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APPROXIMATION OF FUNCTIONS OF TWO VARIABLES WITH MODIFIED SPLINE FUNCTIONS OF TYPE (0,2)

N.A.A. Rahman (Budapest, Hungary)

In honor of the 75th birthday of Prof. János Balázs

1. Introduction

The issue of approximation with spline functions of several variables was studied by a number of mathematicians in the recent years. Concerning the references, see, e.g. the following monographs: [1], [2], [7], [8], [3], [9]. The spline functions of two variables of Hermitian type, and the spline functions of n variables of reduced Hermitian type were studied and interesting theorems were proved by M. Lénárd in her papers [4], [5], [6] recently.

In this paper we give a construction of a modified spline function of type (0,2). We prove existence, uniqueness and convergence theorems. We verify the order of approximation, and also give error estimation. The construction is the simplest possible one. It is expected that the results published here will be well applicable to the partial differential equations and to their applications. We note that the results discussed in this paper can be extended to the case of more than two variables.

2. The construction of modified spline functions of type (0,2). Existence and uniqueness

In the closed, finite rectangle domain $D := \{(x,y) : a_1 \leq x \leq b_1,\ a_2 \leq \leq y \leq b_2\}$ let the system of interpolation nodes

(2.1) $\quad \Delta : \{P_{i,j}\}, \qquad i = \overline{0,n_1},\ j = \overline{0,n_2}, \qquad n_1 = 2,3,\cdots;\ n_2 = 2,3,\cdots$

be given, where $P_{i,j} = (x_i, y_j)$, $i = \overline{0, n_1}$, $j = \overline{0, n_2}$.

To the system of interpolation nodes (2.1) let the arbitrary real numbers

$$(2.2) \qquad u^{(i,j)}, \quad \alpha^{(i,j)}, \quad \beta^{(i,j)}, \quad i = \overline{0, n_1}, \quad \overline{0, n_2}$$

be given.

Let $D_{i,j}$ denote the closed subdomain in D defined by $P \in D_{i,j} = \{(x, y) : x_i \le x \le x_{i+1},\ y_i \le y \le y_{i+1}\}$. Furthermore, set

$$h_i = (x_{i+1} - x_i), \quad i = \overline{0, n_1 - 1}; \quad k_j = (y_{j+1} - y_j), \quad j = \overline{0, n_2 - 1}.$$

On the system of interpolation nodes (2.1) the spline function of type (0,2) $S_\Delta(P) = S_\Delta(x, y)$ constructed with the values (2.2) is defined as follows.

For all $P \in D_{i,j}$, $i = \overline{0, n_1 - 1}$, $j = \overline{0, n_2 - 1}$, we have

$$(2.3) \qquad S_\Delta(P) \equiv S_{i,j}(P) =$$

$$= \frac{1}{h_i k_j} \left[p_{i+1,j+1} \cdot q_{i,j} + p_{i,j+1} \cdot q_{i+1,j} + p_{i+1,j} \cdot q_{i,j+1} + p_{i,j} \cdot q_{i+1,j+1}\right],$$

where

$$(2.4) \qquad \begin{aligned} p_{i+1,j+1} &= (x_{i+1} - x)(y_{j+1} - y), & p_{i,j+1} &= (x - x_i)(y_{j+1} - y), \\ p_{i+1,j} &= (x_{i+1} - x)(y - y_j), & p_{i,j} &= (x - x_i)(y - y_j) \end{aligned}$$

and

(2.5)

$$\begin{aligned} q_{i,j} &= u^{(i,j)} + \frac{1}{4}\left[(x_i - x)h_i\alpha^{(i,j)} + (y_j - y)k_j\beta^{(i,j)}\right], \\ q_{i+1,j} &= u^{(i+1,j)} + \frac{1}{4}\left[(x - x_{i+1})h_i\alpha^{(i+1,j)} + (y_j - y)k_j\beta^{(i+1,j)}\right], \\ q_{i,j+1} &= u^{(i,j+1)} + \frac{1}{4}\left[(x_i - x)h_i\alpha^{(i,j+1)} + (y - y_{j+1})k_j\beta^{(i,j+1)}\right], \\ q_{i,j} &= u^{(i+1,j+1)} + \frac{1}{4}\left[(x - x_{i+1})h_i\alpha^{(i+1,j+1)} + (y - y_{j+1})k_j\beta^{(i+1,j+1)}\right]. \end{aligned}$$

In consequence of (2.4), (2.5) the function $S_\Delta(P)$ is a polynomial of two variables and of second degree also in the variables x and y in every $D_{i,j}$, but there is no x^2y^2 term in it.

The spline function $S_\Delta(P)$ interpolates at the interpolation nodes $P_{i,j}$, $i = \overline{0, n_1}$, $j = \overline{0, n_2}$. Namely, by (2.4), (2.5) susbstituting $x = x_i$, $(i = \overline{0, n_1})$, $y = y_j$, $(j = \overline{0, n_2})$ into the expression of $S_\Delta(P)$ (5.2.3), we obtain

$$
\begin{aligned}
S_\Delta(P_{i,j}) &= S_{i,j}(x_i, y_j) = u^{(i,j)}, \\
S_\Delta(P_{i+1,j}) &= S_{i,j}(x_{i+1}, y_j) = u^{(i+1,j)}, \\
S_\Delta(P_{i,j+1}) &= S_{i,j}(x_i, y_{j+1}) = u^{(i,j+1)}, \\
S_\Delta(P_{i+1,j+1}) &= S_{i,j}(x_{i+1}, y_{j+1}) = u^{(i+1,j+1)}.
\end{aligned} \tag{2.6}
$$

If $P \in D_{i,j}$, $i = \overline{0, n_1 - 1}$, $j = \overline{0, n_2 - 1}$, then differentiating the identity (2.3) twice, by (2.4), (2.5) we obtain the equality

$$
\frac{\partial^2 S_\Delta(P)}{\partial x^2} \equiv \frac{\partial S_{i,j}(P)}{\partial x^2} = \tag{2.7}
$$

$$
= \frac{1}{k_j}\left[(y_{j+1} - y)\frac{\alpha^{(i,j)} + \alpha^{(i+1,j)}}{2} + (y - y_j)\frac{\alpha^{(i,j+1)} + \alpha^{(i+1,j+1)}}{2}\right].
$$

Differentiating the identity (2.3) twice, by (2.4), (2.5) we obtain the equality

$$
\frac{\partial^2 S_\Delta(P)}{\partial y^2} \equiv \frac{\partial S_{i,j}(P)}{\partial y^2} = \tag{2.8}
$$

$$
= \frac{1}{h_i}\left[(x_{i+1} - x)\cdot\frac{\beta^{(i,j)} + \beta^{(i,j+1)}}{2} + (x - x_i)\frac{\beta^{(i+1,j)} + \beta^{(i+1,j+1)}}{2}\right].
$$

From (2.7), if $y = y_j$ or $y = y_{j+1}$, $j = \overline{0, n_2 - 1}$, we obtain the equality

$$
\begin{aligned}
\left.\frac{\partial^2 S_\Delta(P)}{\partial x^2}\right|_{y=y_j} &= \left.\frac{\partial^2 S_{i,j}(P)}{\partial x^2}\right|_{y=y_j} = \frac{1}{2}\left(\alpha^{(i,j)} + \alpha^{(i+1,j)}\right), \\
\left.\frac{\partial^2 S_\Delta(P)}{\partial x^2}\right|_{y=y_{j+1}} &= \left.\frac{\partial^2 S_{i,j}(P)}{\partial x^2}\right|_{y=y_{j+1}} = \frac{1}{2}\left(\alpha^{(i,j+1)} + \alpha^{(i+1,j+1)}\right).
\end{aligned} \tag{2.9}
$$

From (2.8), since $x_{i+1} - x_i = h_i$, $i = \overline{0, n_1 - 1}$, the following equalities hold

$$
\begin{aligned}
\left.\frac{\partial^2 S_\Delta(P)}{\partial y^2}\right|_{x=x_i} &= \left.\frac{\partial^2 S_{i,j}(P)}{\partial y^2}\right|_{x=x_i} = \frac{1}{2}\left(\beta^{(i,j)} + \beta^{(i,j+1)}\right), \\
\left.\frac{\partial^2 S_\Delta(P)}{\partial y^2}\right|_{x=x_{i+1}} &= \left.\frac{\partial^2 S_{i,j}(P)}{\partial y^2}\right|_{x=x_{i+1}} = \frac{1}{2}\left(\beta^{(i+1,j)} + \beta^{(i+1,j+1)}\right).
\end{aligned} \tag{2.10}
$$

By (2.3), (2.4), (2.5), if $y = y_j$, $j = \overline{1, n_2 - 1}$, and $x_i \leq x \leq x_{i+1}$, $i = \overline{0, n_1 - 1}$, we obtain that

$$
\begin{aligned}
S_\Delta(x, y_j) &= S_{i,j}(x, y_j) = \\
&= \frac{1}{h_i}(x_{i+1} - x)\left[u^{(i,j)} + \frac{1}{4}h_i(x_i - x)\alpha^{(i,j)}\right] + \\
&+ \frac{1}{h_i}(x - x_{i+1})\left[u^{(i+1,j)} + \frac{1}{4}h_i(x - x_{i+1})\alpha^{(i+1,j)}\right] = \\
&= S_{i,j-1}(x, y_j),
\end{aligned}
\tag{2.11}
$$

and if $x = x_i$, $i = \overline{1, n_1 - 1}$, $y_j \leq y \leq y_{j+1}$, $j = \overline{0, n_2 - 1}$, then in consequence of (2.3), (2.4) and (2.5) we have the equality

$$
S_\Delta(x_i, y) = S_{i,j}(x_i, y) = S_{i+1,j}(x_i, y). \tag{2.12}
$$

From (2.11) and (2.12) we obtain that the function S_Δ is continuous in the closed domain D, i.e. $S_\Delta \in C(D)$. However, the partial derivatives of $S_\Delta(P)$ are not continuous at the interpolation nodes $P_{i,j}$.

S_Δ is not a modified spline function of type (0,2), because at the interpolation nodes $P_{i,j}$ $i = \overline{0, n_1}$, $j = \overline{0, n_2}$ the second order partial derivatives of S_Δ interpolate the values

$$
\frac{1}{2}\left(\alpha^{(i,j)} + \alpha^{(i+1,j)}\right), \quad \text{respectively} \quad \frac{1}{2}\left(\beta^{(i,j)} + \beta^{(i,j+1)}\right)
$$

$$
(i = \overline{0, n_1 - 1}, \quad j = \overline{0, n_2 - 1}),
$$

instead of the values $\alpha_{i,j}$, $\beta_{i,j}$.

We verify that S_Δ in (2.3) is a unique modified spline function of type (0,2) with the properties that in every subdomain $D_{i,j}$, $i = \overline{0, n_1 - 1}$, $j = \overline{0, n_2 - 1}$ it is identical with a second degree polynomial in x and y variables including no x^2y^2 term and satisfies the equalities (2.6) and (2.10).

Namely, let us suppose there is another spline function S^*_Δ satisfying the equalities (2.6), (2.9), (2.10). Then every difference is of the form

$$
\begin{aligned}
&S_\Delta(P) - S^*_\Delta(P) = S_{i,j}(P) - S^*_{i,j}(P) = \\
&= c_1 + c_2x + c_3y + c_4xy + c_5x^2 + c_6y^2 + c_7xy^2 + c_8x^2y.
\end{aligned}
\tag{2.13}
$$

However, since according to the assumption and by (2.9) we have

$$\left.\frac{\partial^2 S_{i,j}(P)}{\partial x^2}\right|_{y=y_j} - \left.\frac{\partial^2 S^*_{i,j}(P)}{\partial x^2}\right|_{y=y_j} = 0,$$

$$\left.\frac{\partial^2 S_{i,j}(P)}{\partial x^2}\right|_{y=y_{j+1}} - \left.\frac{\partial^2 S^*_{i,j}(P)}{\partial x^2}\right|_{y=y_{j+1}} = 0,$$

so from (2.13) we obtain

$$2c_5 + 2c_8 y_j = 0$$
$$2c_5 + 2c_8 y_{j+1} = 0,$$

and since $y_{j+1} - y_j = h_j \neq 0$, so $c_8 = c_5 = 0$. Similarly, from (2.10) it comes that $c_7 = c_6 = 0$. By (2.6) and from (2.13) for the coefficients c_i, $i = 1,2,3,4$ we obtain a homogeneous system of linear equations. Its determinant is $D = h_i^2 k_j^2 \neq 0$, this is easy to evaluate. Thus $c_i = 0$, $i = \overline{0,8}$, so in consequence of (2.13) we have $S_\Delta \equiv S^*_\Delta$.

By reason of the foregoing we can assert the following

Theorem 1. *For the given system of interpolation nodes (2.1) and the arbitrary real numbers (2.2), S_Δ in (2.3) is a unique modified spline function of type (0,2) which satisfies equalities (2.6), (2.9) and (2.10).*

3. Approximation theorems

In the closed, finite rectangle domain $D : \{(x,y) : a_1 \leq x \leq b_1, a_2 \leq y \leq b_2\}$ let the function of two variables u be given for which $u \in C^2(D)$, i.e. the functions $u, u_x, u_y, u_{xx}, u_{yy}$ and $u_{xy} = u_{yx}$ in D are continuous.

The modulus of continuity of the functions u_{xx}, u_{yy} and u_{yx} is denoted by $\omega_1(h), \omega_2(h), \omega_3(h)$, respectively, i.e. if $\overline{P_1P_2} \leq h$, $P_1, P_2 \in D$, then

$$(3.1) \qquad \begin{aligned} \omega_1(h) &= \max_h |u_{xx}(P_1) - u_{xx}(P_2)|, \\ \omega_2(h) &= \max_h |u_{yy}(P_1) - u_{yy}(P_2)|, \\ \omega_3(h) &= \max_h |u_{xy}(P_1) - u_{xy}(P_2)|. \end{aligned}$$

Obviously, $\omega_i(h)$, $i = 1,2,3$ are monotone increasing functions, i.e. if $h_2 \geq h_1$, then $\omega_i(h_2) \geq \omega_i(h_1)$. By the continuity if $h \to 0$, then $\omega_i(h) \to 0$.

For the domain D let the system of interpolation nodes

$$(3.2) \qquad \Delta :$$

$$\begin{cases} a_1 = x_0 < x_1 < \cdots < x_i < x_{i+1} < \cdots < x_{n_1-1} < x_{n_1} = b_1, & n_1 = 2, 3, \cdots \\ a_2 = y_0 < y_1 < \cdots < y_j < y_{j+1} < \cdots < y_{n_2-1} < y_{n_2} = b_2, & n_2 = 2, 3, \cdots \end{cases}$$

be given, for which we use the following notations, too:

$$h_i = x_{i+1} - x_i, \quad \max_i h_i = h, \quad \min_i h_i = h^*$$

$$k_j = y_{i+1} - y_i, \quad \max_j k_j = k, \quad \min_j k_j = k^*,$$

and

$$(3.3) \qquad \frac{h}{k^*} = c_1 \qquad \frac{k}{h^*} = c_2, \quad c = \max(c_1, c_2).$$

In the following $D_{i,j}$ $i = \overline{0, n_1 - 1}$, $j = \overline{0, n_2 - 1}$ denotes the subdomains of the closed domain D defined by

$$(3.4) \qquad D_{i,j} : \{(x, y) : x_i \leq x \leq x_{i+1}, \ y_j \leq y \leq y_{j+1}\}.$$

Here after we put the following notations

$$u(x_i, y_j) = u^{(i,j)}, \quad \alpha^{(i,j)} = u_{xx}(x_i, y_j) = u_{xx}^{(i,j)}, \quad \beta^{(i,j)} = u_{yy}(x_i, y_j) = u_{yy}^{(i,j)},$$

$$(3.5) \qquad i = \overline{0, n_1}, \quad j = \overline{0, n_2}.$$

For the system of interpolation nodes (3.2) and for

$$P \in D_{i,j}, \qquad i = \overline{0, n_1 - 1}, \ \ j = \overline{0, n_2 - 1}$$

the modified spline function of type (0,2) $S_\Delta(P, u) = S_\Delta(P)$ corresponding to the function $u(P)$ is defined by (2.3)-(2.5).

If into equations (2.7)-(2.10) we write the values $u_{xx}^{(i,j)}$ instead of $\alpha^{(i,j)}$, $u_{yy}^{(i,j)}$ instead of $\beta^{(i,j)}$, $i = \overline{0, n_1 - 1}$, $j = \overline{0, n_2 - 1}$, then we obtain the following equalities:

$$(3.6) \qquad \frac{\partial^2 S_\Delta(P)}{\partial x^2} \equiv \frac{\partial^2 S_{i,j}(P)}{\partial x^2} =$$

$$= \frac{1}{k_j}\left[(y_{j+1}-y)\frac{u_{xx}^{(i,j)}+u_{xx}^{(i+1,j)}}{2}+(y-y_j)\frac{u_{xx}^{(i,j+1)}+u_{xx}^{(i+1,j+1)}}{2}\right],$$

(3.7)
$$\frac{\partial^2 S_\Delta(P)}{\partial y^2}\equiv\frac{\partial^2 S_{i,j}(P)}{\partial y^2}=$$

$$= \frac{1}{h_i}\left[(x_{i+1}-x)\cdot\frac{u_{yy}^{(i,j)}+u_{yy}^{(i,j+1)}}{2}+(x-x_i)\frac{u_{yy}^{(i+1,j)}+u_{yy}^{(i+1,j+1)}}{2}\right],$$

and

(3.8)
$$\left.\frac{\partial^2 S_\Delta(P)}{\partial x^2}\right|_{y=y_j}=\left.\frac{\partial^2 S_{i,j}(P)}{\partial x^2}\right|_{y=y_j}=\frac{1}{2}\left(u_{xx}^{(i,j)}+u_{xx}^{(i+1,j)}\right),$$
$$\left.\frac{\partial^2 S_\Delta(P)}{\partial x^2}\right|_{y=y_{j+1}}=\left.\frac{\partial^2 S_{i,j}(P)}{\partial x^2}\right|_{y=y_{j+1}}=\frac{1}{2}\left(u_{xx}^{(i,j+1)}+u_{xx}^{(i+1,j+1)}\right);$$

furthermore, we have

(3.9)
$$\left.\frac{\partial^2 S_\Delta(P)}{\partial y^2}\right|_{x=x_i}=\left.\frac{\partial^2 S_{i,j}(P)}{\partial y^2}\right|_{x=x_i}=\frac{1}{2}\left(u_{yy}^{(i,j)}+u_{yy}^{(i,j+1)}\right),$$
$$\left.\frac{\partial^2 S_\Delta(P)}{\partial y^2}\right|_{x=x_{i+1}}=\left.\frac{\partial^2 S_{i,j}(P)}{\partial y^2}\right|_{x=x_{i+1}}=\frac{1}{2}\left(u_{yy}^{(i+1,j)}+u_{yy}^{(i+1,j+1)}\right).$$

We note that also equations similar to (2.11) and (2.12) are satisfied, i.e. $S_\Delta(u) = S_\Delta \in C(D)$. And by Theorem 1 $S_\Delta(P)$ (3.6) is a unique modified spline function satisfying also the required conditions.

In the proofs we will need the finite Taylor expansion of the functions $u_x(P), u_x(P), u_y(P)$, if $P \in D_{i,j}$, $i=\overline{0,n_1-1}$, $j=\overline{0,n_2-1}$:

(3.10)
$$u(P)=u^{(i,j)}+u_x^{(i,j)}(x-x_i)+u_y^{(i,j)}(y-y_j)+$$

$$+u_{xx}(P_1)\frac{(x-x_i)^2}{2}+u_{xy}(P_1)(x-x_i)(y-y_j)+u_{yy}(P_1)\frac{(y-y_j)^2}{2},$$

where

$$P_1\in D_{i,j},\qquad u_x^{(i,j)}=u_x(x_i,y_j),\qquad u_y^{(i,j)}=u_y(x_i,y_j);$$

and

$$u_x(P) = u_x^{(i,j)} + u_{xx}(P_2)(x - x_i) + u_{xy}(P_2)(y - y_j), \tag{3.11}$$

and

$$u_y(P) = u_y^{(i,j)} + u_{yx}(P_3)(x - x_i) + u_{yy}(P_3)(y - y_j), \tag{3.12}$$

where $P_2, P_3 \in D_{i,j}$.

In the following we always have $P_k \in D_{i,j}$, $k = 4, 5, \ldots$

We prove the following

Theorem 2. *Let $u \in C^2(D)$ and denote $S_\Delta(\cdot, u)$ the spline function corresponding to the nodes (3.2) and generated by the function u. Then for all $P \in D$*

$$\left|u_{xx}(P) - \frac{\partial^2 S_\Delta(P)}{\partial x^2}\right| = \left|u_{xx}(P) - \frac{\partial^2 S_{i,j}}{\partial x^2}\right| \le \omega_1(h + k), \tag{3.13}$$

$$\left|u_{yy}(P) - \frac{\partial^2 S_\Delta(P)}{\partial y^2}\right| = \left|u_{xx}(P) - \frac{\partial^2 S_{i,j}}{\partial y^2}\right| \le \omega_2(h + k), \tag{3.14}$$

where $\omega_1(h)$ is the modulus of continuity of the function u_{xx}, and $\omega_2(h)$ is that of the function of u_{yy} in D.

Proof. By (3.6) for $P \in D_{(i,j)}$ we have

$$\left|u_{xx}(P) - \frac{\partial^2 S_{i,j}(P)}{\partial x^2}\right| =$$

$$= \left|u_{xx}(P) - \frac{1}{k_j}\left[(y_{j+1} - y)\frac{u_{xx}^{(i,j)} + u_{xx}^{(i+1,j)}}{2} + (y - y_j)\frac{u_{xx}^{(i,j+1)} + u_{xx}^{(i+1,j+1)}}{2}\right]\right|.$$

Since $u_{xx}(P)$ is continuous in D, we obtain

$$\frac{u_{xx}^{(i,j)} + u_{xx}^{(i+1,j)}}{2} = u_{xx}(P_4), \qquad \frac{u_{xx}^{(i,j+1)} + u_{xx}^{(i+1,j+1)}}{2} = u_{xx}(P_5),$$

where $P_4 = (x_i + t_4 h_i, y_j)$, $P_5 = (x_i + t_5 h_i, y_{j+1})$ and $0 \le t_4, t_5 \le 1$. If $y = y_j + t k_j$, $0 \le t \le 1$, then we get

$$\left|u_{xx}(P) - \frac{\partial^2 S_{i,j}(P)}{\partial x^2}\right| = |u_{xx}(P) - [(1 - t)u_{xx}(P_4) + t u_{xx}(P_5)]| \le$$

$$\leq |u_{xx}(P) - u(P_6)| \leq \omega_1(\overline{PP_6}) \leq \omega_1(h+k).$$

Namely, since $P, P_6 \in D_{i,j}$ we have $\overline{PP_6} \leq \sqrt{h_i^2 + k_j^2} = \delta \leq \sqrt{h^2+k^2} \leq h+k$.

In a similar manner the equation

$$\left| u_{yy}(P) - \frac{\partial^2 S_{i,j}(P)}{\partial y^2} \right| \leq \omega_2(h+k)$$

can be verified by (3.7), since $u_{yy}(P)$ is continuous in D. It is obvious that $\omega_1(h+k)$, $\omega_2(h+k) \to 0$, if $h, k \to 0$, i.e. if $n_1 \to \infty$, $n_2 \to \infty$.

Theorem 3. *Let $u \in C^2(D)$. Then the spline function (2.3) corresponding to the nodes (3.2) satisfies*

$$(3.15) \qquad \left| u_{xx}(P) - \frac{\partial^2 S_\Delta(P)}{\partial x \partial y} \right| = \left| u_{xy}(P) - \frac{\partial^2 S_{i,j}(P)}{\partial x \partial y} \right| \leq (1+2c)\omega(\delta),$$

where $\omega_3(\delta)$ is the modulus of continuity of the function u_{xy}, and $\omega_2(\delta)$ is that of the function u_{yy} in D, furthermore

$$\omega(\delta) = \max[\omega_1(\delta), \omega_2(\delta), \omega_3(\delta)],$$

c is the constant at (3.3).

Proof. By Taylor expansion

$$\begin{aligned} u^{(i+1,j+1)} &= u^{(i,j)} + u_x^{(i,j)} h_i + u_y^{(i,j)} k_j + \\ &\quad + u_{xx}(P_7)\frac{h_i^2}{2} + u_{xy}(P_7) h_i k_j + u_{yy}(P_7)\frac{k_j^2}{2}, \\ u^{(i+1,j)} &= u^{(i,j)} + u_x^{(i,j)} h_i + u_{xx}(P_8)\frac{h_i^2}{2}, \\ u^{(i,j+1)} &= u^{(i,j)} + u_y^{(i,j)} k_j + u_{yy}(P_9)\frac{k_j^2}{2}, \end{aligned}$$

where $P_7, P_8, P_9 \in D_{i,j}$. Differentiating the spline function (2.3) first with respect to x, then to y, by (3.10)-(3.12) after a simple calculation and considering

$$x = x_i + t^* h_i, \quad 0 \leq t^* \leq 1, \qquad y = y_j + t k_j, \quad 0 \leq t \leq 1,$$

we obtain that

$$(3.16) \qquad \frac{\partial^2 S_{i,j}(P)}{\partial x \partial y} =$$

$$= u_{xy}(P_7) + [u_{xx}(P_7) - u_{xx}(P_8)]\frac{h_i}{2k_j} + [u_{yy}(P_7) - u_{yy}(P_9)]\frac{k_j}{2h_i} +$$

$$+\frac{[2x-(x_i+x_{i+1})]}{4k_j}\left[\left(u_{xx}^{(i,j)} - u_{xx}^{(i,j+1)}\right) + \left(u_{xx}^{(i+1,j+1)} - u_{xx}^{(i+1,j)}\right)\right] +$$

$$+\frac{[2y-(y_j+y_{j+1})]}{4h_i}\left[\left(u_{yy}^{(i,j)} - u_{yy}^{(i+1,j)}\right) + \left(u_{yy}^{(i+1,j+1)} - u_{yy}^{(i,j+1)}\right)\right].$$

By (3.16) and (3.3), applying the triangle inequality we have

$$\left|u_{xy}(P) - \frac{\partial^2 S_{i,j}(P)}{\partial x \partial y}\right| \le \omega_3(\delta) + c[\omega_1(\delta) + \omega_2(\delta)]. \tag{3.17}$$

Since we supposed that $u_{xy} = u_{yx}$, so for the functions u_{yx} and $\frac{\partial S_{i,j}}{\partial y \partial y}$ we obtain also inequality (3.17). If

$$\omega(\delta) = \max[\omega_1(\delta), \omega_2(\delta), \omega_3(\delta)],$$

then by (3.17) Theorem 3 holds.

Theorem 4. *Under the conditions of Theorem 2 the following inequalities hold:*

$$\left|u_x(P) - \frac{\partial S_\Delta(P)}{\partial x}\right| = \left|u_x(P) - \frac{\partial S_{i,j}(P)}{\partial x}\right| \le (3+2c)\omega(\delta)\delta, \tag{3.18}$$

and

$$\left|u_y(P) - \frac{\partial S_\Delta(P)}{\partial y}\right| = \left|u_y(P) - \frac{\partial S_{i,j}(P)}{\partial y}\right| \le (3+2c)\omega(\delta)\delta, \tag{3.19}$$

where c is the constant at (3.3).

Proof. Applying the finite Taylor expansion we have the inequality

$$\left|u_x(P) - \frac{\partial S_{i,j}(P)}{\partial x}\right| =$$

$$= \left| \left[u_x - \frac{\partial S_{i,j}}{\partial x}\right]_{\substack{x=x_i\\ y=y_j}} + \left[u_{xx}(P_{10}) - \frac{\partial^2 S_{i,j}}{\partial x^2}\Bigg|_{P_{10}}\right](x-x_i) + \right. \tag{3.20}$$

$$\left. + \left[u_{xy}(P_{10}) - \frac{\partial^2 S_{i,j}}{\partial x \partial y}\Bigg|_{P_{10}}\right](y-y_j)\right|,$$

where $P_{10} \in D_{i,j}$.

Differentiating the function $S_\Delta(P) \equiv S_{i,j}(P)$ with respect to x and substituting $x = x_i$, $y = y_j$, then applying the finite Taylor expansion, we obtain the equality

$$\left.\frac{\partial S_{i,j}}{\partial x}\right|_{\substack{x=x_i\\ y=y_j}} = \frac{1}{h_i}(u^{(i+1,j)} - u^{(i,j)}) - (u_{xx}^{(i+1,j)} + u_{xx}^{(i,j)})\frac{h_i}{4} =$$

$$= u_x^{(i,j)} + [u_{xx}(P_8) - u_{xx}(P_{11})]\frac{h_i}{2}.$$

Substituting this into equality (3.20), then applying the triangle inequality, since

$$x - x_i \le h_i \le h, \qquad y - y_j \le k_j \le k,$$

by Theorems 2 and 3, we have the inequality

$$\left|u_x(P) - \frac{\partial S_{i,j}(P)}{\partial x}\right| \le$$

$$\le 2\omega_1(\delta)h + \omega_3(\delta)k + c[\omega_1(\delta) + \omega_2(\delta)]k = (2h + ck)\omega_1(\delta) + c\omega_2(\delta)k + \omega_3(\delta)k.$$

Since $h, k < \delta = \sqrt{h^2 + k^2}$, and if

$$\omega(\delta) = \max[\omega_1(\delta), \omega_2(\delta), \omega_3(\delta)],$$

we have the inequality

$$\left|u_x(P) - \frac{\partial S_{i,j}(P)}{\partial x}\right| \le \left(\frac{5}{2} + 2c\right)\omega(\delta)\delta.$$

In a similar manner it can be verified that the following inequality holds

$$\left|u_y(P) - \frac{\partial S_{i,j}(P)}{\partial y}\right| \le \left(\frac{5}{2} + 2c\right)\omega(\delta)\delta.$$

Theorem 5. *Under the conditions of Theorem 2 the following inequality holds:*

(3.21) $$|u(P) - S_\Delta(P)| = |u(P) - S_{i,j}(P)| \le (7 + 6c)\omega(\delta)\delta^2,$$

where c is the constant at (3.3).

Proof. Applying the finite Taylor expansion, and since

$$u(x_i, y_j) = u^{(i,j)}, \qquad S_{i,j}(x_i, y_j) = u^{(i,j)}, \qquad P_i = P(x_i, y_j),$$

we obtain that

$$|u(P) - S_{i,j}(P)| = \left| \left[u_x - \frac{\partial S_{i,j}}{\partial x}\right]\Big|_{P_{i,j}} (x - x_i) + \left[u_y - \frac{\partial S_{i,j}}{\partial y}\right]\Big|_{P_{i,j}} (y - y_j) + \right.$$

$$+ \left[u_{xx} - \frac{\partial^2 S_{i,j}}{\partial x^2}\right]\Big|_{P_{12}} \frac{(x - x_i)^2}{2} + \left[u_{xy} - \frac{\partial^2 S_{i,j}}{\partial x \partial y}\right]\Big|_{P_{12}} (x - x_i)(y - y_j) +$$

$$\left. + \left[u_{yy} - \frac{\partial^2 S_{i,j}}{\partial y^2}\right]\Big|_{P_{12}} \frac{(y - x_j)^2}{2} \right|.$$

Applying the triangle inequality and Theorems 2-4 and since

$$x - x_i \le h_i \le h < \delta, \qquad y - y_j \le k_j \le k\delta,$$

we have the inequality

$$|u(P) - S_{i,j}(P)| \le (7 + 6c)\omega(\delta)\delta^2.$$

The convergence follows from Theorems 2-3 if $\delta \to 0$, i.e. if $n_1 \to \infty$, $n_2 \to \infty$.

The convergence is of "Jackson order" if we compare it with the polynomial approximation. The convergence is simultaneous, since the modified spline functions S_Δ simultaneously approximate the functions $u, u_x, u_y, u_{xx}, u_{yy}$ and u_{xy} "Jackson order".

By means of S_Δ an extremely good integral approximation procedure can be given. Namely, by Theorem 5 if $u \in C^2(D)$, then the following inequality holds:

$$\left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} [u(x, y)dydx - S_\Delta(x, y)]\, dydx \right| \le$$

$$\le \int_{a_1}^{b_1} \int_{a_2}^{b_2} |u(x, y) - S_\Delta(x, y)|\, dydx \le \tag{3.22}$$

$$\le (b_1 - a_1)(b_2 - a_2)(7 + 6c)\omega(\delta)\delta^2.$$

For the sake of simplicity let us put $a_1 = a_2 = 0$, let the system of interpolation nodes be of equal distances, i.e.

$$x_i = i\frac{b_1}{n_1}, \qquad i = \overline{0, n_1}, \qquad n_1 = 2, 3, \cdots ,$$

$$y_j = j\frac{b_2}{n_2}, \qquad j = \overline{0, n_2}, \qquad n_2 = 2, 3, \cdots .$$

In case of this system of interpolation nodes, integrating the function $S_\Delta(P) = = S_\Delta(x, y)$ (2.3) and since

$$x_{i+1} - x_i = h = \frac{b_1}{n_1}, \qquad y_{j+1} - y_j = k = \frac{b_2}{n_2},$$

we obtain that

$$\int_0^{b_1}\int_0^{b_2} S_\Delta(x, y) dy dx =$$

$$\frac{b_1 b_2}{4n_1 n_2} \sum_{i=0}^{n_1-1} \Big\{ \sum_{j=0}^{n_2-1} \left[u^{(i,j)} + u^{(i+1,j)} + u^{(i,j+1)} + u^{(i+1,j+1)} \right] -$$

$$(3.23) \qquad -\frac{b_1^2}{12n_1^2} \left[u_{xx}^{(i,j)} + u_{xx}^{(i+1,j)} + u_{xx}^{(i,j+1)} + u_{xx}^{(i+1,j+1)} \right] -$$

$$-\frac{b_2^2}{12n_2^2} \left[u_{yy}^{(i,j)} + u_{yy}^{(i+1,j)} + u_{yy}^{(i,j+1)} + u_{yy}^{(i+1,j+1)} \right] \Big\}.$$

If the system of interpolation nodes is not of equal distances, then (3.23) is a little more complicated, but the integration is easy, because the function S_Δ is simple.

Example 1. Let us put

$$u = u(x, y) = xe^y, \qquad a_1 = a_2 = 0, \quad b_1 = b_2 = 1, \quad h = k = \frac{1}{10},$$

then by (3.23) the following can be easily evaluated

$$\int_0^1\int_0^1 S_\Delta(x, y) dy dx =$$

$$= \left(\frac{1}{40} - \frac{1}{48 \cdot 10^3}\right)(1 + e^{0,1})\left(1 + \sum_{j=1}^{9} e^{0,j}\right) = \underline{0,85914019}.$$

On the other hand

$$\int_0^1 \int_0^1 xe^y \, dydx = \frac{1}{2}(e - 1) = \underline{0.8591408} \cdots .$$

Thus, the approximation is extremely good.

By Theorem 2 the approximation of the Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{3.24}$$

of a function $u \in C^2(D)$ becomes possible. If the Laplacian of the spline function $S_\Delta(P; u) = S_\Delta(P)$ (2.3) belonging to the function u is

$$\Delta S_\Delta(P) = \frac{\partial^2 S_\Delta(P)}{\partial x^2} + \frac{\partial^2 S_\Delta(P)}{\partial y^2} = \frac{\partial^2 S_{i,j}(P)}{\partial x^2} + \frac{\partial^2 S_{i,j}(P)}{\partial y^2}$$

$$P \in D_{i,j}, \qquad i = \overline{0, n_1 - 1}, \quad j = \overline{0, n_2 - 1}. \tag{3.25}$$

Then by Theorem 2 we have the inequality

$$|\Delta(u - S_\Delta)| \le$$

$$\le \left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 S_{i,j}(P)}{\partial x^2}\right| + \left|\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 S_{i,j}(P)}{\partial y^2}\right| \le$$

$$\le \omega_1(\delta) + \omega_2(\delta).$$

Example 2. If $u(x, y) = \frac{1}{2}x^2 e^y$, then

$$\Delta u(x, y) = \left(1 + \frac{x^2}{2}\right) e^y,$$

and its value at $P = P\left(\frac{19}{20}, \frac{19}{20}\right)$

$$\Delta u \Big|_{\left(x = \frac{19}{20}, y = \frac{19}{20}\right)} = 3,752511202 \cdots .$$

Let

$$D = \{0 \le x \le 1, \ 0 \le y \le 1\}, \qquad x_i = i\frac{1}{10}, \ y_j = j\frac{1}{10}, \ i,j = \overline{0,10},$$

and $h = k = 0,1$.

In case of this system of approximation nodes the value of the Laplace operator evaluated with the spline function S_Δ belonging to the function u at $P = \left(\frac{19}{20}, \frac{19}{20}\right)$ is

$$\Delta S_\Delta\left(\frac{19}{20}, \frac{19}{20}\right) = \underline{3,75}72073.$$

This is a quite good approximation, since the evaluation was made with the polynomial $S_\Delta(P) \equiv S_{9,9}(x,y)$ belonging to the last subdomain

$$D_{9,9} = \{(x,y): \ 0,9 \le x \le 1, \ 0,9 \le y \le 1\}.$$

It is obvious that here it is where the approximation is the worst. If the value of $h = k$ is smaller, e.g. $h = k = 0,01$, then the approximation is much better.

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N.A.A. Rahman
Department of Numerical Analysis
Eötvös Loránd University
VIII. Múzeum krt. 6-8.
H-1088 Budapest, Hungary