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INTEGRABILITY AND L^1–CONVERGENCE OF TRIGONOMETRIC AND WALSH SERIES

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Dedicated to Professor János Balázs on the occasion of his 75th birthday

Abstract. In this paper we summarize the recent results on integrability and L^1- convergence of trigonometric, Walsh and Vilenkin series by giving a unified approach. We investigate this question from the aspect of the coefficient sequence. Several conditions sufficient to conclude the integrability and L^1-convergence of the corresponding series will be presented for the coefficient sequence. Since this is in close relation with the so called Sidon-type inequalities a summary about them is also given. We focus our attention onto this relation. Connections, similarities and differences will be pointed out concerning the different approaches and systems. We also show examples for consequences, generalizations and applications.

1. Introduction

Let (u_n) represent the cosine, the sine, the trigonometric, the Walsh or the Vilenkin system defined on $[-\pi,\pi]$, or $[0,1]$. Furthermore, let L^p $(1 < p \leq \leq \infty)$ stand for the function spaces $L^p[-\pi,\pi]$, $L^p[0,1]$ defined as usual, with the corresponding norm denoted by $\| \, \|_p$. Throughout this paper k, j, n, ℓ will denote arbitrary natural numbers if not specified otherwise.

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We will consider the integrability and L^1-convergence of the series

$$\sum_{n=0}^{\infty} a_n u_n , \tag{1.1}$$

where (a_n) is a sequence of real (or complex) numbers.

Set $\hat{L}_u = \{(\hat{f}_n)_u : f \in L^1\}$, where $(\hat{f}_n)_u$ denotes the sequence of Fourier coefficients of f with respect to the system (u_n). Then, the problem of integrability of (1.1) is to decide whether $(a_n) \in \hat{L}_u$. Unfortunately, there is no characterization known for the elements of $\hat{L}_u$ expressed by the terms of the sequences. However, there exist conditions described in terms of sequences alone, such that the classes induced by them are proper subsets of $\hat{L}_u$. They are called *integrability classes* with respect to (u_n).

The other question investigated in this paper is the L^1-convergence of (1.1). It is known that $(a_n) \in \hat{L}_u$ alone does not guarantee the convergence of (1.1). Similarly to the integrability we are interested in conditions for (a_n) which imply the L^1-convergence of the series. Connecting the two problems we introduce the following concept. A class of sequences is said to be an *integrability and L^1-convergence class* (with respect to (u_n)) if it is an integrability class and for each element (a_n) of it the corresponding series converges in L^1-norm if and only if $\lim\limits_{n\to\infty} |a_n| \, \|U_n\|_1 = 0$, where $U_n = \sum\limits_{k=0}^{n} u_k$. Let us now show the connection of the above problems with upper estimations for $\left\| \sum\limits_{k=0}^{n} c_k U_k \right\|_1$ expressed in terms of coefficients c_j. They are called *Sidon-type inequalities*. The connection of such inequalities with integrability and L^1-convergence problems can be made clear by an Abel transformation. Namely,

$$\sum_{k=0}^{n} a_k u_k = \sum_{k=0}^{n-1} \Delta a_k \, U_k + a_n U_n \tag{1.2}$$

$(\Delta a_k = a_k - a_{k+1})$. Compare the last term with the definition of integrability and L^1-convergence classes.

Now we show how some classical results for the cosine system can be interpreted by the above mentioned relation.

Let (u_n) be the cosine system, i.e.

$$u_0(x) = \frac{1}{2} \quad , \quad u_k(x) = \cos kx \qquad (k > 0,\ x \in [-\pi, \pi]),$$

Then U_k is the trigonometric Dirichlet kernel D_k. It is well known (see e.g. [44]) that the null sequences satisfying the condition

$$(1.3) \qquad \sum_{k=0}^{\infty} |\Delta a_k| (\log k + 1) < \infty$$

form an integrability and L^1-convergence class. Obviously, (1.3) is a consequence of (1.2) and of the estimation for the Lebesgue constants of the trigonometric system

$$(1.4) \qquad \|D_n\|_1 \leq C(\log n + 1)$$

(C denotes an absolute, and C_p an only on p depending positive constant may be not the same in different occurences.) It is clear that (1.4) is equivalent to the Sidon-type inequality

$$(1.5) \qquad \frac{1}{n+1}\left\|\sum_{k=0}^{n} c_k D_k\right\|_1 \leq C\frac{\log n + 1}{n+1}\sum_{k=0}^{n}|c_k|.$$

Another classical example is due to Young [43] in 1913. Namely, he showed that the convex null sequences, i.e. those for which

$$\Delta^2 a_k \geq 0, \qquad \lim_{k\to\infty} a_k = 0$$

hold ($\Delta^2 a_k = \Delta a_k - \Delta a_{k+1}$), form an L^1-convergence class. Kolmogorov [19] obtained that the same holds for the set of quasiconvex null sequences ($\sum_{k=0}^{\infty} k\,|\Delta^2 a_k| < \infty$).

We note that this follows from Young's result by considering that the linear space spanned by the convex sequences is exactly the set of quasiconvex sequences. On the other hand, it is easy to see that Kolmogorov's result can be deduced from Fejér's estimation for the arithmetic means of the Dirichlet kernels

$$(1.6) \qquad \frac{1}{n+1}\left\|\sum_{k=0}^{n} D_k\right\|_1 \leq C.$$

Indeed, using the notation $K_k = \frac{1}{k+1}\sum_{j=0}^{k} D_k$, we have similarly to (1.2) that

$$\sum_{k=0}^{n} a_k \cos kx = \sum_{k=0}^{n-2} \Delta^2 a_k (k+1) K_k(x) + \Delta a_{n-1} n K_{n-1} + a_n D_n .$$

Then by (1.6) the result of Kolmogorov follows from the estimation

$$\int_0^\pi \left|\sum_{k=0}^n a_k \cos kx\right| dx \le$$

$$\le C\left(\sum_{k=0}^{n-1}(k+1)|\Delta^2 a_k| + n|\Delta a_{n-1}| + (n+1)(\log n+1)\right).$$

As we can see both (1.5) and (1.6) induce L^1-convergence classes. They can be considered as special Sidon-type inequalities, i.e. upper estimations for the integral norm of linear combinations of Dirichlet kernels.

Sidon [33] investigated this problem in general in 1939 and proved the following inequality named after him

$$\frac{1}{n+1}\left\|\sum_{k=0}^n c_k D_k\right\|_1 \le \max_{0\le k\le n}|c_k|. \tag{1.7}$$

We note that (1.7) is a generalization of (1.6) but not of (1.5).

In Section 2 we summarize the recent results on Sidon-type inequalities with respect to several systems. In Section 3 we deal with different types of integrability and L^1-convergence classes.

2. Sidon-type inequalities

2.1. The cosine and the Walsh case

We will consider upper estimations for

$$\frac{1}{n+1}\left\|\sum_{k=0}^n c_k D_k\right\|_1, \tag{2.1}$$

where D_k denotes either the trigonometric or the Walsh-Dirichlet kernel.

Let us start with the trigonometric case. Concerning the improvements of Sidons's inequality we first mention the result of Bojanic and Stanojević [7]. They proved that

$$\frac{1}{n+1}\left\|\sum_{k=0}^{n} c_k D_k\right\|_1 \le C_p \left(\frac{1}{n+1}\sum_{k=0}^{n}|c_k|^p\right)^{1/p} \qquad (p>1). \tag{2.2}$$

It is easy to see that (2.2) is not valid for $p = 1$. Indeed, if $c_n = 1$ and $c_k = 0 \ (k < n)$ then the left side of (2.2) is of order $\log n/n$ while the right side is of order $1/n$ with $n \to \infty$. Still, it is possible to balance the $p > 1$ and the $p = 1$ cases. Namely, Tanović-Miller showed [36] that (2.2) can be improved as follows

(2.3)

$$\frac{1}{n+1}\left\|\sum_{k=0}^{n} c_k D_k\right\|_1 \le C_p \left(\frac{\log\alpha}{n+1}\sum_{k=0}^{n}|c_k| + \alpha^{-1/q}\Big(\frac{1}{n+1}\sum_{k=0}^{n}|c_k|^p\Big)^{1/p}\right)$$

$(\alpha \ge 1,\ 1 < p \le 2,\ 1/p + 1/q = 1)$. We note that (1.5) does not follow even from this estimation.

The above results can be written in a unified form. Indeed, if the coefficient vector $(c_k)_0^{2^n-1}$ is associated with the step function Γ_n defined on $[0,1]$ as follows

$$\Gamma_n = \sum_{k=0}^{2^n-1} c_k\, \chi_{[k2^{-n},(k+1)2^{-n})}$$

(where χ_A denotes the characteristic function of the set of real numbers A), then (1.5), (1.7), (2.2), and (2.3) can be expressed in terms of the L^p-norm $(1 \le p \le \infty)$ of Γ_n. For the indices 2^n the right sides of (1.5), (1.7) and (2.2) are respectively $n\,\|\Gamma_n\|_1$, $\|\Gamma_n\|_\infty$ and $\|\Gamma_n\|_p$ $(p > 1)$, and (2.3) corresponds to the mixed norm $\log\alpha\,\|\Gamma_n\|_1 + \alpha^{-1/q}\|\Gamma_n\|_p$ $(\alpha \ge 1,\ 1 < p \le 2)$.

Schipp [31] improved the above results by showing

$$\frac{1}{2^n}\left\|\sum_{k=0}^{2^n-1} c_k D_k\right\|_1 \le \|\Gamma_n\|_{\mathcal{H}}, \tag{2.4}$$

where $\mathcal{H}$ denotes the *non-periodic Hardy space* (see e.g. [18]). $\mathcal{H}$ can be defined in several ways. One of them is based on so called ***atomic decompositions.*** A function $f \in L^\infty[0,1]$ is said to be an *atom* if either $h \equiv 1$ or there exists an interval $I \subset [0,1)$ such that

$$i)\ \operatorname{supp} h \subset I, \qquad ii)\ \int_0^1 h = 0, \qquad iii)\ \|h\|_\infty \le |I|^{-1}.$$

Then $\mathcal{H}$ is the collection of functions f that can be decomposed as $f = \sum_{k=0}^{\infty} \lambda_k h_k$, where the h_k's are atoms, λ_k's are real numbers and $\sum_{k=0}^{\infty} |\lambda_k| < \infty$. The norm is defined as

$$\|f\|_{\mathcal{H}} = \inf \sum_{k=0}^{\infty} |\lambda_k|$$

by taking the infimum over all such decompositions.

Schipp [31], [32] observed that an inequality similar to Sidon's original one for coefficients having zero sum and the uniform boundedness of the L^1-norm of the Fejér kernels imply that

$$\left\| \sum_{k=0}^{2^n-1} c_k D_k \right\|_1 \le C \|\Gamma_n\|_1$$

holds if Γ_n is an atom. From here (2.3) follows by atomic decomposition. Schipp showed that (2.3), consequently also (1.7) and (2.2), can be deduced from (2.3).

Observe, that in (1.7), (2.2), (2.3) the given upper estimations do not depend on the order of the coefficients. In other words, they define *rearrangement invariant norms* in the n-dimensional eucledian space. The only exception is (2.3). However, the Hardy norm is not rearrangement invariant but still has a certain translation invariance property. On the other hand $\|\Gamma_n\|_{\mathcal{H}}$ depends on the order of c_k's $(0 \le k < 2^n)$ in a quite different way as $\left\| \sum_{k=0}^{2^n-1} c_k D_k \right\|_1$ (see [15] for details). Unlike the previous cases $\|\Gamma_n\|_H$ is complicated to express directly by the c_k's. In order to derive a formula for the c_k's from (2.3) the author [15] used the well known estimation (see e.g. [30])

$$\|f\|_{\mathcal{H}} \le C \left(\int_0^1 |f| \log^+ |f| + 1 \right) \qquad (f \in \mathcal{H})$$

to obtain

$$(2.5)\qquad \left\|\sum_{k=0}^{n} c_k D_k\right\|_1 \le C \sum_{k=0}^{n} |c_k| \left(1 + \log^+ \frac{|c_k|}{(n+1)^{-1} \sum\limits_{j=0}^{n} |c_j|}\right).$$

Seeing the above outlined process of the improvements of Sidon's inequality it is natural to ask how close are the recent results to the best possible one. It is clear that (2.1) defines a norm in the $n+1$-dimensional eucledian space. In this setting the best result would be the characterization, uniformly in n, of this norm by a known norm. This problem was partially answered by the author in [15]. Namely, it was shown that

$$(2.6)\qquad \max_{p \in P_n} \left\|\sum_{k=0}^{n} c_{p_k} D_k\right\|_1 \ge C \sum_{k=0}^{n} |c_k| \left(1 + \log^+ \frac{|c_k|}{(n+1)^{-1} \sum\limits_{j=0}^{n} |c_j|}\right).$$

(Here, P_n denotes the set of permutations of $\{0, 1, \dots, n\}$.) (2.6) means that the best rearrangement invariant Sidon-type inequality is the one given in (2.5). It was also shown that

$$\frac{1}{2^n} \sum_{k=0}^{2^n-1} |c_k| \left(1 + \log^+ \frac{|c_k|}{2^{-n} \sum\limits_{j=0}^{2^n-1} |c_j|}\right)$$

is equivalent to $\|\Gamma_n\|_M$, where $\|\ \|_M$ is the *Orlicz-norm* generated by the N-function

$$M(x) = \begin{cases} 1/2|x|^2 & \text{if} \quad 0 \le |x| < 1, \\ 1/2 + |x| \log^+ |x| & \text{if} \quad |x| \ge 1. \end{cases}$$

Consequently, the best rearrangement invariant norm that can be used for an upper estimation for (2.1) is an Orlicz-type norm. It was also shown in [15] that the Orlicz space in question is isomorph with a rearrangement invariant Hardy-type norm.

Sidon-type inequalities have also been investigated with respect to the *Walsh system*. This is often referred to as the dyadic case. The Walsh system ordered in Paley's sense can be defined as follows. Let r_k represent the k-th Rademacher function, i.e.

$$r_0(x) = \begin{cases} +1 & \text{if } 0 \le x < 1/2, \\ -1 & \text{if } 1/2 \le x < 1 \end{cases}$$

periodic with 1, and

$$r_k(x) = r_0(2^k x) \quad (k > 0).$$

Then the n-th Walsh function is defined by

$$w_n = \prod_{k=0}^{\infty} r_k^{n_k},$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$ $(n_k = 0$ or $1)$, and the Dirichlet kernels are defined by the sum

$$D_k = \sum_{j=0}^{k-1} w_j.$$

The Walsh equivalent of (2.2) has been proved by Móricz and Schipp [26]. Schipp [31] proved the dyadic version of (2.3). Here, $\|\Gamma_n\|_{\mathcal{H}}$ has to be replaced by the so-called dyadic Hardy norm of Γ_n. The *dyadic Hardy space* H and its norm can be defined by means of dyadic atomic decompositions in a similar way as $\mathcal{H}$. The only difference is that by interval we mean dyadic intervals in the dyadic case, i.e. sets of the form $[k2^{-n}, (k+1)2^{-n})$ $(0 \le k < 2^n)$. (In connection with the dyadic Hardy space we refer to [30].) The author [16] showed that (2.5) and (2.6) are also valid for the Walsh case. As a summary we may say that the same program has been carried out for the Walsh system as for the cosine system.

We note that the results for the Walsh system can be generalized for Vilenkin systems of bounded type. (See Section 3.3 for the definition of Vilenkin systems.) However, this is not true for the unbounded case. Indeed, as it was proved by Price [29], the integral norms of the Fejér kernels with respect to a Vilenkin system of unbounded type are not uniformly bounded. Consequently, Sidon's inequality fails to hold for these systems.

2.2. Generalizations

First we mention a result of Móricz that is useful in several applications. He [21] generalized (2.2) by showing

$$\int_{\gamma}^{\pi}\left|\sum_{k=0}^{n} c_k D_k\right| \le \frac{C}{(p-1)^{1/p}}\gamma^{-1/q}\left(\sum_{k=0}^{n}|c_k|^p\right)^{1/p} \tag{2.7}$$

$$(0<\gamma<\pi, 1<p\le 2,\ 1/p+1/q=1)$$

for the trigonometric Dirichlet kernels. He obtained similar results for the *modified conjugate trigonometric kernel*

$$\overline{D}_k(x) = -\frac{\cos(k+1/2)x}{2\sin(x/2)}.$$

His result for $\overline{D}_k$ is a generalization of the one due to Telyakovskiĭ [39].

Another possibility to generalize the inequalities of Section 2.1 is to take the so called *shifted variant* of (2.1), i.e.

$$\left\|\sum_{k=K}^{N} c_k D_k\right\|_1 \qquad (K\le N).$$

In connection with it Móricz [21] showed that

$$\begin{aligned}\left\|\sum_{k=K}^{N} c_k D_k\right\|_1 &\le C\left(1+\log\frac{N+1}{N-K+1}\right)\sum_{k=K}^{N}|c_k|+ \\ &+ C_p\,(N-K+1)\left(\sum_{k=K}^{N}\frac{|c_k|^p}{N-K+1}\right)^{1/p}\end{aligned} \tag{2.8}$$

holds for the trigonometric Dirichlet kernels. Similar inequalities have been proved for the modified conjugate ([21]) and for the *modified complex Dirichlet kernels* ([22])

$$E_k^*(x) = \sum_{k=0}^{n} e^{ikx} + \frac{1}{e^{ix}-1} = \frac{\exp\left(i(n+1/2)x\right)}{2i\sin(x/2)}.$$

(2.8) has been improved by Buntinas and Tanović-Miller in [9] as follows

$$\left\|\sum_{k=K}^{N} c_k D_k\right\|_1 \le C_p\left(\log\frac{\mu}{\nu}\sum_{k=K}^{N}|c_k| + \nu^{1/q}\Big(\sum_{k=K}^{N}|c_k|^p\Big)^{1/p}\right) \tag{2.9}$$

$$(1 < p \le 2,\quad 1/p + 1/q = 1,\quad 0 < \nu \le \mu,\quad \mu \ge N).$$

Finally, (2.9) was improved by the author in [15]. Namely, it is shown there that

$$\begin{aligned}\left\|\sum_{k=K}^{N} c_k D_k\right\|_1 \le C\Bigg(&\log\frac{N}{N-K+1}\left|\sum_{k=K}^{N} c_k\right| + \\ &+\sum_{k=K}^{N}|c_k|\left(1+\log^+\frac{|c_k|}{(N-K+1)^{-1}\sum_{j=K}^{N}|c_j|}\right)\Bigg).\end{aligned} \tag{2.10}$$

The fact that (2.10) is an improvement of both (2.8) and (2.9) is a consequence of the following inequality proved in [15]

(2.11)

$$\begin{aligned}\max_{p\in P_{N-K}}\left\|\sum_{k=0}^{N-K} c_{K+p_k} D_k\right\|_1 \ge C\Bigg(&\log\frac{N}{N-K+1}\left|\sum_{k=K}^{N} c_k\right| + \\ &+\sum_{k=K}^{N}|c_k|\left(1+\log^+\frac{|c_k|}{(N-K+1)^{-1}\sum_{j=K}^{N}|c_j|}\right)\Bigg).\end{aligned}$$

(Again, P_{N-K} denotes the set of permutations of $\{0, 1, \dots, N-K\}$.) Consequently, (2.10) can be considered as the best rearrangament invariant shifted Sidon-type inequality for the cosine system. Clearly, (2.10) and (2.11) are generalizations of (2.5) and (2.6).

Concerning the Walsh system we note that an inequality similar to (2.7) was proved by Móricz in [23]. The dyadic equivalents of (2.10) and (2.11) have also been obtained by the author in [16].

Inequalities (1.7), (2.2) and their dyadic versions have been generalized for the multidimensional case by Telyakovskiĭ [40], A.A. Fomin [12], and by Móricz and Schipp [27] respectively. For their shifted trigonometric versions see [21].

Concerning other systems we mention that Schipp [32] was able to extend his results (2.3) for several systems satisfying certain properties, that he called F- and S-properties. We say that a system (u_n) defined on $[0,1]$ satisfies the *F-property* (Fejér-property) if

$$\frac{1}{2^n}\left\|\sum_{k=0}^{2^n-1} U_k\right\|_1 \le C,$$

where $U_n = \sum_{k=0}^{n} u_k$. Similarly, we say that the system (u_n) has the *strong S-property* (Sidon-property) if

$$\frac{1}{2^n}\left\|\sum_{k=0}^{2^n-1} c_k U_{k+\ell}\right\|_1 \le C \max_{0\le k<2^n} |c_k| \tag{2.12}$$

holds for any ℓ and $\sum_{k=0}^{2^n-1} c_k = 0$. Observe that (2.12) is a shifted version of Sidon's inequality (1.7) for the system (u_n). However, it is required to be satisfied only for coefficients with zero sum. It is easy too see ([32]) that if (u_n) has the character property $u_k u_\ell = u_{k+\ell}$ then the inequality corresponding to (1.7) implies (2.12).

If (2.12) holds only for $\ell = j2^n$ then (u_n) is said to satisfy the *dyadic S-property*. As we mentioned in the previous section, these properties are in strong connection with the atomic and dyadic atomic decompositions of the corresponding Γ_n step function. On the basis of this connection Schipp [32] proved that if (u_n) has the F-property and the strong S-property then the following Sidon-type inequality holds for it

$$\frac{1}{2^n}\left\|\sum_{k=1}^{2^n} U_k\right\|_1 \le C\,\|\Gamma_n\|_{\mathcal{H}}.$$

(Recall that $\mathcal{H}$ stands for the non-periodic Hardy space.)

On the other hand, if (u_n) has the F-property and the dyadic S-property then

$$\frac{1}{2^n}\left\|\sum_{k=1}^{2^n} D_k\right\|_1 \le C\,\|\Gamma_n\|_{H}. \tag{2.13}$$

(Recall that H stands for the dyadic Hardy space.)

Several examples are given in [32] for systems satisfying the F- and S-properties. For instance, the cosine system has the F- and the strong S-property. The complex trigonometric system has the strong S-property but not the F-property. This is the reason why modified kernels (see e.eg [21], [22]) are used for the complex trigonometric and for the sine system (conjugate kernels). The obvious example for a system having the F- and the dyadic S-property is the Walsh system. It is the product system of the Rademacher system, which can be considered as a special UDMD system. The concept of *UDMD systems* can be defined as follows. Let (φ_n) be a unitary dyadic martingal difference (UDMD) sequence. That is, φ_n is a complex function defined on $[0,1)$ with

i) $|\varphi_n| \equiv 1$,

ii) $\int\limits_I \varphi = 0$ for all $I \in \mathcal{I}_n$,

iii) φ_n is constant on every interval belonging to $\mathcal{I}_{n+1}$,

where $\mathcal{I}_n$ is the set of dyadic intervals of the form $[k2^{-n},(k+1)2^{-n})$ $(0 \leq \leq k < 2^n)$. Then the product system (ψ_n) of (φ_n) is defined as follows

$$\psi_n = \prod_{k=0}^{\infty} \varphi_k^{n_k},$$

where $n = \sum\limits_{k=0}^{\infty} n_k 2^n$ is the binary form of n. Schipp [32] proved that the product system of any UDMD system satisfies the F- and the dyadic S-property. Consequently, (2.13) holds for these systems.

Our last example ([32]) for a system satisfying the F- and the dyadic S-property is the *Ciesielski system*. For the definition see e.g. [29]. Here we only note that it relates to the Franklin system as the Walsh system relates to the Haar system.

3. Integrability and L^1-convergence classes

3.1 Sidon-type classes for cosine and Walsh series

In this section we will use the concepts of integrability and L^1-convergence classes defined in the introductory part of this paper. We will show how Sidon-type inequalities generate such classes.

Telyakovskiĭ [39] was the first to discover the connection between Sidon's inequality and L^1-convergence classes. On the basis of (1.7) he proved that the collection of null sequences (a_k) for which there exists a monotonically decreasing nonnegative sequence (A_k) such that $\sum_{k=0}^{\infty} A_k < \infty$, and $|\Delta a_k| \leq$ $\leq A_k$, is an integrability and L^1-convergence class for even trigonometric series. This class is often referred to as Telyakovskiĭ class. It is easy to see that this class contains the set of quasiconvex null sequences properly, but does not contain all of the sequences satisfying (1.3).

We note that using (2.2) and Telyakovskiĭ's method Č.V. Stanojević and V.B. Stanojević [34] enlarged this class by showing that the condition

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p} < C \qquad (1 < p < \infty)$$

also generates an integrability and L^1-convergence class.

The above classes have been extended by several authors by a method based on Sidon-type inequalities. The general construction goes as follows. Let (u_k) denote the cosine or the Walsh system, and let D_k be the corresponding Dirichlet kernel. Since (see e.g. [44], [29])

$$|D_k(x)| < C\,\frac{1}{x} \qquad (x \neq 0) \tag{3.1}$$

we have by (1.2) that the pointwise convergence of

$$\sum_{k=0}^{\infty} a_k\, u_k \tag{3.2}$$

is equivalent to the pointwise convergence of $\sum_{k=0}^{\infty} \Delta a_k D_k$ provided $\lim_{n\to\infty} a_n = 0$. Then we have by (3.1) that if $\sum_{k=0}^{\infty} |\Delta a_k| < \infty$, i.e. if (a_k) is of bounded variation, and $\lim_{n\to\infty} a_n = 0$ then (3.2) converges pointwise except possibly at $x = 0$.

Now let X be one of the spaces: L^p $(1 < p \leq \infty)$, $\log\alpha\, L^1+$ $+\alpha^{-1/q}L^p$ $(\alpha \geq 1,\ 1 < p \leq 2,\ 1/p + 1/q = 1)$, $\mathcal{H}$ (in trigonometric case),

H (in Walsh case), L_M (the Orlicz space defined by the Young function M (see Section 2.1)). By the results presented in Section 2.1 we have

$$\frac{1}{2^n}\left\|\sum_{k=2^n}^{2^{n+1}-1} \Delta a_k D_k\right\|_1 \leq C_X \,\|\Omega_n\|_X, \tag{3.3}$$

where $\Omega_n = \sum_{k=0}^{2^n-1} \Delta a_{2^n+k}\, \chi_{[k2^{-n},(k+1)2^{-n})}$.

Clearly,

$$\sum_{n=0}^{\infty} 2^n \|\Omega_n\|_X < \infty \tag{3.4}$$

implies $\sum_{n=0}^{\infty} 2^n \|\Omega_n\|_1 < \infty$, that is (a_k) is of bounded variation. Then (3.4) and $\lim_{n\to\infty} a_k = 0$ are sufficient to conclude that the pointwise limit of (3.2) exists. It is also easy to see that under these conditions the limit function is integrable and its Fourier series is (3.2). In other words, the class generated by these two conditions is an integrability class. Let us denote it by $\mathcal{X}$.

In order to prove that $\mathcal{X}$ is a convergence class suppose for a moment that $X \neq H, \mathcal{H}$. Let (a_k) satisfy (3.4) and $\lim_{n\to\infty} a_n = 0$. The following is a simple property of the X-norm. If g is a restriction of $h \in X$, i.e. $\operatorname{supp} g \subset \subset \operatorname{supp} h$, $g(x) = h(x)$ $(x \in \operatorname{supp} g)$, then $\|g\|_X \leq \|h\|_X$. Especially,

$$\left\|\sum_{k=\ell}^{2^n-1} \Delta a_{2^n+k}\, \chi_{[k2^{-n},(k+1)2^{-n})}\right\|_X \leq \|\Omega_n\|_X \qquad (0 \leq \ell < 2^n). \tag{3.5}$$

From here the L^1-convergence of $\sum_{k=0}^{\infty} \Delta a_k D_k$ follows immediately. Then we have by (1.2) that the L^1-convergence of (3.2) is equivalent to $\lim_{n\to\infty} a_n \|D_n\|_1 = = 0$. This means that $\mathcal{X}$ is an L^1-convergence class.

The class corresponding to $X = L^p$ $(1 < p < \infty)$ was introduced by Fomin [13] for the cosine system and now called as Fomin class. Its Walsh version was obtained by Móricz and Schipp [26]. These classes were extended by Tanović-Miller [36] by taking $X = \log\alpha\, L^1 + \alpha^{-1/q} L^p$ $(\alpha \geq 1,\ 1 < p \leq \leq 2,\ 1/p + 1/q = 1)$. The case $X = L_M$ determines an even larger class, which was proved by the author for the cosine and also for the Walsh system [15], [16]. Among these classes the latter is the only one that subsumes the class determined by (1.3). We note that in all these cases the condition that defines $\mathcal{X}$ can directly be expressed by the coefficients.

The above consideration does not work whithout changes for $X = H, \mathcal{H}$. The reason behind it is that (3.5) holds for Hardy spaces only in a modified form

(3.6)

$$\left\| \sum_{k=\ell}^{2^n-1} |\Delta a_{2^n+k}| \, \chi_{[k2^{-n},(k+1)2^{-n})} \right\|_X \leq \|\Omega_n\|_X \qquad (X = H, \mathcal{H},\ 0 \leq \ell < 2^n).$$

Using (3.6) convergence classes, larger than the ones listed above (see Schipp [32]) but subsets of $\mathcal{X}$ for $X = H, \mathcal{H}$, can be defined for the cosine and for the Walsh system by

$$\sum_{n=0}^{\infty} 2^n \| \, |\Omega_n| \, \|_X < \infty \qquad (X = H, \mathcal{H}), \qquad \lim_{n\to\infty} a_k = 0.$$

A simple observation shows that the first part of the above condition can be relaxed as follows

$$\sum_{n=0}^{\infty} 2^n \|\Omega_n\|_X < \infty, \qquad \lim_{n\to\infty} 2^n \| \, |\Omega_n| \, \|_X = 0 \qquad (X = H, \mathcal{H}).$$

Unfortunately, as it was mentioned before, unlike the other cases $\|\Delta_n\|_H$ and $\|\Delta_n\|_{\mathcal{H}}$ are difficult to express by the Δa_k's.

Observe that in the construction of $\mathcal{X}$ the indices 2^n play a special role. These indices come from the inequality (3.3), which is a consequence of (1.7), (2.2), (2.3), (2.5), (2.3), and their dyadic versions. It is clear that a similar construction is possible if the indices 2^n are replaced by a sequence (n_k) for which $n_{k+1}/(n_{k+1} - n_k) < C$. In order to enlarge the above classes by getting rid of this restriction for the indices one has to take the shifted inequalities (2.8), (2.9), (2.10). The one that corresponds to (2.9) was introduced by Buntinas and Tanović-Miller [9] for the cosine system. A larger class induced by (2.10) was defined by the author in [15] for the cosine system. It is the collection of null sequences (a_k) for which there exists a strictly increasing sequence of natural numbers (N_j) such that

$$\sum_{j=0}^{\infty}\left(\log\frac{N_{j+1}}{N_{j+1}-N_j}\left|\sum_{k=N_j}^{N_{j+1}-1}\Delta a_k\right|+\right.$$

$$\left.+\sum_{k=N_j}^{N_{j+1}-1}|\Delta a_k|\left(1+\log^+\frac{|\Delta a_k|}{(N_{j+1}-N_j)^{-1}\sum\limits_{\ell=N_j}^{N_{j+1}-1}|\Delta a_\ell|}\right)\right)<\infty \tag{3.7}$$

and

$$\lim_{j\to\infty}\log\frac{N_{j+1}}{N_{j+1}-N_j}\sum_{k=N_j}^{N_{j+1}-1}|\Delta a_k|=0\,.$$

The dyadic version of conditions (3.7) is given in [16].

3.2. Connections with other classes

In the literature there are several examples for integrability and L^1-convergence classes which do not rely on Sidon-type inequalities. In this section we cannot present the history of the development of such type of classes. We only deal with three of them which subsume many of the others and are of special interest. Some historical hints will also be given.

First we mention the class due to Telyakovskiĭ [37]. Namely, he proved that the conditions

$$\sum_{n=2}^{\infty}\left|\sum_{k=1}^{[n/2]}\frac{\Delta a_{n-k}-\Delta a_{n+k}}{k}\right|<\infty,\qquad \sum_{n=1}^{\infty}|\Delta a_n|<\infty,\qquad \lim_{n\to\infty}a_n=0 \tag{3.8}$$

($[x]$ denotes the integer part of x) induce an integrability and L^1-convergence class for the cosine system. The speciality of (3.8) is that however it was introduced in 1964, there have no proof been given, that any of the convergence classes constructed since then is an extension of it. We note that (3.8) is originated from a condition of Boas [6].

Very recently Buntinas and Tanović-Miller [10], and Aubertin and Fournier [1] showed independently that the class of null sequences (a_k) satisfying

$$\sum_{m=0}^{\infty}\left(\sum_{j=1}^{\infty}\left(\sum_{n=j2^m}^{(j+1)2^m-1}|\Delta a_n|\right)^2\right)^{1/2}<\infty \tag{3.9}$$

is an integrability and L^1-convergence class for even trigonometric series. The same result was proved for the Walsh system by Aubertin and Fournier in [2]. In connection with (3.9) we note that it is the result of a several steps refinement process of conditions of similar type. The class determined by (3.9) subsumes many of the previously known convergence classes, such as (1.3), $\mathcal{X}$ when $X = L^p$ $(1 < p \leq \infty)$ or $\log\alpha\, L^1 + \alpha^{-1/q}L^p$ $(\alpha \geq 1,\ 1 < p \leq 2,\ 1/p+1/q = 1)$.

Our last example is an extension of (3.9) given by Buntinas and Tanović-Miller in [10]. They showed that the condition of (3.9) can be relaxed as follows

$$\sum_{m=0}^{\infty}\max_{0\leq k<2^m}\left(\sum_{j=1}^{\infty}|a_{j2^m+k}-a_{(j+1)2^m}|^2\right)^{1/2}<\infty. \tag{3.10}$$

Next we show that the classes determined by (3.7) and (3.8) are incomparable with those corresponding to (3.9), (3.10). Aubertin and Fournier [2] gave an example for showing that (3.9) cannot be deduced from (3.8). Namely, an easy computation shows that the sequence $a_k = 1/(n+1)$ $(2^n < k \leq 2^{n+1})$ satisfies the condition of (3.9) but not of (3.8). It is also easy to see that this sequence does not satisfy (3.7). More generally, for sequences (a_k) which are constant on lacunary blocks $(n_k, n_{k+1}]$ (3.8) reduces to $\sum_{k=0}^{\infty}\log n_k\,|\Delta a_{n_k}| < \infty$. We note that similar estimation can be obtained for (3.7) (see [16]). On the other hand (3.9) is equivalent to $\sum_{k=0}^{\infty}\left(\sum_{j=k}^{\infty}|\Delta a_{n_j}|^2\right)^{1/2} < \infty$. Clearly, the latter is a weaker condition. Consequently, (3.9) performs better for this type of sequences than (3.8) and (3.7).

The counterpart of the above situation was given by the author in [16]. Let us take a lacunary sequence, i.e. $a_k \neq 0$ if and only if $k = n_j$ for some fixed lacunary sequence (n_j). Then, for such sequences (3.8) and (3.7) are equivalent to the trivial integrability condition $\sum_{j=0}^{\infty}|a_{n_j}| < \infty$, while (3.9) is $\sum_{k=0}^{\infty}\left(\sum_{j=k}^{\infty}|a_{n_j}|^2\right)^{1/2} < \infty$. We note that the same holds for (3.10) if for instance $n_j = 2^j$. Consequently, (3.8) and (3.7) are better to use for lacunary sequences than (3.9) or (3.10). Unfortunately, we do not know what the relation is between the classes induced by (3.7) and (3.8).

At the end we note that weaker conditions can be constructed for the L^1-convergence of cosine and Walsh series if it is assumed a priori that the series represents an integrable function. For such type of conditions see [35], [23].

3.3. Generalizations

In this section we show some of the generalizations of the results of the previous section, i.e. we show integrability and L^1-convergence conditions for the sine, the complex trigonometric and for the Vilenkin series.

Let us start with the sine system. Since the sine system can be originated from the cosine system by differentiation, several integrability conditions (see e.g. [17], [20], [40]) have been constructed for the sine system based on this relation. It turned out that the conditions given for the cosine system are usually not sufficient for the integrability of the corresponding sine series. Therefore the problem was to find an additional condition that together with the one given for the cosine system already guarantee the integrability of the sine series. Now we show a result of Telyakovskiĭ [39] as an example.

Suppose (a_k) belongs to the Telyakovskiĭ class (see the definition in Section 3.1). Then $\sum_{k=1}^{\infty} a_k \sin kx$ is a Fourier series if and only if

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty. \tag{3.11}$$

Here, the additional condition is $\sum_{k=1}^{\infty} |a_k|/k < \infty$. A similar result holds with respect to the Fomin class [13].

In order to see that this is the general situation consider $\sum_{k=1}^{\infty} a_k \sin kx$ as the conjugate series of $\sum_{k=1}^{\infty} a_k \cos kx$. If the latter one is the Fourier series of an integrable function f then the integrability of the conjugate series implies (see e.g. [18]) $f \in \mathcal{H}$. Then (3.11) follows from Hardy's inequality (see e.g. [44])

with respect to the Fourier coefficients of functions in $\mathcal{H}$. Consequently, (3.11) is necessary to the integrability of $\sum_{k=1}^{\infty} a_k \sin kx$.

If (a_k) satisfies a Sidon-type integrability condition for the cosine system, i.e. for instance $(a_k) \in \mathcal{X}$ $(X = L^p \ (1 < p \le \infty),\ \log \alpha L^1 + \alpha^{-1/q} L^p \ (\alpha \ge 1,\ 1 < p \le 2,\ 1/p + 1/q = 1),\ L_M)$, $\mathcal{H}$, H then the sufficiency of (3.11) can also be proved. The method is given in detail in [15] for the case $X = L_M$. It is based on the following restricted Sidon-type inequality for the complex trigonometric system with nonnegative indices

$$\frac{1}{2^n} \int_0^{\pi} \left| \sum_{k=K}^{K+2^n-1} a_k \sum_{j=0}^{k} e^{ijx} \right| dx \le \|\Gamma_n\|_{\mathcal{H}},$$

provided $\sum_{k=K}^{K+2^n-1} a_k = 0$. Here, $\Gamma_n = \sum_{k=0}^{2^n-1} a_{k+K}\, \chi_{[k2^{-n},(k+1)2^{-n})}$.

We note that the above inequality can be deduced from the results of Schipp [32], however it is not stated explicitly there. Using this inequality the proof can be completed by decomposing (a_k) into the sum of two sequences, the first term of which is constant in dyadic blocks and the other one is having blockwise zero sum.

Obviously the method described above can be applied for constructing conditions which imply convergence in Hardy norm. Such conditions were given by the author in [15]. For the dyadic case see [16].

It is clear that proper combinations of the results given for the cosine and sine series lead to various integrability and L^1-convergence conditions for the trigonometric and for the complex trigonometric series. In connection with the trigonometric and the complex trigonometric case we refer to [11], [14], [22], [27], [35]. Generalizations for the multidimensional case can be found in [26], [41], [40].

A possible way to generalize the integrability and L^1-convergence conditions for the Walsh system is to extend them for the Vilenkin system. It means usually only technicalities if the Vilenkin system is of bounded type but this is not the situation for the unbounded case. For instance, as it was shown by Nurpeisov in [28], there exist Vilenkin systems of unbounded type for which the quasiconvex and even the convex sequences do not form an integrability class. Consequently, the dyadic results cannot be transferred automatically for unbounded Vilenkin systems. In order to show positive results let us define the concept of Vilenkin systems. For the sake of simplicity, here we only give the definition of the so called multiplicative case. For the general definition we refer to [4], [42].

Let (m_k) be a sequence of integers which satisfy $m_k \geq 2$, and let Z_{m_k} represent the discrete cyclic group of order m_k. Then the *multiplicative Vilenkin group* G_m generated by (m_k) is defined as the direct product of the cyclic groups Z_{m_k}. The Vilenkin system (ψ_n) is the character system of G_m ordered in Paley's sense. The Vilenkin system is called to be of bounded type if the generating sequence (m_k) is bounded. Set $M_0 = 1$, $M_{k+1} = m_k M_k$.

An integrability condition that can be considered as modified quasiconvexity was proved by Nurpeisov [28]. Namely, he showed that if $\lim_{k\to\infty} a_k = 0$ and

$$\sum_{n=0}^{\infty} \log b_n \sum_{k=M_n}^{M_{n+1}-1} k\,|\Delta^2 a_k| < \infty, \tag{3.12}$$

where $b_n = \max_{1\leq j\leq n} m_j$, then $\sum_{n=0}^{\infty} a_n\psi_n$ is a Vilenkin-Fourier series. He also showed that if the system is of unbounded type then for certain type of sequences condition (3.12) is not only sufficient but also necessary for the integrability of $\sum_{n=0}^{\infty} a_n\psi_n$.

Similarly, Avdispahić and Pepić [5] introduced an integrability class by a modified Fomin type condition. That is the collection of null sequences which satisfy

$$\sum_{n=1}^{\infty} \left(\left(\sum_{j=1}^{n} m_{j-1} M_j^{1/q} \right) \left(\sum_{k=M_n}^{M_{n+1}-1} |\Delta a_k|^p \right)^{1/p} \right) < \infty$$

$$(1 < p \leq 2,\ 1/p + 1/q = 1)$$

and

$$\sum_{n=0}^{\infty} \log m_n \sum_{k=M_n}^{M_{n+1}-1} |\Delta a_k| < \infty.$$

We note that a Fomin type integrability and L^1-convergence class for the bounded case was introduced earlier by Bloom and Fournier [8].

Recently, Aubertin and Fournier [3] were able to find the Vilenkin version of the condition (3.9). They proved that (3.9) together with a simmetry condition, which is similar to the ones used for the complex trigonometric system, imply the integrability of the corresponding Vilenkin series. Their approach reflects to the connection between the Vilenkin and the trigonometric case.

Finally, we mention the papers of Móricz and Schipp [24], concerning the generalization of Fomin's condition for multidimensional Walsh series.

References

[1] **Aubertin B. and Fournier J.**, An integrability theorem for Walsh series, *Boll. Un. Mat. Ital.* (to appear)

[2] **Aubertin B. and Fournier J.**, Integrability theorems for trigonometric series, *Studia Math.*, **107** (1) (1993), 33-59.

[3] **Aubertin B. and Fournier J.**, An integrability theorem for unbounded Vilenkin systems (to appear)

[4] **Агаев Г.Н., Виленкин Н.Я., Джафарли Г.М. и Рубинштейн А.Н.**, *Мультипликативные системы функций и гармонический анализ на нуль-мерных группах*, ЭЛМ, Баку, 1981. (Agaev G.N., Vilenkin N.Ya., Džafarli G.M. and Rubinštein A.N., *Multiplicative systems of functions and harmonic analysis on zero-dimensional groups*, ELM, Baku, 1981.)

[5] **Avdispahić M. and Pepić M.**, An integrability theorem on unbounded Vilenkin groups, *J. Math. Anal. Appl.*, **175** (1993), 438-447.

[6] **Boas R.P.**, Absolute convergence and integrability of trigonometric series, *J. Rat. Mech. and Anal.*, **5** (1956), 621-632.

[7] **Bojanic R. and Stanojević Č.**, A class of L^1 convergence, *Trans. Amer. Math. Soc.*, **269** (1982), 677-683.

[8] **Bloom W. and Fournier J.**, Smoothness conditions and integrability theorems on bounded Vilenkin groups, *J. Austral. Math. Soc. 5 (Ser. A)*, **4** (1988), 46-61.

[9] **Buntinas M. and Tanović-Miller N.**, New integrability and L^1 convergence classes for even trigonometric series, *Radovi Mat.*, **6** (1990), 149-170.

[10] **Buntinas M. and Tanović-Miller N.**, New integrability and L^1 convergence classes for even trigonometric series II., *"Approximation Theory" Coll. Math. Soc. J. Bolyai* **68**, eds. J.Szabados and K.Tandori, North-Holland, 1991, 103-125.

[11] **Chang-Pao Chen**, L^1-convergence of Fourier series, *J. Austral. Math. Soc. (Ser. A)*, **41** (1986), 376-390.

[12] **Фомин А.А.**, О линейных методах суммирования двойных рядов Фурье, *Изв. вузов Математика*, **2(81)** (1969), 106-116. (Fomin A.A., Linear methods of summing double Fourier series, *Izv. Vyssh. Uchebn. Zaved. Mat.*, 2(81) (1969), 106-116.)

[13] **Фомин Г.А.**, Об одном классе тригонометрических рядов, *Мат. заметки,* **23** (1978), 213-222. (Fomin G.A., A class of trigonometric series, *Mat. Zametki,* 23 (1978), 213-222.)

[14] **Fournier J and Self W.**, Some sufficient conditions for uniform convergence of Fourier series, *J. Math. Anal. Appl.,* **126** (1987), 355-374.

[15] **Fridli S.**, An inverse Sidon type inequality, *Studia Math.,* **105** (3) (1993), 283-308.

[16] **Fridli S**, An inverse Sidon type inequality for the Walsh system, *J. Math. Anal. Appl.* (to appear)

[17] **Takeshi Kano**, Coefficients of some trigonometric series, *J. Fac. Sci. Shinshu Univ.,* **3** (1968), 153-162.

[18] **Кашин Б.С. и Саакян А.А.**, *Ортогональные ряды,* Наука, Москва, 1984. (Kašin B.S. and Saakjan A.A., Orthogonal series, Nauka, Moscow, 1984.)

[19] **Kolmogorov S.A.**, Sur l'ordre de grandeur des coefficients de la série de Fourier-Lebesgue, *Bull. Acad. Polon. Sci. (A), Sci. Math.,* (1923), 83-86.

[20] **Móricz F.**, On the integrability and L^1-convergence of sine series, *Studia Math.,* **92** (1989), 187-200.

[21] **Móricz F.**, Sidon type inequalities, *Publ. de L'Inst. Math.,* **48 (62)** (1990), 101-109.

[22] **Móricz F.**, Lebesgue integrability and L^1-convergence of trigonometric series with special coefficients, *"Approximation Theory " Coll. Math. Soc. J. Bolyai* **68**, eds. J. Szabados and K. Tandori, North Holland, 1991, 513-536.

[23] **Móricz F.**, On L^1-convergence of Walsh-Fourier series II., *Acta Math. Hungar.,* **58** (1-2) (1991), 203-210.

[24] **Móricz F.**, On the integrability and L^1-convergence of double trigonometric series, *Studia Math.,* **98** (1991), 203-225.

[25] **Móricz F.**, On the integrability and L^1-convergence of complex trigonometric series, *Proc. Amer. Math. Soc.,* **113** (1991), 53-64.

[26] **Móricz F. and Schipp F.**, On the integrability and L^1-convergence of Walsh series with coefficients of bounded variation, *J. Math. Anal. Appl.,* **146** (1990), 99-109.

[27] **Móricz F. and Schipp F.**, On the integrability and L^1-convergence of double Walsh series, *Acta Math Hungar.,* **57** (3-4) (1991), 371-380.

[28] **Nurpeisov R.Ž.**, On the uniform $(C,1)$-summability of Fourier series and integrability of series with respect to multiplicative orthonormal systems, *Analysis Math.,* **15** (1989), 127-143.

[29] **Price J.J.**, Sparse subsets of orthonormal systems, *Proc. Amer. Math. Soc.,* **35** (1972), 161-164.

[30] **Schipp F., Wade W.R., Simon P. and Pál J.**, *Walsh series,* Adam Hilger, Bristol-New York, 1990.

[31] **Schipp F.**, On Sidon type inequalities, *"Approximation Theory " Coll. Math. Soc. J. Bolyai* **68**, eds J. Szabados and K. Tandori, North Holland, 1991, 603-614.

[32] **Schipp F.**, Sidon-type inequalities, *Approximation Theory (Lecture notes in pure and applied mathematics* **138***)*, ed. G.A. Anastassiou, Marcel Dekker, 1991, 421-436.

[33] **Sidon S.**, Hinreichende Bedingungen für den Fourier-Charakter einer trigonometrischen Reihe, *J. London Math. Soc.,* **14** (1939), 158-160.

[34] **Stanojević Č.V. and Stanojević V.B.**, Generalizations of the Sidon-Telyakovskii theorem, *Proc. Amer. Math. Soc.,* **101** (1987), 679-684.

[35] **Stanojević Č.V.**, Structure of Fourier and Fourier-Stieltjes coefficients of series with slowly varying convergence moduli, *Bull. Amer. Math. Soc. (N.S.),* **19** (1988), 283-286.

[36] **Tanović-Miller N.**, On integrability and L^1-convergence of cosine series, *Boll. Un. Mat. It. (7),* **4-B** (1990), 499-516.

[37] **Теляковский С.А.**, Условия интегрируемости тригонометрических рядов и их приложение к изучению линейных методов суммирования рядов Фурье, *Изв. АН СССР, Сер. мат.,* **28** (1964), 1209-1236. (Telyakovskiĭ S.A., Integrability conditions of trigonometric series and their applications to the study of linear methods of summing Fourier series, *Izv. Akad. Nauk. SSSR, Ser. Mat.,* **28** (1964), 1209-1236.)

[38] **Теляковский С.А.**, Некоторые оценки для тригонометрических рядов с квазивыпуклыми коэффициентами, *Мат. сб.,* **63(105)** (1964), 426-444. (Telyakovskiĭ S.A., Some estimates for trigonometric series with quasi-convex coefficients, *Mat. Sb.,* **63(105)** (1964), 426-444.)

[39] **Теляковский С.А.**, Об одном достаточном условии Сидона интегрируемости тригонометрических рядов, *Мат. зам.,* **14** (1973), 317-328. (Telyakovskiĭ S.A., On a sufficient condition of Sidon for the integrability of trigonometric series, *Mat. Zametki,* **14** (1973), 317-328.)

[40] **Telyakovskiĭ S.A.**, On the integrability of sine series, *Proc. Steklov Inst. Mat.,* **4** (1985), 269-273.

[41] **Telyakovskiĭ S.A.**, On conditions for integrability of multiple trigonometric series, *Proc. Steklov Inst. Mat.,* **2** (1985), 205-215.

[42] **Виленкин Н.Я.**, Об одном классе полных ортогональных систем, *Изв. АН СССР, Сер. мат.,* **11** (1947), 363-400. (Vilenkin N.Ya., On a class of orthogonal series, *Izv. Akad. Nauk. SSSR Ser. Mat.,* **11** (1947), 363-400.)

[43] **Young W.H.,** On the Fourier series of bounded functions, *Proc. London Math. Soc.,* **(2) 12** (1913), 41-70.

[44] **Zygmund A.,** *Trigonometric series,* University Press, Cambridge, 1959.

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