Annales Univ. Sci. Budapest., Sect. Comp. 16 (1996) 283-289

A_p–WEIGHTS IN MECHANICS

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Dedicated to Professor János Balázs on his 75-th birthday

1. Introduction

Weights satisfying the so-called A_p-condition introduced by R.Hunt, B.Muckenhoupt and R.Wheeden [2] play important role in many fields of mathematical analysis, e.g. in Fourier analysis, theory of operators, approximation theory and differential equations (see [1-4]). In order to generalize a Riesz theorem concerning the trigonometric conjugate operator $Tf = \tilde{f}$ H.Helson and G.Szegő [1] have found that T is bounded in a weighted space $L_v^2[2\pi]$ if and only if

$$v(x) = \exp\{u_1(x) + \tilde{u}_2(x)\}, \tag{1}$$

where u_1 and u_2 are 2π-periodic functions, whose sup-norms satisfy $\|u_1\| < \infty$ and $\|u_2\| < \pi/2$. Later it turns out that weights given by (1) coincide with the weights satisfying the A_2-condition, a special case of A_p-condition $(1 < p < \infty)$ introduced by [2]. We remember here the definition of A_p-condition. Weights (in this paper) mean arbitrary functions defined on a finite or infinite interval (a, b) which are positive and finite a.e. on (a, b). Let $1 < p < \infty$. We say that a weight $m(t)$ satisfies the A_p-condition on (a, b) if the inequality

$$\left(\frac{1}{d-c}\int_c^d m(t)dt\right)\left(\frac{1}{d-c}\int_c^d [m(t)]^{-\frac{1}{p-1}}dt\right)^{p-1} \leq M \tag{2}$$

holds for all finite subintervals (c, d) of (a, b), M is a constant depending only on p. The class of such weights will be denoted by $A_M^p(a, b)$.

The aim of this paper is to characterize the A_p-weights by using some fundamental quantities of mechanics. Our result shows that the A_p-condition has close connection with the fundamental law of mechanics $L = \frac{mv^2}{2}$.

2. A_p-weights and mechanical systems

All preliminaries on the Hamilton-principle used in the future can be found for example in [5].

Let $m(t)$ be a weight on (a, b) and let $1 < p < \infty$. Assume that for any finite subinterval (c, d) of (a, b)

$$\int_c^d m^{-\frac{1}{p-1}}(t)dt < \infty. \tag{3}$$

Let $a < t_0 < b$ be a given number. Consider the mechanical system

$$\{L, q\} \tag{4}$$

given by

$$q(t) = \int_{t_0}^{t} [m(z)]^{\frac{1}{p}\left(\frac{p'}{p}-1\right)} dz, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \quad t \in (a, b), \tag{5}$$

and the Lagrange function

$$L = L(q, \dot q, t) = |\dot q(t)|^p m(t). \tag{6}$$

Using the Hölder inequality, from (3) we have that the integral on the right-hand side of (5) is finite.

In order to see that (4) with (5) and (6) is really a mechanical system we have to show that it satisfies the Hamilton principle, which means in this case that for any $a < t_1 < t_2 < b$

$$\min_{g \in AC[t_1, t_2],\ g(t_1) = q(t_1),\ g(t_2) = q(t_2)} \int_{t_1}^{t_2} |\dot g(t)|^p m(t)dt = \int_{t_1}^{t_2} |\dot q(t)|^p m(t)dt, \tag{7}$$

where $AC[t_1, t_2]$ denotes the class of all functions absolutely continuous on $[t_1, t_2]$. In other words, the function $q(t)$ is a solution of the min-problem given on the left-hand side of (7). We shall show that $q(t)$ is the unique solution of (7).

Indeed, let $g \in AC[t_1, t_2]$ be an arbitrary function satisfying $g(t_1) = q(t_1)$ and $g(t_2) = q(t_2)$. We prove if $g \not\equiv q$, i.e. there exists a $t^* \in (t_1, t_2)$ such that

$$g(t^*) \neq q(t^*), \tag{8}$$

then

$$\int_{t_1}^{t_2} |\dot{g}(t)|^p m(t) > \int_{t_1}^{t_2} |\dot{q}(t)|^p m(t)dt. \tag{9}$$

As we shall see, from (8) follows the existence of a set $E \subset (t_1, t_2)$ having Lebesgue-measure $|E| > 0$ such that

$$\dot{g}(t) \neq \dot{q}(t) \qquad (t \in E). \tag{10}$$

Indeed, suppose that

$$\dot{g}(t) = \dot{q}(t) \qquad \text{for almost all} \quad t \in (t_1, t_2). \tag{11}$$

Then from

$$g(t) = \int_{t_1}^{t} \dot{g}(z)dz + \lambda \qquad (t \in [t_1, t_2]),$$

$$q(t) = \int_{t_1}^{t} \dot{q}(z)dz + \mu \qquad (t \in [t_1, t_2])$$

with some constants λ and μ, we have

$$g(t) - q(t) = \lambda - \mu \qquad (t \in [t_1, t_2]).$$

Setting $t = t_1$ and using the condition $g(t_1) = q(t_1)$ we get $\lambda = \mu$, hence $g(t) = q(t)$ for all $t \in [t_1, t_2]$, which contradicts to (8). We have proved (10).

Now by (5) using the Hölder equality (see [7]) we have

$$q(t_2) - q(t_1) = \int_{t_1}^{t_2} \dot{q}(t) m^{1/p}(t) m^{-1/p}(t)dt = \int_{t_1}^{t_2} m^{-p'/p^2}(t) m^{-1/p}(t)dt =$$

$$(12)\qquad = \left(\int_{t_1}^{t_2} m^{-p'/p}(t)dt\right)^{1/p} \left(\int_{t_1}^{t_2} m^{-p'/p}(t)dt\right)^{1/p'} =$$

$$= \left(\int_{t_1}^{t_2} [q(t)]^p m(t)dt\right)^{1/p} \left(\int_{t_1}^{t_2} m^{-p'/p}(t)dt\right)^{1/p'}$$

We estimate the analogous integral of g. For this purpose let us introduce

$$h(t) := |g(t)| m^{1/p}(t), \qquad k(t) := m^{-1/p}(t).$$

From (5) and (10) for some $E' \subset [t_1, t_2]$, $|E'| > 0$, we have

$$(13)\qquad h^p(t) \neq k^{p'}(t) \qquad (t \in E').$$

Hence by the Hölder inequality (the case of strict inequality) we get

$$(14)\qquad q(t_2) - q(t_1) = g(t_2) - g(t_1) = \int_{t_1}^{t_2} g(t)dt \leq$$

$$\leq \int_{t_1}^{t_2} |g(t)|dt = \int_{t_1}^{t_2} h(t)k(t)dt < \left(\int_{t_1}^{t_2} h^p(t)dt\right)^{1/p} \left(\int_{t_1}^{t_2} k^{p'}(t)dt\right)^{1/p'} =$$

$$= \left(\int_{t_1}^{t_2} |g(t)|^p m(t)dt\right)^{1/p} \left(\int_{t_1}^{t_2} m^{-p'/p}(t)dt\right)^{1/p'}$$

Combining (12) and (14) we have (9).

Remark. When $m(t)$ is continuously differentiable on (a, b) we can prove (7) by using Euler's method. If $m(t)$ is not differentiable, then the min-problem on left side of (7), as we showed above, has also solution, but there is no Euler's equation. In this case (4) with (5) and (6) may also be considered as a mechanical system in the sense of the Hamilton principle.

Now let us turn to the main problem. We want to answer the question: what does A_p-condition mean in mechanics?

a) Consider first the simple case when $m(t) = m_0/2$, $p = 2$, where m_0 is a positive constant. Remark that in this case $m(t)$ satisfies the A_p-condition. Here we use the notation $\{L_0, q_0\}$ instead of $\{L, q\}$. By (5) we have

$$q_0\,(t) = \text{constant} \qquad (:= v) \tag{15}$$

and

$$L_0 := L(\dot{q}_0, t) = \frac{m_0 v^2}{2}. \tag{16}$$

The equality (16) states a fundamental law of mechanics.

b) The general case

Put

$$S(t_1, t_2) := \int_{t_1}^{t_2} L(q, t)dt. \tag{17}$$

Let furthermore

$$S(t) := \begin{cases} S(t_1, t) & \text{for } t > t_1, \\ S(t, t_1) & \text{for } t < t_1. \end{cases} \tag{18}$$

We introduce the following average quantities

$$\begin{cases} S_h(t) = \dfrac{S(t+h) - S(t)}{h}, \\ m_h(t) = \dfrac{1}{h} \displaystyle\int_t^{t+h} m(z)dz, \\ v_h(t) = \dfrac{q(t+h) - q(t)}{h} \end{cases} \tag{19}$$

$$(t \in (a,b), \quad h > 0, \quad h + t \in (a,b)).$$

The A_p-weights can be characterized by using the physical quantities (19). Since the following statement is a generalization of the fundamental law presented in Part a), it may be called an A_p-law.

Theorem (A_p-law). *Assume that $m(t)$ satisfies the condition (3). Let $\{L, q\}$ be the mechanical system given by (5) and (6). Then the following statements are equivalent:*

(*i*) $$m \in \mathcal{A}^p_M(a,b);$$

(ii) For any $t \in (a,b)$, $h > 0$, $t + h \in (a,b)$

(20) $$S_h(t) \geq M^{-1} m_h(t) v_h^p(t).$$

Specially, $m \in \mathcal{A}^2_1$ iff

(21) $$S_h(t) \geq m_h(t) v_h^2(t).$$

Proof. By (5), (6) and (19)

(22) $$S_h(t) = \frac{1}{h} \int_t^{t+h} m^{-p'/p}(t)dt.$$

Therefore (20) is equivalent to

(23)
$$\frac{1}{h}\int_t^{t+h} m^{-p'/p}(z)dz \geq M^{-1}\left(\frac{1}{h}\int_t^{t+h} m(z)dz\right)\left(\frac{1}{h}\int_t^{t+h}[m(z)]^{-p'/p^2-1/p}dz\right)^p$$

By (12) we get

(24) $$\int_t^{t+h}[m(z)]^{-p'/p^2-1/p}dz = \int_t^{t+h}[m(z)]^{-p'/p}dz.$$

This implies that (23) is equivalent to

$$\left(\frac{1}{h}\int_t^{t+h} m(z)dz\right)\left(\frac{1}{h}\int_t^{t+h}[m(z)]^{-\frac{1}{p-1}}dz\right)^{p-1} \leq M,$$

which is just the inequality defining A_p-condition. We have proved our theorem.

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