Annales Univ. Sci. Budapest., Sect. Comp. 16 (1996) 263-272

ON A MODIFIED BI–QUADRATIC SPLINE FUNCTION OF HERMITE–TYPE

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Dedicated to Professor J. Balázs on his 75-th birthday

Abstract. In two dimensional case on a rectangular subdivision of a rectangular domain there are given several biquadratic, rational biquadratic and bicubic spline constructions (e.g. [1], [3], [6], [7], [8], [9]). The objective of this paper is to define a continuous quadratic spline function (i.e. a piecewise quadratic polynomial in each variable) which interpolates the unknown function at the knots and gives the best order of approximation assured by the Jackson theory. One of the main purpose of this paper was to decrease the number of operations and to get a formula for the approximant as simple as possible. In the construction we use only the function values and the first order partial derivatives at the knots. However the spline function is only continuous, also its partial derivatives approximate the partial derivatives of the unknown function on the subrectangles of the subdivision with the best order of approximation depending on the smoothness of the function. Comparing with the methods given in [3], [4], this spline construction needs fewer algebraic operations while the order of approximation is the same.

In the first part of this paper we give the construction of the spline function and in the second part approximation theorems are proved.

1. Construction of the spline function

Let $\{(x_i, y_j), 0 \le i \le N, 0 \le j \le M\}$ be a subdivision of $\Omega = [a_1, b_1]\times$ $\times[a_2, b_2]$ with $h_i = x_{i+1} - x_i$, $l_j = y_{j+1} - y_j$, furthermore let $\{u_{i,j}\}$, $\{\alpha_{i,j}\}$

Research supported by Kuwait University Grant SM-127.

and $\{\beta_{i,j}\}$ $(i = 0, \dots, N,\ j = 0, \dots, M)$ be given systems of real numbers. Let $h = \max_{0 \le i \le N} h_i$, $l = \max_{0 \le j \le M} l_j$, and let $d = \sqrt{h^2 + l^2}$ denote the diameter of the subdivision. Furthermore $\omega(h; u)$ denotes the modulus of continuity of the function u.

We are to determine the *modified bi-quadratic spline function S of reduced Hermite-type* in two variables with the following properties:

$$S(x,y) = S_{i,j}(x,y) = \sum_{\substack{\alpha,\beta=0 \\ \alpha+\beta\le 3}}^{2} A_{i,j}^{(\alpha,\beta)} (x - x_i)^\alpha (y - y_j)^\beta \tag{1.1}$$

for $(x, y) \in \Omega_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$,

$$S(x_i, y_j) = u_{i,j} \tag{1.2}$$

for $i = 0, \dots, N$, $j = 0, \dots, M$, and

$$\begin{aligned} \partial_1^2 S_{i,j}(x, y_p) &= \frac{\alpha_{i+1,p} - \alpha_{i,p}}{h_i} p = j, j+1, \\ \partial_2^2 S_{i,j}(x_q, y) &= \frac{\beta_{q,j+1} - \beta_{q,j}}{l_j} q = i, i+1. \end{aligned} \tag{1.3}$$

Theorem 1.1. *There exists a unique modified bi-quadratic spline function of reduced Hermite-type and it is continuous on Ω.*

Proof. Introducing the notations $t = \dfrac{x - x_i}{h_i}$ and $v = \dfrac{y - y_j}{l_j}$ let for $(x, y) \in \Omega_{i,j}$

$$S(x,y) = S_{i,j}(x,y) = (1 - v)(1 - t)\left(u_{i,j} + \frac{1}{2}[\alpha_{i,j} h_i t + \beta_{i,j} l_j v]\right) +$$

$$\begin{aligned} &+ v(1 - t)\left(u_{i,j+1} + \frac{1}{2}[\alpha_{i,j+1} h_i t + \beta_{i,j+1} l_j (v - 1)]\right) + \\ &+ (1 - v)t\left(u_{i+1,j} + \frac{1}{2}[\alpha_{i+1,j} h_i (t - 1) + \beta_{i+1,j} l_j v]\right) + \end{aligned} \tag{1.4}$$

$$+vt\left(u_{i+1,j+1} + \frac{1}{2}[\alpha_{i+1,j+1} h_i (t - 1) + \beta_{i+1,j+1} l_j (v - 1)]\right).$$

By an easy calculation we can verify that this function S satisfies the conditions (1.1)-(1.3). Also the uniqueness can be shown easily in a standard way.

Using the construction (1.4) we can see that

$$S_{i,j}(x, y_{j+1}) = S_{i,j+1}(x, y_{j+1}) \quad x_i \leq x \leq x_{i+1}$$

and

$$S_{i,j}(x_{i+1}, y) = S_{i+1,j}(x_{i+1}, y) \quad y_j \leq y \leq y_{j+1},$$

which proves the continuity of the spline function.

2. Approximation properties of the spline function

Lemma 2.1. *Let $f : [x_i, x_{i+1}] \to \mathbf{R}$ be differentiable and for $x \in [x_i, x_{i+1}]$*

$$S_i(x) = \frac{x_{i+1} - x}{h}\left[f_i + f_i'\frac{x - x_i}{2}\right] + \frac{x - x_i}{h}\left[f_{i+1} + f_{i+1}'\frac{x - x_{i+1}}{2}\right], \tag{2.1}$$

where $h = x_{i+1} - x_i$, $f_i = f(x_i)$, $f_i' = f'(x_i)$.
If $f \in C^1([x_i, x_{i+1}])$, then

$$|f(x) - S_i(x)| \leq \frac{3}{8}h\omega(h; f'),$$
$$|f'(x) - S_i'(x)| \leq \frac{3}{2}\omega(h; f'),$$

if $f \in C^2([x_i, x_{i+1}])$, then

$$|f(x) - S_i(x)| \leq \frac{1}{27}h^2\omega(h; f''),$$
$$|f'(x) - S_i'(x)| \leq \frac{1}{4}h\omega(h; f''),$$
$$|f''(x) - S_i''(x)| \leq \omega(h; f'')$$

for all $x \in [x_i, x_{i+1}]$, where $\omega(h; f)$ denotes the modulus of continuity of the function f.

Proof. Introducing the notation $v = \dfrac{x - x_i}{h}$ we can rewrite (2.1) in the following form

$$S_i(x) = (1-v)f_i + vf_{i+1} + v(v-1)(f'_{i+1} - f'_i)\frac{h}{2},$$

and

$$(2.2)\quad S_i(x) - f(x) = [f_i - f(x)](1-v) + [f_{i+1} - f(x)]v + [f'_{i+1} - f'_i]v(v-1)\frac{h}{2}.$$

If $f \in C^1([x_i, x_{i+1}])$, then by the Lagrangian theorem on the interval $[x_i, x_{i+1}]$

$$f(x) = f_i + f'(\xi_1)vh,$$

$$f(x) = f_{i+1} + f'(\xi_2)(v-1)h,$$

where $\xi_1, \xi_2 \in (x_i, x_{i+1})$, hence

$$R(x) = S_i(x) - f(x) = hv(1-v)\Big([f'(\xi_2) - f'(\xi_1)] + [f'_i - f'_{i+1}]/2\Big),$$

and

$$|S_i(x) - f(x)| \le hv(1-v)\frac{3}{2}\omega(h; f') \le \frac{3}{8}h\omega(h; f').$$

Furthermore for the derivative we get

$$|R'(x)| = |S'_i(x) - f'(x)| =$$

$$= \left|[(f_{i+1} - f_i)/h - f'(x)] + \frac{2v-1}{2}[f'_{i+1} - f'_i]\right| \le \frac{3}{2}\omega(h; f').$$

If $f \in C^2([x_i, x_{i+1}])$, then by the second order Taylor formula for $x \in$ $\in [x_i, x_{i+1}]$

$$f_i - f(x) = -vhf'(x) + \int_x^{x_i} (x_i - s)f''(s)ds,$$

$$f_{i+1} - f(x) = (1-v)hf'(x) + \int_x^{x_{i+1}} (x_{i+1} - s)f''(s)ds,$$

and

$$f_i' = f'(x) + \int_x^{x_i} f''(s)ds,$$

and substituting into (2.2) we have

$$R(x) =$$

$$= \int_{x_i}^{x} (1-v)\left[s - x_i - v\frac{h}{2}\right] f''(s)ds + \int_{x}^{x_{i+1}} v\left[x_{i+1} - s + (v-1)\frac{h}{2}\right] f''(s)ds =$$

$$(2.3) \qquad = h^2 \left[\int_0^v \psi_1(v,\tau) f''(x_i + \tau h)d\tau + \int_v^1 \psi_2(v,\tau) f''(x_i + \tau h)d\tau\right],$$

where $\tau = (s - x_i)/h$ and

$$\psi_1(v,\tau) = (1-v)\left(\tau - \frac{v}{2}\right),$$
$$\psi_2(v,\tau) = v\left(\frac{v+1}{2} - \tau\right).$$

The functions ψ_1 and ψ_2 as functions in τ change the sign only at $\tau = \tau^* = v/2$ and $\tau = \tau^{**} = (v+1)/2$, respectively. Applying the mean value theorem on the appropriate subintervals

$$R(x) = h^2 \Bigg[f''(\zeta_1) \int_0^{\tau^*} \psi_1(v,\tau)d\tau + f''(\zeta_2) \int_{\tau^*}^{v} \psi_1(v,\tau)d\tau +$$

$$+ f''(\zeta_3) \int_v^{\tau^{**}} \psi_2(v,\tau)d\tau + f''(\zeta_4) \int_{\tau_{**}}^{1} \psi_2(v,\tau)d\tau \Bigg] =$$

$$= h^2 \left[f''(\zeta_2) - f''(\zeta_1)\right] \frac{v^2(1-v)}{8} + h^2 \left[f''(\zeta_3) - f''(\zeta_4)\right] \frac{v(1-v)^2}{8},$$

where $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in (t_i, t_{i+1})$. At last

$$|R(x)| \le \frac{1}{27} h^2 \omega(h; f'').$$

In order to approximate the derivatives we differentiate (2.2) with respect to x by the rule for parametric integrals

$$R'(x) = hf''(x)\left[\psi_1(v,v) - \psi_2(v,v)\right] +$$

$$+h\left[\int_0^v \frac{\partial\psi_1}{\partial v}(v,\tau)f''(x_i+\tau h)d\tau + \int_v^1 \frac{\partial\psi_2}{\partial v}(v,\tau))f''(x_i+\tau h)d\tau\right] =$$

$$= h\left[\int_0^v \left(v-\tau-\frac{1}{2}\right) f''(x_i+\tau h)d\tau + \int_v^1 \left(v-\tau+\frac{1}{2}\right) f''(x_i+\tau h)d\tau\right].$$

If $v < \frac{1}{2}$, the function $\frac{\partial\psi_1}{\partial v}$ does not change sign for $\tau \in [0,v]$ and $\frac{\partial\psi_2}{\partial v}$ changes the sign only at the point $\tau = \bar{\tau} = v + 1/2$, hence we can use the mean value theorem on the appropriate subintervals

$$R'(x) = h\left[f''(\xi)\int_0^v \left(v-\tau-\frac{1}{2}\right)d\tau +\right.$$

$$\left. +f''(\xi_1)\int_v^{\bar{\tau}} \left(v-\tau+\frac{1}{2}\right)d\tau + f''(\xi_2)\int_{\bar{\tau}}^1 \left(v-\tau+\frac{1}{2}\right)d\tau\right] =$$

$$= \left[\frac{v(v-1)}{2}f''(\xi) + \frac{1}{8}f''(\xi_1) - \frac{1}{2}\left(v-\frac{1}{2}\right)f''(\xi_2)\right] =$$

$$= h\left[\frac{v(v-1)}{2}(f''(\xi) - f''(\xi_2)) + \frac{1}{8}(f''(\xi_1) - f''(\xi_2))\right],$$

where $\xi, \xi_1, \xi_2, \in (x_i, x_{i+1})$. Hence for $x \in [x_i, x_i + h/2]$

$$|R'(x)| \le \frac{1}{4}h\omega(h; f'').$$

In a similar way we get the same estimation for $x \in [x_i + h/2, x_{i+1}]$.

For the estimation of $|R''(x)|$ we get immediately

$$|R''(x)| = \left|\frac{1}{h}(f'_{i+1} - f'_i) - f''(x)\right| = |f''(\eta_i) - f''(x)| \le \omega(h; f''),$$

where $\eta_i \in (x_i, x_{i+1})$ and the lemma is proved.

Despite the fact, that the above defined spline function S is only continuous on Ω, we can define its partial derivatives on the subrectangles of the subdivision. In what follows, we will use the notation $\partial_1 S(x,y) = \partial_1 S_{i,j}(x,y)$ for $(x,y) \in \Omega_{i,j}$, etc.

Theorem 2.2. *If $u \in C^{1,1}(\Omega)$, and $u_{i,j} = u(x_i, y_j)$, $\alpha_{i,j} = \partial_1 u(x_i, y_j)$, $\beta_{i,j} = \partial_2 u(x_i, y_j)$ for $i = 0, \dots, N, j = 0, \dots, M$, then*

$$|u(x,y) - S(x,y)| \le \frac{3}{8}\Big(h\omega(h;\partial_1 u) + l\omega(l;\partial_2 u)\Big),$$

$$|\partial_1 u(x,y) - \partial_1 S(x,y)| \le \frac{3}{2}\omega(d;\partial_1 u) + \frac{1}{4}\bar{\delta}\omega(d;\partial_2 u),$$

$$|\partial_2 u(x,y) - \partial_2 S(x,y)| \le \frac{3}{2}\omega(d;\partial_2 u) + \frac{1}{4}\bar{\bar{\delta}}\omega(d;\partial_1 u),$$

where $\bar{\delta} = \max\limits_{i,j} \dfrac{l_j}{h_i}$, $\bar{\bar{\delta}} = \max\limits_{i,j} \dfrac{h_i}{l_j}$.

Proof. For $(x,y) \in \Omega_{i,j}$ we can write (1.4) in the following form

$$S_{i,j}(x,y) = \tag{2.3}$$

$$= (1-v)\left[(1-t)\left(u_{i,j} + \frac{1}{2}\alpha_{i,j}h_i t\right) + t\left(u_{i+1,j} + \frac{1}{2}\alpha_{i+1,j}h_i(t-1)\right)\right] +$$

$$+v\left[(1-t)\left(u_{i,j+1} + \frac{1}{2}\alpha_{i,j+1}h_i t\right) + t\left(u_{i+1,j+1} + \frac{1}{2}\alpha_{i+1,j+1}h_i(t-1)\right)\right] +$$

$$+(1-v)v\frac{l_j}{2}\Big[(1-t)(\beta_{i,j} - \beta_{i,j+1}) + t(\beta_{i+1,j} - \beta_{i+1,j+1})\Big],$$

where $t = (x - x_i)/h_i$, $v = (y - y_j)/l_j$.

As $u \in C^{(1,1)}$, we can use the Taylor formula and applying Lemma 2.1 we have

$$|S_{i,j}(x,y) - u(x,y)| \le \frac{3}{8}h_i\omega(h_i;\partial_1 u) +$$

$$+\Big|(1-v)\big[u(x,y_j) - u(x,y)\big] + v\big[u(x,y_{j+1}) - u(x,y)\big]\Big| +$$

$$+(1-v)v\frac{l_j}{2}\Big|(1-t)(\beta_{i,j} - \beta_{i,j+1}) + t(\beta_{i+1,j} - \beta_{i+1,j+1})\Big| \le$$

$$\le \frac{3}{8}h_i\omega(h_i;\partial_1 u) + (1-v)vl_j\omega(l_j;\partial_2 u) + (1-v)v\frac{l_j}{2}\omega(l_j;\partial_2 u) \le$$

$$\le \frac{3}{8}\Big(h_i\omega(h_i;\partial_1 u) + l_j\omega(l_j;\partial_2 u)\Big).$$

Now we differentiate (2.3) with respect to y and applying Lemma 2.1 for $(x,y)\in$ $\in \Omega_{i,j}$ we have

$$|\partial_2 S_{i,j}(x,y)-\partial_2 u(x,y)|=$$

$$=\left|-\frac{1}{l_j}\left[(1-t)u_{i,j}+tu_{i+1,j}+\frac{t(t-1)}{2}h_i(\alpha_{i+1,j}-\alpha_{i,j})\right]+\right.$$

$$+\frac{1}{l_j}\left[(1-t)u_{i,j+1}+tu_{i+1,j+1}+\frac{t(t-1)}{2}h_i(\alpha_{i+1,j+1}-\alpha_{i,j+1})\right]+$$

$$\left.+\frac{1-2v}{2}\left[(1-t)(\beta_{i,j}-\beta_{i,j+1})+t(\beta_{i+1,j}-\beta_{i+1,j+1})\right]-\partial_2 u(x,y)\right|=$$

$$=\left|(1-t)\frac{u_{i,j+1}-u_{i,j}}{l_j}+t\frac{u_{i+1,j+1}-u_{i+1,j}}{l_j}-\partial_2 u(x,y)\right|+$$

$$+\frac{t(1-t)}{2}\frac{h_i}{l_j}\left|\alpha_{i+1,j+1}-\alpha_{i,j+1}-\alpha_{i+1,j}+\alpha_{i,j}\right|+$$

$$+\left|\frac{1-2v}{2}\right|\left|(1-t)(\beta_{i,j}-\beta_{i,j+1})+t(\beta_{i+1,j}-\beta_{i+1,j+1})\right|\le$$

$$\le\omega(d;\partial_2 u)+\frac{1}{4}\frac{h_i}{l_j}\omega(d;\partial_1 u)+\frac{1}{2}\omega(d;\partial_2 u),$$

which proves the theorem.

Theorem 2.3. *If* $u\in C^{2,2}(\Omega)$, *and* $u_{i,j}=u(x_i,y_j)$, $\alpha_{i,j}=\partial_1 u(x_i,y_j)$, $\beta_{i,j}=\partial_2 u(x_i,y_j)$ *for* $i=0,\dots,N, j=0,\dots,M$, *then*

$$|u(x,y)-S(x,y)|\le\frac{4}{27}\left(h^2\omega(d;\partial_1^2 u)+l^2\omega(d;\partial_2^2 u)\right),$$

$$|\partial_k u(x,y)-\partial_k S(x,y)|\le\frac{1}{4}d\omega(d;\partial_k^2 u)+\frac{3}{8}d\omega(d;\partial_1\partial_2 u),$$

$$|\partial_k^2 u(x,y)-\partial_k^2 Su(x,y)|\le\omega(d;\partial_k^2 u),$$

for $k=1,2$.

Proof. As in the previous theorem, using Lemma 2.1 and the second order Taylor formula from the form (1.5) we have

$$|S_{i,j}(x,y)-u(x,y)|\le\frac{1}{27}h_i^2\omega(h_i;\partial_1^2 u)+$$

$$+\Big|(1-v)\big[u(x,y_j)-u(x,y)\big]+v\big[u(x,y_{j+1})-u(x,y)\big]+$$

$$+(1-v)v\frac{l_j}{2}\left[(1-t)(\beta_{i,j}-\beta_{i,j+1})+t(\beta_{i+1,j}-\beta_{i+1,j+1})\right]\Big| \le$$

$$\le \frac{1}{27}h_i^2\omega(h_i;\partial_1^2 u)+(1-v)vl_j\Big|-\partial_2 u(x,y_j)-\frac{1}{2}vl_j\partial_2^2 u(x,\eta_1)+\partial_2 u(x,y_{j+1})-$$

$$-\frac{1}{2}(1-v)l_j\partial_2^2 u(x,\eta_2)-\frac{1}{2}(1-t)l_j\partial_2^2 u(x_i,\eta_3)-\frac{1}{2}tl_j\partial_2^2 u(x_i,\eta_4)\Big| =$$

$$= \frac{1}{27}h_i^2\omega(h_i;\partial_1^2 u)+(1-v)vl_j^2\left|\partial_2^2 u(x,\eta_5)-\partial_2^2 u(\xi,\eta_6)\right| \le$$

$$\le \frac{1}{27}h_i^2\omega(h_i;\partial_1^2 u)+\frac{1}{4}l_j^2\omega(d;\partial_2^2 u),$$

where $\xi \in (x_i, x_{i+1})$, $\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6 \in (y_j, y_{j+1})$. Using the symmetry of the formula (1.5), we can interchange the two variables and so

$$|S_{i,j}(x,y)-u(x,y)| \le \frac{1}{27}l_j^2\omega(l_j;\partial_2^2 u)+\frac{1}{4}h_i^2\omega(d;\partial_1^2 u),$$

and from the two estimations we get

$$|S_{i,j}(x,y)-u(x,y)| \le \frac{4}{27}\left(h^2\omega(d;\partial_1^2 u)+l^2\omega(d;\partial_2^2 u)\right).$$

To prove the estimations for the partial derivatives we differentiate the formula (2.3) with respect to x and applying Lemma 2.1 we have for $(x,y) \in$ $\in \Omega_{i,j}$

$$|\partial_1 S_{i,j}(x,y)-\partial_1 u(x,y)| \le \frac{1}{4}h_i\omega(h_i;\partial_1^2 u)+$$

$$+\left|(1-v)(\partial_1 u(x,y_j)-\partial_1 u(x,y))+v(\partial_1 u(x,y_{j+1})-\partial_1 u(x,y))\right|+$$

$$+\frac{(1-v)v}{2}\frac{l_j}{h_i}|\beta_{i+1,j+1}-\beta_{i,j+1}-\beta_{i+1,j}+\beta_{i,j}| \le$$

$$\le \frac{1}{4}h_i\omega(h_i;\partial_1^2 u)+(1-v)vl_j|\partial_1\partial_2 u(x,\eta_2)-\partial_1\partial_2 u(x,\eta_1)|+$$

$$+\frac{(1-v)v}{2}l_j|\partial_1\partial_2 u(\xi_1,y_{j+1})-\partial_1\partial_2 u(\xi_2,y_j)| \le$$

$$\le \frac{1}{4}h\omega(h;\partial_1^2)+\frac{1}{4}l\omega(l;\partial_1\partial_2 u)+\frac{1}{8}l\omega(d;\partial_1\partial_2 u),$$

where $\xi_1,\xi_2 \in (x_i,x_{i+1})$, $\eta_1,\eta_2 \in (y_j,y_{j+1})$.

At last for the second order derivatives we have for $(x,y) \in \Omega_{i,j}$

$$|\partial_1^2 S_{i,j}(x,y)-\partial_1^2 u(x,y)| =$$

$$= \left|(1-v)\frac{\alpha_{i+1,j}-\alpha_{i,j}}{h_i}+v\frac{\alpha_{i+1,j+1}-\alpha_{i,j+1}}{h_i}-\partial_1^2 u(x,y)\right| =$$

$$= \left|(1-v)\partial_1^2 u(\xi_1,y_j)+v\partial_1^2 u(\xi_2,y_{j+1})-\partial_1^2 u(x,y)\right| \le \omega(d;\partial_1^2 u).$$

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