\tilde{C}_k\ (u_k)) \to C(u) \quad \text{weakly in} \quad L^q(0,T;X').$$

(2.17), (2.18) imply

$$\limsup_{l\to\infty} [B_{k_l}(u_{k_l}),\ u_{k_l} - M_{k_l}(\varphi_{k_l} u)] \le 0.$$

Thus, from Theorem 3 we obtain that

$$\widetilde{B}_{k_l}(u_{k_l}) \to B(u) \quad \text{weakly in} \quad L^q(0,T;X'),$$

whence by (2.19) we find

$$A_{k_l}(u_{k_l}) \to A(u) \quad \text{weakly in} \quad L^q(0,T;X').$$

Since $\widetilde{A}_k(u_k)$ is bounded in $L^q(0,T;X')$, thus we have also

$$\widetilde{A}_k(u_k) \to A(u) \quad \text{weakly in} \quad L^q(0,T;X').$$

Examples

1. Let operators C_k be defined by

$$[C_k(u),\ \nu] = \sum_{|\alpha|\le m-1} \int_0^T \left[\int_{\Omega_r} g_\alpha^k(t,x,u,\dots,D_x^\gamma u,\dots) D_x^\alpha \nu dx \right] dt +$$

$$+ \sum_{|\alpha|\le m-1} \int_0^T \left\{ \int_0^t \left[\int_{\Omega_r} h_\alpha^k(t,\tau,x,u(\tau,x),..,D_x^\gamma u(\tau,x),..) D_x^\alpha \nu(t,x) dx \right] d\tau \right\} dt,$$

where $|\gamma| \le m-1$, the functions $g_\alpha^k,\ h_\alpha^k$ satisfy the Carathéodory conditions and

$$|g_\alpha^k(t,x;\eta)| \le c_3'|\eta|^{\rho-1} + k_3(t,x), \quad |h_\alpha^k(t,\tau,x,\eta)| \le c_3'|\eta|^{\rho-1} + k_3(t,x)$$

with some constant c_3' and $k_3 \in L^q(Q_T^r)$;

finally

$$g_\alpha^k(t,x,\eta) \to g_\alpha(t,x,\eta), \quad h_\alpha^k(t,\tau,x,\eta) \to h_\alpha(t,\tau,x,\eta)$$

as $k \to \infty$ uniformly in $\eta \in G$ for any bounded $G \subset \mathbb{R}^M$ and a.e. (t,x) resp. (t,τ,x). (Such functional differentional operators have been considered in [9].)

Then it is easy to show that operators C_k satisfy (2.13)-(2.15) with $\delta = 1$ and by using Vitali's theorem we find (2.16) with C defined by

$$[C(u),\nu] = \sum_{|\alpha|\le m-1} \int_0^T \left[\int_{\Omega_r} g_\alpha(t,x,u,\dots,D_x^\gamma u,\dots) D_x^\alpha \nu dx \right] dt +$$

$$+\sum_{|\alpha|\le m-1}\int_0^T\left\{\int_0^t\left[\int_{\Omega_r} h_\alpha(t,\tau,x,u(\tau,x),..,D_x^\gamma u(\tau,x),..)D_x^\alpha \nu dx\right]d\tau\right\}dt.$$

2. Assume that $m=1$ and the boundary of Ω, $\partial\Omega$ is bounded and continuously differentiable. Let operators C_k be defined by

$$[C_k(u),\nu]= \tag{2.20}$$

$$=\int_0^T\left[\int_{\partial\Omega} g^k(t,x,u)\nu d\sigma_x\right]dt+\int_0^T\left\{\int_0^t\left[\int_{\partial\Omega} h^k(t,\tau,x,u(\tau,x))\nu(t,x)d\sigma_x\right]d\tau\right\}dt,$$

where the functions g^k, h^k satisfy the Carathéodory conditions and

$$|g^k(t,x,\eta)|\le c_4'|\eta|^{\rho-1}+k_4(t,x),\quad |h^k(t,\tau,x,\eta)|\le c_4'|\eta|^{\rho-1}+k_4(t,x) \tag{2.21}$$

with some constant c_4' and $k_4\in L^q((0,T)\times\partial\Omega)$; further

$$g^k(t,x,\eta)\to g(t,x,\eta),\quad h^k(t,\tau,x,\eta)\to h(t,\tau,x,\eta) \tag{2.22}$$

as $k\to\infty$ uniformly in $\eta\in G$ for any bounded $G\in\mathbb{R}$ and a.e. (t,x) resp. (t,τ,x).

We shall show that these operators C_k satisfy (2.13), (2.16) with C defined by

$$[C(u),\nu]=$$

$$=\int_0^T\left[\int_{\partial\Omega} g(t,x,u)\nu d\sigma_x\right]dt+\int_0^T\left\{\int_0^t\left[\int_{\partial\Omega} h(t,\tau,x,u(\tau,x))\nu(t,x)d\sigma_x\right]d\tau\right\}dt.$$

The solutions of problems (1.1), (1.2) with $A_k=B_k+C_k$, $A=B+C$ (operators B_k, B are defined in A), $m=1$) are weak solutions of second order nonlinear parabolic equations satisfying certain third boundary condition with delay. The existence of solutions of problems (1.1) follows e.g. from [1].

In order to prove (2.13)-(2.16) apply Hölder's inequality, assumption (2.21) and the boundedness of the trace operator $W_p^{1-\delta}(\Omega_r)\to L^{\widetilde{p}}(\partial\Omega)$ with $\widetilde{p}=$ $=(\rho-1)q<p$, sufficiently small $\delta>0$ and sufficiently great $r>0$:

$$\left|\int_{\partial\Omega} g^k(t,x,u)\nu d\sigma_x\right|+\left|\int_0^t\left[\int_{\partial\Omega} h^k(t,\tau,x,u(\tau,x))\nu(t,x)d\sigma_x\right]d\tau\right|\le \tag{2.23}$$

$$\leq \left\{ \int_{\partial\Omega} \left[c_4' |u(t,x)|^{\rho-1} + k_4(t,x) \right]^q d\sigma_x \right\}^{1/q} \cdot \left\{ \int_{\partial\Omega} |\nu(t,x)|^p d\sigma_x \right\}^{1/p} +$$

$$+ \int_0^T \left\{ \int_{\partial\Omega} \left[c_4' |u(\tau,x)|^{\rho-1} + k_4(\tau,x) \right]^q d\sigma_x \right\}^{1/q} d\tau \cdot \left\{ \int_{\partial\Omega} |\nu(t,x)|^p d\sigma_x \right\}^{1/p} \leq$$

$$\leq c_5' [\|u(t,.)\|^{\rho-1}_{W^1_p(\Omega_k)} + c_6'] \|\nu(t,.)\|_{W^{1-\delta}_p(\Omega_r)} +$$

$$+ c_5' \left[\int_0^T \|u(\tau,.)\|^{\rho-1}_{W^1_p(\Omega_k)} d\tau + c_6' \right] \|\nu(t,.)\|_{W^{1-\delta}_p(\Omega_r)}.$$

Thus by Hölder's inequality one obtains

$$|[C_k(u),\nu]| \leq$$

$$\leq c_7' \left\{ \left[\int_0^T \|u(t,.)\|^p_{W^1_p(\Omega_k)} dt \right]^{(\rho-1)/p} + c_8' \right\} \cdot \left\{ \int_0^T \|\nu(t,.)\|_{W^{1-\delta}_p(\Omega_r)} dt \right\}^{1/p} +$$

$$+ c_7' \left\{ \left[\int_0^T \|u(\tau,.)\|^p_{W^1_p(\Omega_k)} d\tau \right]^{(\rho-1)/p} + c_8' \right\} \cdot \left\{ \int_0^T \|\nu(t,.)\|_{W^{1-\delta}_p(\Omega_r)} dt \right\}^{1/p},$$

which implies (2.13)-(2.15).

If $(N_k u_k) \to u$ weakly in $L^p(0,T;X)$ and $\frac{du_k}{dt}$ is bounded in $L^q(0,T;X_k')$, then for any $\delta > 0$ there is a subsequence of (u_k) which is convergent in $L^p(0,T;W^{1-\delta}_p(\Omega_r))$ and consequently (choosing sufficiently small $\delta > 0$) also in $L^p(0,T;L^p(\partial\Omega))$. Thus we can choose a subsequence (u_{k_l}) for which $(u_{k_l}) \to u$ a.e. on $(0,T) \times \partial\Omega$. By using (2.22), Vitali's theorem and estimations similar to (2.23) (considering measurable subsets of $\partial\Omega$ instead of $\partial\Omega$) we find

$$(\widetilde{C}_{k_l}(u_{k_l})) \to C(u) \quad \text{weakly in} \quad L^q(0,T;X'),$$

whence we obtain (2.16).

C) Theorem 1 can be applied to nonlinear parabolic equations with third boundary condition on $S_k = \{x \in \mathbb{R}^n : \ |x| = k\}$ when $\partial\Omega$ is bounded. Let the operator D_k be defined by

$$[D_k(u), \nu] = \sum_{|\alpha| \le m-1} \int_0^T \left[\int_{S_k} g_\alpha^k(t, x, u, \dots, D_x^\gamma u, \dots) D_x^\alpha \nu d\sigma_x \right] dt, \tag{2.24}$$

where $|\gamma| \le m-1$, the functions g_α^k satisfy the Carathéodory conditions,

$$|g_\alpha^k(t, x, \eta)| \le c_3' |\eta|^{p-1} + k_3|_{S_k}(t, x) \tag{2.25}$$

for a.e. (t, x) with some $k_3 \in L^q(0, T; W_q^1(\Omega))$ ($k_3|_{S_k}$ denotes the trace of $x \to k_3(t, x)$ on S_k which is defined for a.e. t); further

$$\sum_{|\alpha| \le m-1} \left[g_\alpha^k(t, x, \eta) - g_\alpha^k(t, x, \eta') \right] (\xi_\alpha - \xi_\alpha') \ge 0. \tag{2.26}$$

Theorem 5. *Let operators B_k, B be defined in A) and D_k by (2.24)-(2.26). Then operators $A_k = B_k + D_k$, $A = B$ satisfy II-IV.*

Proof. By using the transformation

$$\int_{S_k} |g|^p d\sigma_x = \int_{S_1} |g(ky)|^p k^{p-1} d\sigma_y$$

and the continuity of the trace operator $W_p^1(B_1 \setminus B_{1/2}) \to L^p(S_1)$ it is not difficult to show the inequality

$$\int_{S_k} |g|^p d\sigma \le \text{const} \cdot \|g\|^p_{W_p^1(B_k \setminus B_{k/2})} \tag{2.27}$$

for any $g \in W_p^1(B_k \setminus B_{k/2})$, where the constant is not depending on k.

Thus by (2.24), (2.25) and Hölder's inequality we obtain

$$|[D_k(u), \nu]| \le$$

$$c_3' \left\{ \int_0^T \left[\int_{S_k} |(.., D_x^\gamma u, ..)|^p d\sigma_x + \int_{S_k} |k_3| d\sigma_x \right] dt \right\}^{\frac{1}{q}} \left\{ \int_0^T \left[\int_{S_k} |D_x^\alpha \nu|^p d\sigma_x \right] dt \right\}^{\frac{1}{p}},$$

hence by using (2.27) we find

$$|[D_k(u),\nu]| \le \left[c_4'\|u\|^{p/q}_{L^p(0,T;X_k)} + c_5'\right]\|\nu\|_{L^p(0,T;X_k)},$$

which implies II.

Further, in virtue of (2.26)

$$[D_k(u) - D_k(0),\ u] \ge 0,$$

thus by Hölder's inequality, (2.25), (2.27)

$$[D_k(u),u] \ge -|[D_k(0),u]| \ge -c_6'\|u\|_{L^p(0,T;X_k)}.$$

Consequently, $A_k = B_k + D_k$ satisfy III.

In order to show IV assume that $u_k \in L^p(0,T;X_k)$, $(N_k u_k) \to u$ weakly in $L^p(0,T;X)$ such that $\left\|\frac{du_k}{dt}\right\|_{L^q(0,T;X_k')}$ are bounded and

$$\limsup[A_k(u_k),\ u_k - M_k(\varphi_k u)] \le 0. \tag{2.28}$$

First we show that

$$\liminf[D_k(u_k),\ u_k - M_k(\varphi_k u)] \ge 0. \tag{2.29}$$

Assumption (2.26) implies

$$[D_k(u_k) - D_k(M_k(\varphi_k u)),\ u_k - M_k(\varphi_k u)] \ge 0, \tag{2.30}$$

further,

$$[D_k(M_k(\varphi_k u)),\ u_k - M_k(\varphi_k u)] \to 0 \tag{2.31}$$

for a subsequence because $M_k(\varphi_k u) = 0$ on S_k and so by Hölder's inequality and (2.25), (2.27)

$$|[D_k(M_k(\varphi_k u)),\ u_k - M_k(\varphi_k u)]| = |[D_k(M_k(\varphi_k u)),\ u_k]| \le$$

$$\le c_7'\|k_3\|_{L^q(0,T;W^1_q(B_k\setminus B_{k/2}))}\|u_k\|_{L^p(0,T;X_k)},$$

where $\|u_k\|_{L^p(0,T;X_k)}$ are bounded and there is a subsequence (k_j) such that

$$\lim_{j\to\infty}\|k_3\|_{L^q(0,T;W^1_q(B_{kj}\setminus B_{kj/2}))} = 0$$

since $k_3 \in L^q(0,T;W_q^1(\Omega))$ and $\partial\Omega$ is bounded. From (2.30), (2.31) we obtain (2.29) for a subsequence. By using the above argument one easily gets (2.29) also for the original sequence.

Inequalities (2.28), (2.29) imply

$$\limsup[B_k(u_k),\ u_k - M_k(\varphi_k u)] \le 0.$$

Since operators $B_k,\ B$ satisfy IV, thus

$$\overset{\sim}{B_k}(u_k) \to B(u) \quad \text{weakly in} \quad L^q(0,T;X').$$

Clearly, $\overset{\sim}{D_k}(u_k) = 0$ and so

$$\overset{\sim}{A_k}(u_k) = \overset{\sim}{B_k}(u_k) + \overset{\sim}{D_k}(u_k) \to B(u) \quad \text{weakly in} \quad L^q(0,T;X'),$$

i.e. we have shown that $A_k = B_k + D_k$ and $A = B$ satisfy IV.

D) Now we formulate sufficient conditions for IV.

Assume that we have operators

$$A_k : \quad L^p(0,T;X_k) \to L^q(0,T;W_p^m(\Omega_k)') \tag{2.32}$$

(i.e. operators defined in II are such that for any $u_k \in X_k$ the linear continuous functional $A_k(u_k)$ on $L^p(0,T;X_k)$ has a linear continuous extension to $L^p(0,T;W_p^m(\Omega_k))$.

Then we may define

$$[\hat{A}_k(u_k),\ z] = [A_k(u_k),\ M_k z], \qquad z \in L^p(0,T;X)$$

and $\hat{A}_k(u_k) \in L^q(0,T;X')$.

Further, assume that there exists a hemicontinuous operator

$$A:\ L^p(0,T;X) \to L^q(0,T;X') \quad \text{such that for each} \quad u \in L^p(0,T;X)$$

$$\lim_{k\to\infty} \|\hat{A}_k(M_k(\varphi_k u)) - A(u)\|_{L^q(0,T;X')} = 0. \tag{2.33}$$

(Hemicontinuity of A means that for any fixed $u, \nu, w \in L^p(0,T;X)$

$$\lim_{\lambda\to+0} [A(u-\lambda\nu),\ w] = [A(u), w].$$

(See e.g. [3].)

Finally, for any $R > 0$ there is a continuous function $g_R : [0, +\infty) \to [0, +\infty)$ such that

$$\lim_{\rho\to 0} \frac{g_R(\rho)}{\rho} = 0 \qquad \text{and} \tag{2.34}$$

$u_k, \nu_k \in L^p(0,T;X_k)$, $\|u_k\|_{L^p(0,T;X_k)} \le R$, $\|\nu_k\|_{L^p(0,T;X_k)} \le R$ imply

$$[A_k(u_k) - A_k(\nu_k),\ u_k - \nu_k] \ge -g_R\left(\|u_k - \nu_k\|_{L^p(0,T;W_p^{m-1}(\Omega_r))}\right) \tag{2.35}$$

with some fixed $r > 0$.

Theorem 6. *Assume II and (2.32)-(2.35). Then operators A_k, A satisfy IV.*

Proof. Suppose that $u_k \in L^p(0,T;X_k)$,

$$(N_k u_k) \to u \ \text{ weakly in } \ L^p(0,T;X), \qquad \left\|\frac{du_k}{dt}\right\|_{L^q(0,T;X_k')} \quad \text{are bounded} \tag{2.36}$$

and

$$\limsup[A_k(u_k),\ u_k - M_k(\varphi_k u)] \le 0. \tag{2.37}$$

By II it is sufficient to show that if $\widetilde{A}_k\,(u_k)$ tends to some z weakly in $L^q(0,T;X')$, then $z = A(u)$.

First we show that

$$\lim[A_k(u_k),\ u_k - M_k(\varphi_k u)] = 0. \tag{2.38}$$

According to (2.35)

$$\begin{aligned}&[A_k(u_k) - A_k(M_k(\varphi_k u)),\ u_k - M_k(\varphi_k u)] \ge \\ &\ge -g_R\left(\|u_k - M_k(\varphi_k u)\|_{L^p(0,T;W_p^{m-1}(\Omega_r))}\right).\end{aligned} \tag{2.39}$$

By (2.36) there is a subsequence such that

$$\lim_{l\to\infty} \|u_{k_l} - M_{k_l}(\varphi_{k_l} u)\|_{L^p(0,T;W_\rho^{m-1}(\Omega_r))} = 0,$$

thus (2.34) implies

$$(2.40)\qquad \lim_{l\to\infty} g_R\left(\|u_{k_l} - M_{k_l}(\varphi_{k_l}u)\|_{L^p(0,T;W_p^{m-1}(\Omega_r))}\right) = 0.$$

Further,

$$(2.41)\qquad \begin{aligned}&[A_k(M_k(\varphi_k u)),\ u_k - M_k(\varphi_k u)] = [\hat{A}_k(M_k(\varphi_k u)),\ N_k u_k - \varphi_k u] = \\ &= [\hat{A}_k(M_k(\varphi_k u)) - A(u),\ N_k u_k - \varphi_k u] + [A(u),\ N_k u_k - \varphi_k u] \to 0\end{aligned}$$

because of (2.33), (2.36). From (2.37), (2.39)-(2.41) one gets

$$\lim_{l\to\infty} [A_{k_l}(u_{k_l}),\ u_{k_l} - M_{k_l}(\varphi_{k_l}u)] = 0.$$

By using the above argument it is easy to show that the same holds also for the original sequence, i.e. one has (2.38).

Now consider an arbitrary $w \in L^p(0,T;X)$, by (2.35) we obtain

$$(2.42)\qquad \begin{aligned}&[A_k(u_k) - A_k(M_k(\varphi_k w)),\ u_k - M_k(\varphi_k w)] \geq \\ &\geq -g_R\left(\|u_k - M_k(\varphi_k w)\|_{L^p(0,T;W_\rho^{m-1}(\Omega_r))}\right).\end{aligned}$$

For the left hand side of this inequality we have

$$(2.43)\qquad [A_k(u_k),\ u_k - M_k(\varphi_k u)] + [A_k(u_k),\ M_k(\varphi_k u) - M_k(\varphi_k w)] -$$

$$-[A_k(M_k(\varphi_k w)),\ u_k - M_k(\varphi_k w)] = [A_k(u_k),\ u_k - M_k(\varphi_k u)] + [\widetilde{A}_k(u_k),\ u - w] -$$

$$-[\hat{A}^k(M_k(\varphi_k w)),\ N_k u_k - \varphi_k w] \to [z,\ u - w] - [A(w),\ u - w]$$

by (2.33), (2.36), (2.38) because

$$\widetilde{A}_k(u_k) \to z \quad \text{weakly in} \quad L^q(0,T;X').$$

(2.36) and the continuity of g_R imply that the limit of the right hand side in (2.42) equals

$$-g_R\left(\|u - w\|_{L^p(0,T;W_\rho^{m-1}(\Omega_r))}\right)$$

for a subsequence. Thus (2.42), (2.43) imply

$$(2.44)\qquad [z - A(w),\ u - w] \geq -g_R\left(\|u - w\|_{L^p(0,T;W_\rho^{m-1}(\Omega_r))}\right).$$

Applying (2.44) to $w = u - \lambda\nu$ with an arbitrary $\nu \in L^p(0,T;X)$, $\lambda > 0$ we find

$$[z - A(u - \lambda\nu),\ \nu] \geq -\frac{1}{\lambda} g_R(\lambda \|\nu\|_{L^p(0,T;W^{m-1}_\rho(\Omega_r))}),$$

whence by (2.34) and the hemicontinuity of A we obtain as $\lambda \to +0$

$$[z - A(u),\ \nu] \geq 0.$$

Consequently, $z = A(u)$ which completes the proof of Theorem 6.

Example. Define operators A_k by

$$[A_k(u),\ \nu] = \sum_{|\alpha|\leq m} \int_0^T \left[\int_{\Omega_k} f^k_\alpha(t,x,u,\ldots,D^\beta_x u,\ldots) D^\alpha_x \nu dx \right] dt +$$

$$+ \sum_{|\alpha|\leq m} \int_0^T \left[\int_{\Omega_r} g^k_\alpha(t,x,u,\ldots,D^\gamma_x u,\ldots) D^\alpha_x \nu dx \right] dt +$$

$$+ \sum_{|\alpha|\leq m} \int_0^T \left\{ \int_0^t \left[\int_{\Omega_r} h_\alpha(t,\tau,x,u(\tau,x),\ldots,D^\gamma_x u(\tau,x),\ldots) D^\alpha_x \nu(t,x) dx \right] d\tau \right\} dt$$

for $u,\nu \subset L^p(0,T;X_k)$ where $|\beta| \leq m$ and in the last two terms $|\alpha| + |\gamma| \leq$ $\leq 2m-1$; the functions f^k_α, g^k_α, h^k_α satisfy the Carathéodory conditions and the following inequalities: there exists $p \geq 2$ such that

$$|f^k_\alpha(t,x,\xi)| \leq c_1|\xi|^{p-1} + k_1(t,x) \quad \text{with some} \quad k_1 \in L^q(Q_T); \tag{2.45}$$

$$\sum_{|\alpha|\leq m} [f^k_\alpha(t,x,\xi) - f^k_\alpha(t,x,\xi')](\xi_\alpha - \xi'_\alpha) \geq c_2|\xi - \xi'|^p \tag{2.46}$$

with some constant $c_2 > 0$ and there exists a number ρ with $1 < \rho < p$ such that

$$|g^k_\alpha(t,x,\xi)| \leq c_3|\xi|^{\rho-1} + k_3(t,x), \quad |h^k_\alpha(t,\tau,x,\xi)| \leq c_3|\xi|^{\rho-1} + k_3(t,x),$$

where $k_3 \in L^q(Q_T)$ and

$$\left|\frac{\partial g^k_\alpha}{\partial \xi_\gamma}(t,x,\xi)\right| \leq c_4|\xi|^{\rho-2} + k_4(t,x), \quad \left|\frac{\partial h^k_\alpha}{\partial \xi_\gamma}(t,\tau,x,\xi)\right| \leq c_4|\xi|^{\rho-2} + k_4(t,x),$$

where $k_4 \in L^{p/p-2}(Q_T)$ (in the case $p = 2$, $k_4 \in L^\infty(Q_T)$). Finally, assume that for a.e. (t, x), each ξ

$$f_\alpha^k(t,x,\xi) \to f_\alpha(t,x,\xi), \quad g_\alpha^k(t,x,\xi) \to g_\alpha(t,x,\xi), \quad h_\alpha^k(t,\tau,x,\xi) \to h_\alpha(t,\tau,x,\xi).$$

Then operators A_k satisfy II-IV if operator A is defined by f_α, g_α, h_α similarly to A_k.

The conditions II, III easily follow from our assumptions by using arguments of Example 1 of B). The condition IV follows from Theorem 6 by Young's and Hölder's inequalities (see [11]).

Remark 4. In the special case $g_\alpha^k = 0$, $h_\alpha^k = 0$ the assumption (2.46) implies that the solution of problem (1.2) is unique and for the solution u_k of (1.1) we have

$$\lim_{k\to\infty} \|u_k - M_k(\varphi_k u)\|_{L^p(0,T;X_k)} = 0.$$

Since by (2.46)

$$[A_k(u_k) - A_k(M_k(\varphi_k u)),\ u_k - M_k(\varphi_k u)] \geq c_2' \|u_k - M_k(\varphi_k u)\|^p_{L^p(0,T;X_k)}$$

with some constant $c_2' > 0$ and (2.38), (2.41) imply the left hand side of this inequality converges to 0.

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