Annales Univ. Sci. Budapest., Sect. Comp. 16 (1996) 207-215

A REMARK ON ρ–NORMAL MATRICES

B. Háy (Budapest, Hungary)
L. Pasquini (L'Aquila, Italy)
P. Vértesi (Budapest, Hungary)

Dedicated to Professor J. Balázs on his 75th birthday

1. Introduction. Preliminary results

1.1. Let us consider a triangular interpolatory matrix $X = \{x_{kn}\} \subset [-1, 1]$ defined by

(1.1) $$-1 \le x_{nn} < x_{n-1,n} < \ldots < x_{2n} < x_{1n} \le 1\ , \qquad n = 1, 2 \ldots .$$

The unique Hermite-Fejér (HF) interpolatory polynomial $H_{nm}(f, X, x) \in \in \mathcal{P}_{nm-1}$ of higher order ($m \ge 1$, fixed integer) is defined by

(1.2) $$H_{nm}^{(t)}(f, X, x_{kn}) = \delta_{ot} f(x_{kn})\ , \qquad k = 1, 2, \ldots, n;\ t = 0, 1, \ldots, m-1\ ,$$

where $f \in C$ ($= f$ is continuous on $[-1, 1]$). ($m = 1$: Lagrange-, $m = 2$: the classical HF-interpolation.)

Sometimes we use the Hermite (H) polynomial $\mathcal{H}_{nm} \in \mathcal{P}_{nm-1}$ uniquely defined by

(1.3) $$\mathcal{H}_{nm}^{(t)}(f, X, x_{kn}) = f^{(t)}(x_{kn})\ , \qquad 1 \le k \le n;\ t = 0, 1, \ldots, m-1$$

($f^{(m-1)} \in C$). One can prove the following relations

(1.4) $$\mathcal{H}_{nm}(f, X, x) = \sum_{t=0}^{m-1} \sum_{k=1}^{n} f^{(t)}(x_{kn}) h_{tknm}(X, x)\ ,$$

Research supported by Hungarian National Science Foundation Grant No. T7570 (first and third author) and No. T17425 (third author), and by the MURST Found (Italy, second author)

(1.5) $$H_{nm}(f,X,x)=\sum_{k=1}^{n} f(x_{kn})h_{oknm}(X,x)\ ,$$

where, by obvious short notations, $h_{tk}\in\mathcal{P}_{nm-1}$ satisfy

(1.6) $$h_{tk}^{(r)}(x_{sn})=\delta_{tr}\delta_{ks}$$

and have the form

(1.7) $$\begin{cases} h_{tk}(x)=v_{tk}(x)(x-x_k)^t\ell_k^m(x)\ , \\ v_{tk}(x)=\frac{1}{t!}\sum_{i=0}^{m-1-t} e_{ik}(x-x_k)^i\ , \qquad e_{ik}=\frac{(\ell_k^{-m}(x))^{(i)}_{x=x_k}}{i!}\ , \end{cases}$$

$\ell_k(x)$ are the fundamental polynomials of Lagrange interpolation of the form

(1.8) $$\ell_k(x)=\frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)}\ ,\qquad \omega_n(x)=c_n\prod_{k=1}^{n}(x-x_k)\ ,\qquad c_n\neq 0\ ,$$

$(0\le t,s\le m-1,\ 1\le k,s\le n,\ n=1,2,\dots)$.

When $m=$ odd, a Faber-type result can be proved (cf. J.Szabados [1,[11]] (the reference [11] in the survey paper P.Vértesi [1])).

However for $m=$ even - what will be supposed from now - we can have many matrices X with the good convergence property

(1.9) $$\lim_{n\to\infty}\|H_{nm}(f,X,x)-f(x)\|=0 \qquad \forall\ f\in C\ ,$$

where $\|\cdot\|$ is the maximum norm in $[-1,1]$ (cf. L.Fejér [1,[1]] ($m=2$), R.Sakai and/or P.Vértesi [1, [12], [13], [14]] ($m\ge 2$)).

For the classical case ($m=2$) the idea of ρ-normality was introduced and applied in papers L.Fejér [1,[1]] and G.Grünwald [1,[6]]. In cases $m=$ $=4,6,8,\dots$ the definition was generalized by Y.Shi [1,[2]]. The *modification* of Shi's definition turned out to be very flexible. Namely, let I_{1n} and I_{2n} be two proper disjoint subsets of $J_n:=\{1,2,\dots,n\}$ with $|I_{1n}|=r_{1n}$, $|I_{2n}|=r_{2n}:=$ $:=n-r_{1n}$ with $0\le r_{1n}\le n$.

Definition. Let m be even. X is ρ-normal with parameters r_{1n},r_{2n} and m (shortly X is (ρ,r_{2n}) or (ρ,r_{2n},m)-normal) iff with a proper $\rho>0$ and $n\ge n_0$

(*i*)

$v_{oknm}(x)\ge\rho t!|v_{tknm}(x)|$ for $1\le t\le m-1,\ n\ge n_0,\ |x|\le 1,$ if $k\in I_{1n}$,

(*ii*)

$$\begin{cases} \left\| \sum_{k\in I_{2n}} |h_{oknm}(x)| \right\| = O(1), \quad \lim_{n\to\infty} \left\| \sum_{k\in I_{2n}} |x-x_{kn}|^{\delta} |h_{oknm}(x)| \right\| = 0\ , \\ \text{for every} \quad \delta > 0, \quad \text{moreover} \\ \lim_{n\to\infty} \left\| \sum_{k\in I_{2n}} |h_{tknm}(x)| \right\| = 0, \quad \left\| \sum_{k\in I_{2n}} |v_{tknm}(x)| \ell_{kn}^{m}(x) \right\| = O(1), \\ \qquad 1 \le t \le m-1. \end{cases}$$

This definition was introduced in P. Vértesi [4]; when $r_{2n} = 0$, we get back Shi's original definition. The classical case, treated by L.Fejér and G.Grünwald, corresponds to $m = 2$ and $r_{2n} = 0$; they called the matrices simply ρ-normal. When $r_{2n} = 0$, $(n = 1, 2, \dots)$, the polynomials H_{nm} are positive linear operators; if r_{2n} are "small" we can say that our system X is "practically" ρ-normal.

Using the above definition the following statement holds true for $X^{(\alpha,\beta)} = \{x_{kn}^{(\alpha,\beta)}\}$ (= the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x) \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$, $n = 1, 2, \dots$, $\alpha, \beta \ge -1$, fixed; cf. G.Szegő [2, Ch.4], say).

Theorem A. *Let m be even fixed, and let $\alpha, \beta \ge -1$ satisfy the conditions*

$$(1.10) \qquad C_m := -\frac{1}{2} - \frac{1}{m} < \alpha, \beta < -\frac{1}{2} + \frac{1}{m} := A_m\,.$$

Define $\rho_0 > 0$ by

$$(1.11) \qquad \rho_0 := \min\left(\frac{1}{2} - \frac{m}{4} - \frac{\alpha m}{2},\ \ \frac{1}{2} - \frac{m}{4} - \frac{\beta m}{2} \right).$$

Then for arbitrary fixed $\varepsilon > 0$ with $0 < \varepsilon < \rho_0$ there exists a constant $G = G(\alpha, \beta, m, \varepsilon)$ such that $X^{(\alpha,\beta)}$ is (ρ, G, m)-normal with $\rho = \rho_0 - \varepsilon$. Here ρ_0 cannot be replaced by any $\tilde{\rho}_0 > \rho_0$. Further, if $\Gamma := \max(\alpha, \beta) \ge A_m$ or $\gamma := \min(\alpha, \beta) < C_m$, the statement does not hold true.

The above theorem was proved in P. Vértesi [4]. However, if $m = 2$, even the case $\gamma = -\frac{1}{2} - \frac{1}{m} = -1$ was settled and was proved that $X^{(-1,1)}$ is the only 1-normal matrix; cf. L.Fejér in [1,[8, p. 157 (-3,-5)]] and L. Pasquini [1,[9]] (if $m = 2$, then $v_{1k}(x) = 1$, so (i) yields $v_{ok}(x) \ge \rho$ whence by $1 \equiv v_{ok}(x_k) \ge \rho$ we obtain relation $\rho \le 1$; that means the result is the best possible).

2. The case $\rho = 1$ when $m > 2$

2.1. The first aim of this paper is to settle the case $\gamma = C_m = -\frac{1}{2} - \frac{1}{m}$. We state using notation (1.11)

Theorem 2.1. *Let m be even and let $\gamma = C_m$. Then for arbitrary sequence $\{G_n\}$ with $\lim_{n\to\infty} n^{-2/3}G_n = \infty$, the matrix $X^{(\alpha,\beta)}$ is (ρ, G_n, m)-normal with $\rho = \rho_0 - \varepsilon$, and $0 < \varepsilon < \rho_0$, arbitrary fixed.*

Remarks. 1. If $\alpha = \beta = C_m$, then $\rho_0 = 1$, whence $\rho = 1 - \varepsilon$. On the other hand by $1 = v_{ok}(x_k) \geq \rho(m-1)!\, v_{m-1,k}(x_k) = \rho \quad (v_{m-1,k}(x) \equiv 1/(m-1)!)$, we get $\rho \leq 1$, i.e., again, our result is, in a sense, the best possible.

2. If $m \geq 4$, a possible question is to obtain other $(1-\varepsilon, G_n, m)$ matrices with $G_n = o(n^{2/3})$. (The proof of Theorem 2.1 with a small modification holds when $G_n = An^{2/3}$, $A > 0$ is big enough.)

2.2. Finally, using the *original* definition, i.e. X is ϱ-normal iff

$$(i^*) \qquad v_{oknm}(x) \geq (-1)^{t+1}\varrho t!\, v_{tknm}(x) \quad \text{for} \quad 1 \leq t \leq m-1,\ n \geq n_0,\ |x| \leq 1,\ 1 \leq k \leq n,$$

(cf. Y. Shi [1[2]]), we prove

Theorem 2.2. *If $m \geq 4$, even, then there is no 1-normal matrix.*

Remark. Conditions (i*), using [1[2,(2.8)]], imply $v_{ok}(x) \geq \varrho t!|v_{tk}(x)|$ (cf. (i)).

3. Proofs

3.A. Proof of Theorem 2.1.

3.1. We use many formulae and ideas of papers [1,[12],[13],[3] and [4]]. For sake of simplicity, we suppose $\alpha = \beta = C_m = -\frac{1}{2} - \frac{1}{m}$. First we verify (i) if $I_{1n} := \{k;\ \min(k, n-k+1) := K \geq G_n/2,\ n \geq n_0\}$, whence obviously $|I_{1n}| = G_n$ (for simplicity, G_n=even).

3.2. In [4,(3.26)] we obtained relations

$$v_{ok}(x) \geq (1-\varepsilon)t!\ v_{tk}(x) > 0, \qquad 1 \leq t \leq m-1 \tag{3.1}$$

if $|k-j| \leq c_0$, $K \geq k_0$, $n \geq n_0$, where $|x - x_{jn}| := \min\limits_{1 \leq k \leq n} |x - x_{kn}|$, n_0 is chosen according to the fixed values of c_0 and k_0. Relation (3.1) shows that we have to prove (i) only when $|k-j| \geq c_0$. By [4, (3.27)] we get

$$\begin{aligned} &e_{m-2,k}(x-x_k)^{m-2} + e_{m-1,k}(x-x_k)^{m-1} = \\ &\quad = e_{m-2,k}(x-x_k)^{m-2}\left\{1 - \varepsilon_k \frac{x-x_k}{1-x_k^2}\right\}, \ K \geq k_0, \ n \geq n_0, \end{aligned} \tag{3.2}$$

where $\varepsilon_k = \varepsilon_{kn} = O\left(\frac{1}{K} + \frac{1}{n}\right)$. Let $0 \leq \delta \leq x_k < 1$, $x \geq -1/2$, say. Then for the function $L(x, x_k) := \{\ldots\}$ we get by $2K \geq G_n$

$$L(x, x_k) = 1 + O(1)\left(\frac{1}{k} + \frac{1}{n}\right)\frac{|k+j||k-j|}{k^2} = 1 + O(1)\left(\frac{n^2}{k^3} + \frac{n}{k^2}\right) = 1 + o(1)$$

(we used $|x - x_k| \leq c|k-j||k+j|n^{-2}$), whence

$$\begin{aligned} &e_{m-2,k}(x-x_k)^{m-2} + e_{m-1,k}(x-x_k)^{m-1} = \\ &\quad = (1+o(1))e_{m-2,k}(x-x_k)^{m-2}, \qquad k \in I_{1n}. \end{aligned} \tag{3.3}$$

Another important relation proved in [4,(3.29)] is

$$A\sum_{i=0}^{2t-1} |e_{ik}(x-x_k)|^i \leq e_{2t}(x-x_k)^{2t}, \qquad 2 \leq 2t \leq m-2 \tag{3.4}$$

for any fixed $A > 0$ if c_0 and k_0 are big enough.

Then, by (3.3), (3.4) and (1.7)

$$\frac{t!|v_{tk}(x)|}{v_{ok}(x)} \leq \frac{\sum\limits_{i=0}^{2t-1} |e_{ik}(x-x_k)|^i}{\left(1+\frac{1}{A}\right)e_{m-2,k}(x-x_k)^{m-2} + e_{m-1,k}(x-x_k)^{m-1}} \leq$$

$$\leq \frac{\frac{1}{A}e_{m-2,k}(x-x_k)^{m-2}}{\left(1+\frac{2}{A}\right)e_{m-2,k}(x-x_k)^{m-2}} = \frac{1}{2+A} \leq 1-\varepsilon \quad \text{if} \quad K \in I_{1n}, \quad |k-j| \geq c_0\,.$$

So we verified (i) for $2 \le t \le m-1$. If $t=1$, we write

$$\frac{v_{ok}(x)}{v_{1k}(x)} \ge \frac{\left(1-\frac{2}{A}\right) e_{m-2,k}(x-x_k)^{m-2}}{\left(1+\frac{1}{A}\right) e_{m-2,k}(x-x_k)^{m-2}} > 1-\varepsilon\ , \quad K \in I_{1n}, \quad |k-j| \ge c_0$$

if A is properly chosen (see (3.3) and (3.4)).

3.3. Now we verify relation (ii) for $k \in I_{2n}$. Estimation

$$\left\| \sum_{k \in I_{2n}} |h_{ok}(x)| \right\| = O(1)$$

is an obvious consequence of $\left\| \sum_{k=1}^{n} |h_{ok}(x)| \right\| = O(1)$ which was proved in [3, 3.8 and 3.9]. Now let us prove the second relation in (ii).

We write with $\eta_n \searrow 0$

$$S_1 := \sum_{k \in I_2} |x-x_k|^\delta |h_{ok}(x)| \le \sum_{k=1}^{n} \ldots = \sum_{|x-x_k| \le \eta_n} \ldots + \sum_{|x-x_k| > \eta_n} \ldots := T_1 + T_2.$$

Here by $\left\| \sum_{k=1}^{n} |h_{ok}(x)| \right\| = O(1)$ we obtain relation $T_1 \le \eta_n^\delta \left\| \sum_{k=1}^{n} |h_{ok}(x)| \right\| =$ $= o(1)$. Further, by (1.7)

$$T_2 = \sum_{|x-x_k| \ge \eta_n} \left(\frac{P_n(x)}{P_n'(x_k)} \right)^m \frac{1}{(x-x_k)^{m-\delta}} (1 + |e_{1k}||x-x_k| + \ldots +$$
$$+ |e_{m-1}||x-x_k|^{m-1}) \le \frac{c}{\eta_n^{m-\delta}} \sum_{k=1}^{n} \left\{ \left(\frac{P_n(x)}{P_n'(x_k)} \right)^m \left(\sum_{i=0}^{m-1} |e_{ik}| \right) \right\} .$$

On the other hand, in [4, 3.9. A] we proved that the sum $\sum_{k=1}^{n} \{\ldots\}$ can be estimated by ε_n, where $\varepsilon_n \searrow 0$. So with a proper η_n we obtain $T_2 = o(1)$, whence $S \le T_1 + T_2 = o(1)$ which was to be proven.

To get relations $\left\| \sum_{k \in I_{2n}} |h_{tk}(x)| \right\| = o(1)$ we remark that by standard calculations even the estimations

$$\left\| \sum_{k=1}^{n} |h_{tk}(x)| \right\| \le \begin{cases} c\dfrac{\log n}{n^t} & t \text{ is odd,} \\ \dfrac{c}{n^t}, & t \text{ is even,} \end{cases} \qquad t = 0, 1, \ldots, m-1 \tag{3.5}$$

can be verified (cf. [1, [1, Theorems 1 and 2]] and relations (3.19)-(3.25) in [4]).

Finally, we prove $\left\| \sum_{k\in I_{2n}} |v_{tk}(x)|\ell_k^m(x) \right\| = O(1)$. By the formulae quoted above one can get

$$\sum_{k\in I_{2n}} |v_{tk}(x)|\ell_k^m(x) \leq$$

$$\leq c\sum_{\substack{k=1\\k\neq j}} \frac{k^t}{|k-j|^{t+1}|k+j|^t} \leq c \qquad \text{whenever} \quad 1 \leq t \leq m-1 \ .$$

3.B. Proof of Theorem 2.2.

3.4. We suppose that X is 1-normal. Then, by (i*) we get $v_{ok}(x) \geq v_{1k}(x)$ if $t=1$, whence $v_{ok}(x) - v_{1k}(x) = e_{m-1,k}(x-x_k)^{m-1} \geq 0$. Here $m-1$ is odd, so supposing that $-1 < x_k < 1$, we conclude that

$$e_{m-1,k} = 0 \ , \quad \text{if} \quad |x_k| < 1 \ .$$

Now let $t = 3$. Again, by (i*) $v_{ok}(x) - 3!v_{3k}(x) = e_{m-3,k}(x-x_k)^{m-3} + e_{m-2,k}(x-x_k)^{m-2} + 0 = (x-x_k)^{m-3}\{e_{m-3,k} + e_{m-2,k}(x-x_k)\} \geq 0$. Let $|x_k| < 1$. Then if $e_{m-3,k} > 0$, say, for a proper x with $-1 \leq x < x_k$, $x \approx x_k$, we would get $0.5e_{m-3,k} \leq \{\ldots\}$ whence $(x-x_k)^{m-3}\{\ldots\} < 0$ – a contradiction. The case $e_{m-3,k} < 0$ is similar, i.e. we can conclude

$$e_{m-3,k} = 0 \ , \quad \text{if} \quad |x_k| < 1 \ ,$$

whence by $0 \leq (x-x_k)^{m-3}\{\ldots\} = e_{m-2,k}(x-x_k)^{m-2}$ we get relations

$$e_{m-2,k} \geq 0 \quad \text{if} \quad |x_k| < 1 \ .$$

Using induction we obtain

(3.6) $\qquad e_{tk} = 0 \quad \text{if} \quad |x_k| < 1, \quad t = 1,3,\ldots,m-1, \quad n \geq n_0 \ ,$

and

(3.7) $\qquad e_{tk} \geq 0 \quad \text{if} \quad |x_k| < 1, \quad t = 2,4,\ldots,m-2, \quad n \geq n_0 \ .$

Using relations (3.6) only, we state

$$\ell_{kn}^{(t)}(x_k) = 0, \quad |x_k| < 1, \quad t = 1, 3, \dots, m-1, \quad n \geq n_0 . \tag{3.8}$$

Indeed, from

$$(\ell_k^s(x))^{(t)} = \tag{3.9}$$

$$= \sum_{\substack{i_1+i_2+\dots+i_j=t \\ 1\leq i_1\leq i_2\leq\dots\leq i_j\leq t}} A(I)(s)_j \ell_k^{s-j}(x)\ell_k^{(i_1)}(x)\ell_k^{(i_2)}(x)\dots\ell_k^{(i_j)}(x), \quad t \geq 1,$$

where $A(I) = A(i_1, i_2, \dots, i_j) > 0$, integer, s is real, $(s)_j = s(s-1)\dots(s-j+1)$ (cf. [1 [12, (4.1)]]), using (1.7) we get

$$0 = e_{1k} = (\ell_k^{-m}(x))'_{x=x_k} = -m\ell_k'(x_k) \ , \quad |x_k| < 1,$$

which is (3.8) for $t = 1$. Similarly, with obvious short notations

$$0 = e_{3k} = \frac{(\ell_k^{-m})'''}{6} = \dots(\ell_k')^3 + \dots\ell_k'\ell_k'' - m\ell_k''' = -m\ell_k''' \quad |x_k| < 1,$$

whence we get (3.8) for $t = 3$. Using induction, we get (3.8) for the other values of t, considering that in the sum each term but the last one contains at least one factor $\ell_k^{(i_r)}$ where $1 \leq i_r < t$ and odd, i.e. by the induction condition $\ell_k^{(i_r)} = 0$.

Then relations (3.8) and

$$\ell_k^{(r)}(x_k) = \frac{\omega_n^{(r+1)}(x_k)}{(r+1)\omega_n'(x_k)} \ , \quad 1 \leq k \leq n \quad r = 0, 1, 2, \dots . \tag{3.10}$$

(cf. [3]) applying for $r = t = m-1$ yield that $\omega_n^{(m)}(x_k) = 0$, $|x_k| < 1$. By $m \geq 4$ we obtain that the polynomial $\omega_n^{(m)}(x)$ of degree $n - m \leq n - 4$ has $n - 2$ zeros at least, whence $\omega_n(x) \equiv 0$.

References

[1] **Vértesi P.,** On ρ-normal pointsystems: a survey, IDOMAT Conference in Dortmund, 1995 (to appear).

[2] **Szegő G.**, *Orthogonal polynomials,* 4th ed., AMS Coll. Publ. Vol. **23**, Providence, R.I., 1974.

[3] **Vértesi P.**, Hermite-Fejér interpolation of higher order I., *Acta Math. Hungar.*, **54** (1989), 135-152.

[4] **Vértesi P.**, Practically ρ-normal pointsystems, *Acta Math. Hungar.*, **67** (1995), 237-251.

B. Háy
MTA SZTAKI
XI. Lágymányosi u. 11.
H-1111 Budapest, Hungary

L. Pasquini
L'Aquila University
L'Aquila, Italy

P. Vértesi
Mathematical Institute of the
Hungarian Academy of Sciences
H-1364 Budapest P.O.B. 127
Hungary