

## CONVERGENCE OF THE SPLINE SOLVING (0,m) LACUNARY INTERPOLATION PROBLEM

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*Dedicated to Professor János Balázs on his 75-th birthday*

### 1. Introduction

In 1955 J. Surányi and P. Turán [10] commenced the study of (0,2) lacunary interpolation. Several papers appeared [1-3, 5-9] concentrating on the use of spline polynomials to solve the (0,2) and (0,3) lacunary interpolation problems.

In this paper we show that the convergence of constructed in [4] methods for solving the general lacunary interpolation problem (0,  $m$ ) is the same as the best order of approximation with  $m$ -degree polynomial splines, where  $m$  is integer and  $m \geq 2$ .

### 2. Convergence of the spline polynomials of degrees 5 and 6

In this section we shall study the convergence of the spline polynomials (viz. the (0,5) and (0,6) cases) considered in [4]. Thus, by using the construction of  $S_{\Delta,5}(x)$  given in [4], we introduce the following theorems.

**Theorem 2.1.** *Let  $S_{\Delta,5}(x)$  denote the spline function given in theorem 2.1 [4]. Then the following inequalities*

$$|f^{(j)}(x) - S_{k,5}^{(j)}(x)| \leq C_{k,j} h^{5-j} \omega(D^5 f; h), \quad x_k \leq x \leq x_{k+1}$$

hold for all  $k = 0(1)n - 1$ ,  $j = 0(1)5$ , where  $\omega(D^5 f; h)$  denotes the modulus of continuity of  $f^{(5)}(x)$  and  $C_{k,j}$  are constants independent of  $h$ .

**Proof.** From the spline function  $S_{\Delta,5}(x)$  given in [4] and Taylor expansion

$$\begin{aligned} S_k^{(5)} &= f_k^{(5)}, & S_k^{(0)} &= f_k, \\ S_k^{(4)} &= f_k^{(4)} + \frac{32}{5!} h f_k^{(5)}(\varphi_{k,2}) - \frac{4}{5!} h f_k^{(5)}(\varphi_{k,1}) - \frac{32}{5!} h f_k^{(5)}(\theta_{k,2}) - \frac{4}{5!} h f_k^{(5)}(\theta_{k,1}), \\ S_k^{(3)} &= f_k^{(3)} + \frac{12}{4!} h f_k^{(4)} + \frac{32}{5!} h^2 f_k^{(5)}(\varphi_{k,2}) - \frac{3}{5!} h^2 f_k^{(5)}(\varphi_{k,1}) + \frac{1}{5!} h^2 f_k^{(5)}(\theta_{k,1}) - \\ &\quad - \frac{12}{4!} h S_k^{(4)} - \frac{30}{5!} h^2 S_k^{(5)}, \\ S_k^{(2)} &= f_k^{(2)} + \frac{2}{4!} h^2 f_k^{(4)} + \frac{1}{5!} h^3 f_k^{(5)}(\varphi_{k,1}) - \frac{2}{4!} h^2 S_k^{(4)} - \frac{1}{5!} h^3 f_k^{(5)}(\theta_{k,1}) \end{aligned}$$

and

$$S_k^{(1)} = f_k^{(1)} + \sum_{r=2}^4 \frac{1}{r!} h^{r-1} f_k^{(r)} + \frac{1}{5!} h^4 f_k^{(5)}(\varphi_{k,1}) - \sum_{r=2}^4 \frac{1}{r!} h^{r-1} S_k^{(r)} - \frac{1}{5!} h^4 S_k^{(5)}.$$

From the Taylor expansion and spline function  $S_{\Delta,m}(x)$ , defined in [4], we get

$$\begin{aligned} (2.1) \quad & |f^{(j)}(x) - S_{k,m}^{(j)}(x)| \leq \\ & \leq \sum_{r=j}^{m-1} \frac{1}{(r-j)!} |f_k^{(r)} - S_k^{(r)}| |x - x_k|^{r-j} + \frac{1}{(m-j)!} |f_k^{(m)}(\varphi_{k,j}) - S_k^{(m)}| |x - x_k|^{m-j}. \end{aligned}$$

Using the Taylor expansion and the spline function  $S_{k,5}(x)$  we obtain

$$(2.2) \quad |f_k^{(5)}(\varphi_{k,0}) - S_k^{(5)}| \leq \omega(D^5 f; h),$$

$$\begin{aligned} (2.3) \quad & |f_k^{(4)} - S_k^{(4)}| \leq \\ & \leq \frac{32}{5!} h |f_k^{(5)}(\varphi_{k,2}) - f_k^{(5)}(\theta_{k,2})| + \frac{4}{5!} h |f_k^{(5)}(\varphi_{k,1}) - f_k^{(5)}(\theta_{k,1})| \leq \frac{36}{5!} h \omega(D^5 f; h), \end{aligned}$$

$$\begin{aligned} (2.4) \quad & |f_k^{(3)} - S_k^{(3)}| \leq \\ & \leq \frac{12}{4!} h |f_k^{(4)} - S_k^{(4)}| + \frac{30}{5!} h^2 |f_k^{(5)}(\varphi_{k,2}) - S_k^{(5)}| + \frac{2}{5!} h^2 |f_k^{(5)}(\varphi_{k,2}) - f_k^{(5)}(\varphi_{k,1})| + \end{aligned}$$

$$+\frac{1}{5!}h^2|f_k^{(5)}(\theta_{k,1}) - f_k^{(5)}(\varphi_{k,1})| \leq \frac{51}{5!}h^2\omega(D^5 f; h),$$

$$(2.5) \quad |f_k^{(2)} - S_k^{(2)}| \leq \frac{2}{4!}h^2|f_k^{(4)} - S_k^{(4)}| + \frac{1}{5!}h^3|f_k^{(5)}(\varphi_{k,1}) - f_k^{(5)}(\theta_{k,1})| \leq \frac{4}{5!}h^3\omega(D^5 f; h)$$

and

$$(2.6) \quad |f_k^{(1)} - S_k^{(1)}| \leq \sum_{r=2}^4 \frac{1}{r!}h^{r-1}|f_k^{(r)} - S_k^{(r)}| + \frac{1}{5!}h^4|f_k^{(5)}(\varphi_{k,1}) - S_k^{(5)}| \leq \frac{13}{5!}h^4\omega(D^5 f; h).$$

By substituting (2.2)-(2.6) into (2.1), we get

$$|f^{(j)}(x) - S_{k,5}^{(j)}(x)| \leq C_j h^{5-j}\omega(D^5 f; h),$$

where  $j = 0(1)5$  and the coefficients  $C_j$  are given in the table

$C_0$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$
$\frac{26}{5!}$	$\frac{107}{2!5!}$	$\frac{93}{5!}$	$\frac{147}{5!}$	$\frac{156}{5!}$	1

Thus we complete the proof of convergence of the spline function  $S_{k,5}(x)$  and their derivatives.

**Theorem 2.2.** *Let  $S_{\Delta,6}(x)$  denote the spline function given in Theorem 2.1 in [4], then the following inequalities*

$$|f^{(j)}(x) - S_{k,6}^{(j)}(x)| \leq B_{k,j}h^{6-j}\omega(D^6 f; h), \quad x_k \leq x \leq x_{k+1}$$

hold for all  $k = 0(1)n$ ,  $j = 0(1)6$ , where  $\omega(D^6 f; h)$  denotes the modulus of continuity of  $f^{(6)}(x)$  and  $B_{k,j}$  are constants independent of  $h$ .

**Proof.** From the spline function  $S_{\Delta,6}(x)$ , given in [4], and Taylor expansions we get

$$\begin{aligned} S_k^{(6)} &= f_k^{(6)}, \quad S_k^{(0)} = f_k, \\ S_k^{(5)} &= f_k^{(5)} + \frac{729}{6!}hf_k^{(6)}(\varphi_{k,3}) - \frac{320}{6!}hf_k^{(6)}(\varphi_{k,2}) + \frac{10}{6!}hf_k^{(6)}(\varphi_{k,1}) + \\ &\quad + \frac{5}{6!}hf_k^{(6)}(\theta_{k,1}) - \frac{64}{6!}hf_k^{(6)}(\theta_{k,2}) - \frac{360}{6!}hS_k^{(6)}, \end{aligned}$$

$$\begin{aligned}
S_k^{(4)} &= f_k^{(4)} + \frac{64}{6!} h^2 f_k^{(6)}(\varphi_{k,2}) - \frac{4}{6!} h^2 f_k^{(6)}(\varphi_{k,1}) + \frac{64}{6!} h^2 f_k^{(6)}(\theta_{k,2}) - \\
&\quad - \frac{4}{6!} h^2 f_k^{(6)}(\theta_{k,1}) - \frac{120}{6!} h^2 S_k^{(6)}, \\
S_k^{(3)} &= f_k^{(3)} + \frac{12}{4!} h f_k^{(4)} + \frac{30}{5!} h^2 f_k^{(5)} + \frac{64}{6!} h^3 f_k^{(6)}(\varphi_{k,2}) - \frac{3}{6!} h^3 f_k^{(6)}(\varphi_{k,1}) - \\
&\quad - \frac{1}{6!} h^3 f_k^{(6)}(\theta_{k,1}) - \frac{12}{4!} h S_k^{(4)} - \frac{30}{5!} h^2 S_k^{(5)} - \frac{60}{6!} h^3 S_k^{(6)}, \\
S_k^{(2)} &= f_k^{(2)} + \frac{2}{4!} h^2 f_k^{(4)} + \frac{1}{6!} h^4 f_k^{(6)}(\varphi_{k,1}) + \frac{1}{6!} h^4 f_k^{(6)}(\theta_{k,1}) - \\
&\quad - \frac{2}{4!} h^2 S_k^{(4)} - \frac{2}{6!} h^4 S_k^{(6)}
\end{aligned}$$

and

$$\begin{aligned}
S_k^{(1)} &= f_k^{(1)} + \sum_{r=2}^5 \frac{1}{r!} h^{r-1} f_k^{(r)} + \frac{1}{6!} h^5 f_k^{(6)}(\varphi_{k,1}) - \sum_{r=2}^5 \frac{1}{r!} h^{r-1} S_k^{(r)} - \\
&\quad - \frac{1}{6!} h^5 S_k^{(6)}.
\end{aligned}$$

Using the Taylor expansion and the spline function  $S_{k,6}(x)$  we get

$$(2.8) \quad |f_k^{(6)}(\varphi_{k,0}) - S_k^{(6)}| \leq \omega(D^6 f; h),$$

$$\begin{aligned}
(2.9) \quad &|f_k^{(5)} - S_k^{(5)}| \leq \\
&\leq \frac{360}{6!} h |f_k^{(6)}(\varphi_{k,3}) - S_k^{(6)}| + \frac{320}{6!} h |f_k^{(6)}(\varphi_{k,3}) - f_k^{(6)}(\varphi_{k,2})| + \\
&+ \frac{49}{6!} h |f_k^{(6)}(\varphi_{k,3}) - f_k^{(6)}(\theta_{k,2})| + \frac{16}{6!} h |f_k^{(6)}(\varphi_{k,1}) - f_k^{(6)}(\theta_{k,2})| + \\
&+ \frac{5}{6!} h |f_k^{(6)}(\theta_{k,1}) - f_k^{(6)}(\theta_{k,2})| \leq \frac{744}{6!} h \omega(D^6 f; h),
\end{aligned}$$

$$\begin{aligned}
(2.10) \quad &|f_k^{(4)} - S_k^{(4)}| \leq \\
&\leq \frac{64}{6!} h^2 |f_k^{(6)}(\varphi_{k,2}) - S_k^{(6)}| + \frac{56}{6!} h^2 |f_k^{(6)}(\theta_{k,2}) - S_k^{(6)}| + \frac{4}{6!} h^2 |f_k^{(6)}(\theta_{k,2}) - \\
&- f_k^{(6)}(\varphi_{k,1})| + \frac{4}{6!} h^2 |f_k^{(6)}(\theta_{k,2}) - f_k^{(6)}(\theta_{k,1})| \leq \frac{128}{6!} h^2 \omega(D^6 f; h),
\end{aligned}$$

$$\begin{aligned}
 (2.11) \quad & |f_k^{(3)} - S_k^{(3)}| \leq \\
 & \leq \frac{12}{4!}h|f_k^{(4)} - S_k^{(4)}| + \frac{30}{5!}h^2|f_k^{(5)} - S_k^{(5)}| + \frac{60}{6!}h^3|f_k^{(6)}(\varphi_{k,2}) - S_k^{(6)}| + \\
 & + \frac{3}{6!}h^3|f_k^{(6)}(\varphi_{k,2}) - f_k^{(6)}(\varphi_{k,1})| + \frac{1}{6!}h^3|f_k^{(6)}(\varphi_{k,2}) - f_k^{(6)}(\theta_{k,1})| \leq \\
 & \leq \frac{314}{6!}h^3\omega(D^6 f; h),
 \end{aligned}$$

$$\begin{aligned}
 (2.12) \quad & |f_k^{(2)} - S_k^{(2)}| \leq \\
 & \leq \frac{2}{4!}h^2|f_k^{(4)} - S_k^{(4)}| + \frac{1}{6!}h^4|f_k^{(6)}(\varphi_{k,1}) - S_k^{(6)}| + \frac{1}{6!}h^4|f_k^{(4)}(\theta_{k,1}) - S_k^{(6)}| \leq \\
 & \leq \frac{76}{3!6!}h^4\omega(D^6 f; h)
 \end{aligned}$$

and

$$\begin{aligned}
 (2.13) \quad & |f_k^{(1)} - S_k^{(1)}| \leq \\
 & \leq \sum_{r=2}^5 \frac{1}{r!}h^{r-1}|f_k^{(r)} - S_k^{(r)}| + \frac{1}{6!}h^5|f_k^{(6)}(\varphi_{k,1}) - S_k^{(6)}| \leq \\
 & \leq \frac{8544}{5!6!}h^5\omega(D^6 f; h).
 \end{aligned}$$

Combining (2.1) and (2.8)-(2.13) we get

$$|f^{(j)}(x) - S_{k,6}^{(j)}(x)| \leq C_j h^{6-j} \omega(D^6 f; h),$$

where  $j = 0(1)6$  and the coefficients  $C_j$  are

$C_0$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$
$\frac{17088}{5!6!}$	$\frac{35904}{5!6!}$	$\frac{3268}{3!6!}$	$\frac{934}{6!}$	$\frac{1232}{6!}$	$\frac{1464}{6!}$	1

Thus the proof of Theorem 2.2 is completed.

### 3. The convergence of the general formula $S_{k,m}(x)$

**Theorem 3.1** Let  $S_{k,m}^{(r)}(x)$  be the spline polynomial given in Theorem 3.1 in [4], then the inequalities

$$(2.14) \quad |f^{(r)}(x) - S_{k,m}^{(r)}(x)| \leq C_{km,r} h^{m-r} \omega(D^m f; h), \quad x \in [x_k, x_{k+1}]$$

hold for all  $k=0(1)n$ ,  $r=0(1)m$ , where  $\omega(D^m f; h)$  denotes the modulus of continuity of  $f^{(m)}(x)$  and  $C_{km,r}$  are constants independent of  $h$ .

**Proof.** From Theorems 2.1 and 2.2 we conclude that

(i) For  $m = 1, 2, 3, \dots$ ,  $r = 1, 3, 5, \dots$

$$(2.15) \quad |f_k^{(r)} - S_k^{(r)}| \leq \sum_{n=r+1}^{m-1} C_{rn} |f_k^{(n)} - S_k^{(n)}| h^{n-r} +$$

$$+ \frac{1}{m!} \left| \sum_{L=1}^r C_{rL} f_k^{(m)}(\varphi_{kL}) - C_{n,m-n} S_k^{(m)} \right| h^{m-n} \leq C_{rm} h^{m-n} \omega_m(h),$$

$$\sum_{L=1}^r C_{rL} = C_{n,m-n}.$$

(ii) For  $m = 2, 4, 6, \dots$ ,  $r = 2, 4, 6, \dots$

$$(2.16) \quad |f_k^{(r)} - S_k^{(r)}| \leq \sum_{n=r+2}^{m-2} C_{rn} |f_k^{(n)} - S_k^{(n)}| h^{n-r} +$$

$$+ \frac{1}{m!} \left| \sum_{L=1}^r C_{rL} f_k^{(m)}(\varphi_{kL}) - C_{n,m-n} S_k^{(m)} \right| h^{m-n} \leq C_{rm} h^{m-n} \omega_m(h),$$

$$\sum_{L=1}^r C_{rL} = C_{n,m-n}$$

and

(iii) for  $m = 1, 3, 5, \dots$ ,  $r = 2, 4, 6, \dots$

$$\begin{aligned}
 (2.17) \quad |f_k^{(r)} - S_k^{(r)}| &\leq \sum_{n=r+2}^{m-1} C_{rn} |f_k^{(n)} - S_k^{(n)}| h^{n-r} + \\
 &+ \frac{1}{m!} \left| \sum_{L=1}^{r/2} C_{rL} f_k^{(m)}(\varphi_{kL}) - \sum_{L=\frac{r}{2}+1}^r C_{rL} f_k^{(m)}(\varphi_{kL}) \right| \leq \\
 &\leq C_{rm} h^{m-n} \omega_m(h), \quad \text{where} \quad \sum_{L=1}^{r/2} C_{rL} = \sum_{L=\frac{r}{2}+1}^r C_{rL}.
 \end{aligned}$$

To prove the convergence of  $S_{k,m}$  we have two cases ( $m$  is odd or even).

*Case (1): The convergence of  $S_{k,m}(x)$  when  $m$  is odd*

From Theorem 2.1 we can define the constants  $S_k^{(n)}$  as follows:

$$\begin{aligned}
 (2.18) \quad S_k^{(1)} &= f_k^{(i)} + 1! \left\{ \frac{1}{2!} h f_k^{(2)} + \frac{1}{3!} h^2 f_k^{(3)} + \frac{1}{4!} h^3 f_k^{(4)} + \dots + \frac{1}{m!} h^{m-1} f_k^{(m)}(\varphi_{ki}) \right\} - \\
 &- 1! \left\{ \frac{1}{2!} h S_k^{(2)} + \frac{1}{3!} h^2 S_k^{(3)} + \frac{1}{4!} h^3 S_k^{(4)} + \dots + \frac{1}{m!} h^{m-1} S_k^{(m)} \right\},
 \end{aligned}$$

$$\begin{aligned}
 (2.19) \quad S_k^{(2)} &= f_k^{(2)} + \\
 &+ 2! \left\{ \frac{1}{4!} h^2 f_k^{(4)} + \frac{1}{6!} h^4 f_k^{(6)} + \frac{1}{8!} h^6 f_k^{(8)} + \dots + \frac{1}{(m-1)!} h^{m-3} f_k^{(m-1)} \right\} - \\
 &- 2! \left\{ \frac{1}{4!} h^2 S_k^{(4)} + \frac{1}{6!} h^4 S_k^{(6)} + \frac{1}{8!} h^6 S_k^{(8)} + \dots + \frac{1}{(m-1)!} h^{m-3} S_k^{(m-1)} \right\} + \\
 &+ \frac{1}{m!} h^{m-2} f_k^{(m)}(\varphi_{k1}) - \frac{1}{m!} h^{m-2} f_k^{(m)}(\varphi_{k2}),
 \end{aligned}$$

$$\begin{aligned}
 (2.20) \quad S_k^{(3)} &= f_k^{(3)} + \\
 &+ 3! \left\{ \frac{2}{4!} h f_k^{(4)} + \frac{5}{5!} h^2 f_k^{(5)} + \frac{10}{6!} h^3 f_k^{(6)} + \dots + \frac{1}{m!} \sum_{L=1}^3 C_{3L} h^{m-3} f_k^{(m)}(\varphi_{kL}) \right\} - \\
 &- 3! \left\{ \frac{2}{4!} h S_k^{(4)} + \frac{5}{5!} h^2 S_k^{(5)} + \frac{10}{6!} h^3 S_k^{(6)} + \dots + \frac{1}{m!} C_{3,m-3} h^{m-3} S_k^{(m)} \right\},
 \end{aligned}$$

where

$$\sum_{L=1}^3 C_{3L} = C_{3,m-3},$$

$$(2.21) \quad S_k^{(4)} = f_k^{(4)} +$$

$$+ 4! \left\{ \frac{5}{6!} h^2 f_k^{(6)} + \frac{21}{8!} h^4 f_k^{(8)} + \frac{85}{10!} h^6 f_k^{(6)} + \dots + \frac{1}{(m-1)!} C_{4,m-5} h^{m-5} f_k^{(m-1)} \right\}$$

$$- 4! \left\{ \frac{5}{6!} h^2 S_k^{(6)} + \frac{21}{8!} h^4 S_k^{(8)} + \frac{85}{10!} h^6 S_k^{(6)} + \dots + \frac{1}{(m-1)!} C_{4,m-5} h^{m-5} S_k^{(m-1)} \right\}$$

$$+ \frac{1}{m!} \sum_{L=1,2} C_{4L} h^{m-4} f_k^{(m)}(\varphi_{kL}) + \frac{1}{m!} \sum_{L=3,4} C_{4L} h^{m-4}(\varphi_{kL}),$$

where

$$\sum_{L=1,2} C_{4L} = \sum_{L=3,4} C_{4L},$$

$$(2.22) \quad S_k^{(5)} = f_k^{(5)} +$$

$$+ 5! \left\{ \frac{3}{6!} h f_k^{(6)} + \frac{14}{7!} h^2 f_k^{(7)} + \frac{42}{8!} f_k^{(8)} h^3 f_k^{(8)} + \dots + \frac{1}{m!} \sum_{L=1}^5 C_{5L} h^{m-5} f_k^{(m)}(\varphi_{kL}) \right\} -$$

$$- 5! \left\{ \frac{3}{6!} h S_k^{(6)} + \frac{14}{7!} h^2 S_k^{(7)} + \frac{42}{8!} h^3 S_k^{(8)} + \dots + \frac{1}{m!} C_{5,m-5} h^{m-5} S_k^{(m)} \right\},$$

$$\sum_{L=1}^5 C_{5L} = C_{5,m-5},$$

$$(2.23) \quad S_k^{(6)} = f_k^{(6)} + 6! \cdot \left[ \right.$$

$$\left. \left\{ \frac{14}{8!} h^2 f_k^{(8)} + \frac{147}{10!} h^4 f_k^{(10)} + \frac{1408}{12!} h^6 f_k^{(12)} + \dots + \frac{1}{(m-1)!} C_{8,m-9} h^{m-9} f_k^{(m-1)} \right\} - \right.$$

$$\left. \left\{ \frac{14}{8!} h^2 S_k^{(8)} + \frac{147}{10!} h^4 S_k^{(10)} + \frac{1408}{12!} h^6 S_k^{(12)} + \dots + \frac{1}{(m-1)!} C_{8,m-9} h^{m-9} S_k^{(m-1)} \right\} \right]$$

Thus we can define the general form of  $S_k^{(n)}$ , when  $n$  is odd ( $n = 1, 3, \dots, m-3$ ), as follows:

$$(2.24) \quad S_k^{(n)} = f_k^{(n)} +$$

$$+n! \left\{ \frac{C_{n,1}}{(n+1)!} h f_k^{(n+1)} + \frac{C_{n,2}}{(n+2)!} h^2 f_k^{(n+2)} + \dots + \frac{1}{m!} \sum_{L=1}^n C_{nL} h^{m-n} f_k^{(m)}(\varphi_{kL}) \right\}$$

$$-n! \left\{ \frac{C_{n,1}}{(n+1)!} h S_k^{(n+1)} + \frac{C_{n,2}}{(n+2)!} h^2 S_k^{(n+2)} + \frac{C_{n,3}}{(n+3)!} h^3 S_k^{(n+3)} + \dots + \right.$$

$$\left. + \frac{1}{m!} C_{n,m-n} h^{m-n} S_k^{(n)} \right\}, \quad \text{where} \quad \sum_{L=1}^n C_{nL} = C_{n,m-n}.$$

Also,  $S_k^{(n)}$ , when  $n$  is even ( $n = 2, 4, \dots, m-4$ ), takes on the form:

$$(2.25) \quad S_k^{(n)} = f_k^{(n)} +$$

$$+n! \left\{ \frac{C_{n,1}}{(n+2)!} h^2 f_k^{(n+2)} + \frac{C_{k,2}}{(n+4)!} h^4 f_k^{(n+4)} + \dots + \frac{C_{n,r}}{(n+2r)!} h^{2r} f_k^{(n+2r)} + \dots \right\} -$$

$$-n! \left\{ \frac{C_{n,1}}{(n+2)!} h^2 S_k^{(n+2)} + \frac{C_{n,2}}{(n+4)!} h^4 S_k^{(n+4)} + \dots + \frac{C_{n,r}}{(n+2r)!} h^{2r} S_k^{(n+2r)} + \dots \right\} +$$

$$+ \frac{1}{m!} \sum_{L=1}^{n/2} C_{nL} h^{m-n} f_k^{(m)}(\varphi_{kL}) - \frac{1}{m!} \sum_{L=\frac{n}{2}+1}^n C_{nL} h^{m-n} f_k^{(m)}(\varphi_{kL}),$$

where

$$\sum_{L=1}^{n/2} C_{nL} = \sum_{L=\frac{n}{2}+1}^n C_{nL}.$$

For the other coefficients  $S_k^{(i)}$ ,  $i = m - 2(1)m$  we have

$$(2.26) \quad S_k^{(m-2)} = f_k^{(m-2)} + \frac{C_{m-2,1}}{(m-1)!} h f_k^{(m-1)} + \frac{1}{m!} \sum_{L=1}^{m-2} C_{(m-2)L} f_k^{(m)}(\varphi_{kL}) -$$

$$- \frac{C_{m-2,1}}{(m-1)!} h S_k^{(n-1)} - \frac{C_{m-2,2}}{m!} h^2 S_k^{(n)}, \quad \sum_{L=1}^{m-2} C_{(m-2)L} = C_{m-2,2},$$

$$(2.27) \quad S_k^{(m-1)} = f_k^{(m-1)} +$$

$$+\frac{1}{m!} \sum_{L=1}^{[m-1]/2} C_{(m-1)L} h f_k^{(m)}(\varphi_{kL}) - \frac{1}{m!} \sum_{L=\{[n-1]/2\}+1}^{m-1} C_{(m-1)L} h f_k^{(m)}(\varphi_{kL}),$$

where

$$\sum_{L=1}^{[m-1]/2} C_{(m-1)L} = \sum_{L=\{[m-1]/2\}+1}^{m-1} C_{(m-1)L},$$

$$(2.28) \quad S_k^{(m)} = f_k^{(m)},$$

where the values of the constants  $C_{n,m}$  are given in the following:

(i) when  $n$  is odd:

$$C_{n,1} = \frac{n+1}{2}, \quad n = 1, 3, 5, \dots, \quad C_{1,r} = 1, \quad r = 1, 2, 3, \dots,$$

$$C_{n,r} = \begin{cases} \frac{n+1}{2} C_{n,r-1} + C_{n-2,r}, & r \text{ even} \\ \frac{n+1}{2} C_{n,r-1}, & r \text{ odd}, \end{cases} \quad r = 2, 3, \dots, m-2,$$

$$n = 3, 5, \dots, m-1 \quad (m - \text{even}), \quad n = 3, 5, \dots, m-2 \quad (m - \text{odd});$$

(ii) when  $n$  is even:

$$C_{n,r} = C_{n-1,2r}, \quad r = 1, 2, \dots, \frac{m-n}{2},$$

$$n = 2, 4, \dots, m-2 \quad (m - \text{even}), \quad n = 2, 4, \dots, m-1 \quad (m - \text{odd});$$

$$r = 1, 2, \dots, \frac{m-n}{2} \quad (m - \text{even}), \quad r = 1, 2, \dots, \frac{m-n-1}{3} \quad (m - \text{even}).$$

Using the Taylor expansion and (2.28), we get

$$(2.29) \quad |f_k^{(m)}(\varphi_{k,0}) - S_k^{(m)}| = |f_k^{(m)}(\varphi_{k,0}) - f_k^{(m)}| \leq \omega(D^m f; h).$$

From the Taylor expansion and (27)

$$(2.30) \quad |f_k^{(m-1)} - S_k^{(m-1)}| \leq \frac{h}{m!} \left| \sum_{L=1}^{[m-1]/2} C_{(m-1)L} f_k^{(m)}(\varphi_{kL}) - \sum_{L=\{[m-1]/2\}+1}^{m-1} C_{(m-1)L} f_k^{(m)}(\varphi_{kL}) \right| \leq$$

$$\leq C_A h \omega(D^m f; h),$$

$$m! C_A = \sum_{L=1}^{[m-1]/2} C_{(m-1)L} = \sum_{L=([m-1]/2)+1}^{m-1} C_{(m-1)L}.$$

Combining the Taylor expansion, (2.26), (2.29) and (2.30) we get

$$\begin{aligned} (2.31) \quad & |f_k^{(m-2)} - S_k^{(m-2)}| \leq \\ & \leq \frac{C_{m-2,1}}{(m-1)!} h |f_k^{(m-1)} - S_k^{(m-1)}| + \frac{C_{m-2,2}}{m!} h^2 |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}| \leq \\ & \leq \frac{C_{m-2,1}}{(m-1)!} C_A h^2 \omega_m(h) + \frac{C_{m-2,2}}{m!} h^2 \omega_m(h) = C_B h^2 \omega(D^m f; h), \\ & C_B = \frac{C_{m-2,1}}{(m-1)!} C_A + \frac{C_{m-2,2}}{m!}. \end{aligned}$$

To obtain the estimate  $|f_k^{(n)} - S_k^{(n)}|$  we have two cases.

(a) When  $n$  is odd we use the Taylor expansion and (2.24) as follows:

$$\begin{aligned} (2.32) \quad & |f_k^{(n)} - S_k^{(n)}| \leq \\ & \leq n! \left\{ \frac{C_{n,1}}{(n+1)!} h |f_k^{(n+1)} - S_k^{(n+1)}| + \frac{C_{n,2}}{(n+2)!} h^2 |f_k^{(n+2)} - S_k^{(n+2)}| + \right. \\ & \left. + \frac{C_{n,3}}{(n+3)!} h^3 |f_k^{(n+3)} - S_k^{(n+3)}| + \dots + \frac{C_{n,m-n}}{m!} h^{m-n} |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}| \right\} \leq \\ & \leq C_D h^{m-n} \omega(D^m f; h), \\ & n = 1, 3, \dots, m-4. \end{aligned}$$

(b) When  $n$  is even we use the Taylor expansion and (2.25) as follows:

$$\begin{aligned} (2.33) \quad & |f_k^{(n)} - S_k^{(n)}| \leq \\ & \leq n! \left\{ \frac{C_{n,1}}{(n+2)!} h^2 |f_k^{(n+2)} - S_k^{(n+2)}| + \frac{C_{n,2}}{(n+4)!} h^4 |f_k^{(n+4)} - S_k^{(n+4)}| + \right. \\ & \left. + \frac{C_{n,3}}{(n+6)!} h^6 |f_k^{(n+6)} - S_k^{(n+6)}| + \dots + \frac{C_{n,r}}{(n+2r)!} h^{2r} |f_k^{(n+2r)} - S_k^{(n+2r)}| + \right. \end{aligned}$$

$$\begin{aligned}
& \left. + \frac{1}{m!} h^{m-n} \left| \sum_{L=1}^{n/2} C_{nL} f_k^{(m)}(\varphi_{kL}) - \sum_{L=\frac{n}{2}+1}^n C_{nL} f_k^{(m)}(\varphi_{kL}) \right| \right\} \leq \\
& \leq C_E h^{m-n} \omega(D^m f; h), \\
& n = 2, 4, \dots, m-3.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(2.34) \quad & |f_k^{(4)} - S_k^{(4)}| \leq \\
& \leq 4! \left\{ \frac{5}{6!} h^2 |f_k^{(6)} - S_k^{(6)}| + \frac{21}{8!} h^4 |f_k^{(8)} - S_k^{(8)}| + \frac{85}{10!} h^6 |f_k^{(10)} - S_k^{(10)}| + \right. \\
& \left. + \dots + \frac{1}{m!} h^{m-4} \left| \sum_{L=1}^2 C_{4L} f_k^{(m)}(\varphi_{kL}) - \sum_{L=3}^4 C_{4L} f_k^{(m)}(\varphi_{kL}) \right| \right\} \leq \\
& \leq C_r h^{m-4} \omega(D^m f; h),
\end{aligned}$$

$$\begin{aligned}
(2.35) \quad & |f_k^{(3)} - S_k^{(3)}| \leq \\
& \leq 3! \left\{ \frac{2}{4!} h |f_k^{(4)} - S_k^{(4)}| + \frac{5}{5!} h^2 |f_k^{(5)} - S_k^{(5)}| + \frac{10}{6!} h^3 |f_k^{(6)} - S_k^{(6)}| + \right. \\
& \left. + \frac{1}{m!} h^{m-3} \left| \sum_{L=1}^3 C_{3L} f_k^{(m)}(\varphi_{kL}) - C_{3,m-3} S_k^{(m)} \right| \right\} \leq C_G h^{m-3} \omega(D^m f; h),
\end{aligned}$$

$$\begin{aligned}
(2.36) \quad & |f_k^{(2)} - S_k^{(2)}| \leq \\
& \leq 2! \left\{ \frac{1}{4!} h^2 |f_k^{(4)} - S_k^{(4)}| + \frac{1}{6!} h^4 |f_k^{(6)} - S_k^{(6)}| + \dots + \right. \\
& \left. + \frac{1}{m!} h^{m-2} |f_k^{(m)}(\varphi_{k1}) - f_k^{(m)}(\varphi_{k2})| \right\} \leq C_H h^{m-2} \omega(D^m f; h)
\end{aligned}$$

and

$$(2.37) \quad |f_k^{(1)} - S_k^{(1)}| \leq$$

$$\leq 1! \left\{ \frac{1}{2!} h |f_k^{(2)} - S_k^{(2)}| + \frac{1}{3!} h^2 |f_k^{(3)} - S_k^{(3)}| + \frac{1}{4!} h^3 |f_k^{(4)} - S_k^{(4)}| + \dots + \frac{1}{m!} h^{m-1} |f_k^{(m)}(\varphi_{k1}) - S_k^{(m)}| \right\} \leq C_I h^{m-1} \omega(D^m f; h).$$

By substituting (2.29)-(2.37) into (2.1), when  $m$  is odd, we get

$$\begin{aligned} & |f(x) - S_{k,m}(x)| \leq \\ & \leq \left\{ C_I + \frac{1}{2!} C_H + \frac{1}{3!} C_G + \frac{1}{4!} C_F + \dots + \frac{1}{(m-2)!} C_B + \frac{1}{(m-1)!} C_A + \frac{1}{m!} \right\} \times \\ & \quad \times h^m \omega(D^m f; h) \leq C_M h^m \omega(D^m f; h), \end{aligned}$$

where

$$C_M = C_I + \frac{1}{2!} C_H + \frac{1}{3!} C_G + \frac{1}{4!} C_F + \dots + \frac{1}{(m-2)!} C_B + \frac{1}{(m-1)!} C_A + \frac{1}{m!},$$

$C_I, C_H, C_G, \dots, C_A$  are constants.

Similarly, we can prove the convergence of  $S_{k,m}^{(r)}$ ,  $r = 1(1)m$ .

Thus the proof of the case 1, when  $m$  is odd, is completed.

*Case (2): The convergence of  $S_{k,m}(x)$ , when  $m$  is even*

From Theorem 2.2 we can define the constants  $S_n^{(k)}$ ,  $n = 1(1)m$  by

$$\begin{aligned} (2.38) \quad S_k^{(1)} &= f_k^{(1)} + 1! \left\{ \frac{1}{2!} h f_k^{(2)} + \frac{1}{3!} h^2 f_k^{(3)} + \dots + \frac{1}{m!} h^{m-1} f_k^{(m)}(\varphi_{k1}) \right\} - \\ & - 1! \left\{ \frac{1}{2!} h S_k^{(2)} + \frac{1}{3!} h^2 S_k^{(3)} + \dots + \frac{1}{m!} h^{m-1} S_k^{(m)} \right\}, \end{aligned}$$

$$\begin{aligned} (2.39) \quad S_k^{(2)} &= f_k^{(2)} + 2! \left\{ \frac{1}{4!} h^2 f_k^{(4)} + \frac{1}{6!} h^4 f_k^{(6)} + \dots + \frac{1}{m!} h^{m-2} f_k^{(m)}(\varphi_{k1}) \right\} - \\ & - 2! \left\{ \frac{1}{4!} h^2 S_k^{(4)} + \frac{1}{6!} h^4 S_k^{(6)} + \dots + \frac{1}{m!} h^{m-2} S_k^{(m)} \right\}, \end{aligned}$$

$$(2.40) \quad S_k^{(3)} = f_k^{(3)} +$$

$$\begin{aligned}
& +3! \left\{ \frac{2}{4!} h f_k^{(4)} + \frac{5}{5!} h^2 f_k^{(5)} + \frac{10}{6!} h^3 f_k^{(6)} + \dots + \frac{1}{m!} \sum_{L=1}^3 C_{3L} h^{m-3} f_k^{(m)}(\varphi_{kL}) \right\} - \\
& -3! \left\{ \frac{2}{4!} h S_k^{(4)} + \frac{5}{5!} h^2 S_k^{(5)} + \frac{10}{6!} h^3 S_k^{(6)} + \dots + \frac{C_{3,m-3}}{m!} h^{m-3} S_k^{(m)} \right\}, \\
& \sum_{L=1}^3 C_{3L} = C_{3,m-3},
\end{aligned}$$

(2.41)

$$S_k^{(4)} = f_k^{(4)} +$$

$$\begin{aligned}
& +4! \left\{ \frac{5}{6!} h^2 f_k^{(6)} + \frac{21}{8!} h^4 f_k^{(8)} + \frac{85}{10!} h^6 f_k^{(10)} + \dots + \frac{1}{m!} \sum_{L=1}^4 C_{4L} h^{m-4} f_k^{(m)}(\varphi_{kL}) \right\} - \\
& -4! \left\{ \frac{5}{6!} h^2 S_k^{(6)} + \frac{21}{8!} h^4 S_k^{(8)} + \frac{85}{10!} h^6 S_k^{(10)} + \dots + \frac{1}{m!} C_{4, \frac{m-4}{2}} h^{m-4} S_k^{(m)} \right\}, \\
& \sum_{L=1}^4 C_{4L} = C_{4, \frac{m-4}{2}},
\end{aligned}$$

(2.42)

$$S_k^{(5)} = f_k^{(5)} +$$

$$\begin{aligned}
& +5! \left\{ \frac{3}{6!} h f_k^{(6)} + \frac{14}{7!} h^2 f_k^{(7)} + \frac{42}{8!} h^3 f_k^{(8)} + \dots + \frac{1}{m!} \sum_{L=1}^5 C_{5L} h^{m-5} f_k^{(m)}(\varphi_{kL}) \right\} - \\
& -5! \left\{ \frac{3}{6!} h S_k^{(6)} + \frac{14}{7!} h^2 S_k^{(7)} + \frac{42}{8!} h^3 S_k^{(8)} + \dots + \frac{1}{m!} C_{5,m-5} h^{m-5} S_k^{(m)} \right\}, \\
& \sum_{L=1}^5 C_{5L} = C_{5,m-5},
\end{aligned}$$

(2.43)

$$S_k^{(6)} = f_k^{(6)} +$$

$$\begin{aligned}
& +6! \left\{ \frac{14}{8!} h^2 f_k^{(8)} + \frac{147}{10!} h^4 f_k^{(10)} + \dots + \frac{1}{m!} \sum_{L=1}^6 C_{6L} h^{m-6} f_k^{(m)}(\varphi_{kL}) \right\} - \\
& -6! \left\{ \frac{14}{8!} h^2 S_k^{(8)} + \frac{147}{10!} h^4 S_k^{(10)} + \dots + \frac{1}{m!} C_{6, \frac{m-6}{2}} h^{m-6} S_k^{(m)} \right\},
\end{aligned}$$

$$\sum_{L=1}^6 C_{6L} = C_{6, \frac{m-6}{2}}.$$

Thus, we can define the general form of  $S_k^{(n)}$ , when  $n$  is odd ( $n = 1, 3, \dots, m-3$ ), as follows

$$(2.44) \quad S_k^{(n)} = f_k^{(n)} + n! \left\{ \frac{C_{n,1}}{(n+1)!} h f_k^{(n+1)} + \right. \\ \left. + \frac{C_{n,2}}{(n+2)!} h^2 f_k^{(n+2)} + \frac{C_{n,3}}{(n+3)!} h^3 f_k^{(n+3)} + \dots + \frac{1}{m!} \sum_{L=1}^n C_{nL} h^{m-n} f_k^{(m)}(\varphi_{kL}) \right\} - \\ - n! \left\{ \frac{C_{n,1}}{(n+1)!} h S_k^{(n+1)} + \frac{C_{n,2}}{(n+2)!} h^2 S_k^{(n+2)} + \frac{C_{n,3}}{(n+3)!} h^3 S_k^{(n+3)} + \dots + \right. \\ \left. + \frac{C_{n,m-n}}{m!} h^{m-n} S_k^{(m)} \right\}, \quad \sum_{L=1}^n C_{nL} = C_{n,m-n}.$$

When  $n$  is even [ $n = 2, 4, \dots, m-4$ ],  $S_k^{(n)}$  takes on the form

$$(2.45) \quad S_k^{(n)} = f_k^{(n)} + n! \left\{ \frac{C_{n,1}}{(n+2)!} h^2 f_k^{(n+2)} + \right. \\ \left. + \frac{C_{n,2}}{(n+4)!} h^4 f_k^{(n+4)} + \frac{C_{n,3}}{(n+6)!} h^6 f_k^{(n+6)} + \dots + \frac{1}{m!} \sum_{L=1}^n C_{nL} h^{m-n} f_k^{(m)}(\varphi_{kL}) \right\} - \\ - n! \left\{ \frac{C_{n,1}}{(n+2)!} h^2 S_k^{(n+2)} + \frac{C_{n,2}}{(n+4)!} h^4 S_k^{(n+4)} + \frac{C_{n,3}}{(n+6)!} h^6 S_k^{(n+6)} + \dots + \right. \\ \left. + \frac{1}{m!} C_{n, \frac{m-n}{2}} h^{m-n} S_k^{(m)} \right\}, \quad \sum_{L=1}^n C_{nL} = C_{n, \frac{m-n}{2}}.$$

For the other coefficients  $S_k^{(i)}$ ,  $i = m - 2(1)m$  we have

$$S_k^{(m-2)} = f_k^{(m-2)} + \frac{1}{m!} \sum_{L=1}^{m-2} C_{(m-2)L} h^2 f_k^{(m)}(\varphi_{kL}) - \frac{1}{m!} C_{m-2,1} h^2 S_k^{(m)},$$

$$(2.46) \quad \sum_{L=1}^{m-2} C_{(m-2)L} = C_{m-2,1},$$

$$S_k^{(m-1)} = f_k^{(m-1)} + \frac{1}{m!} \sum_{L=1}^{m-1} C_{(m-1)L} h f_k^{(m)}(\varphi_{kL}) - \frac{1}{m!} C_{m-1,1} h S_k^{(m)},$$

$$(2.47) \quad \sum_{L=1}^{m-1} C_{(m-1)L} = C_{m-1,1}$$

and

$$(2.48) \quad S_k^{(m)} = f_k^{(m)},$$

where the values of the constants  $C_{n,m}$  are given as follow:

(i)  $n$  is odd

$$\begin{aligned} C_{n,1} &= \frac{n+1}{2}, & n &= 1, 3, 5, \dots, & C_{1,r} &= 1, & r &= 1, 2, 3, \dots \\ C_{n,r} &= \left\{ \frac{n+1}{2} \right\} C_{n,r-1} + C_{n-2,r}, & & & & & r &= \text{even}, \\ &= \left\{ \frac{n+1}{2} \right\} C_{n,r-1}, & & & & & r &= \text{odd}, \\ & & n &= 3, 5, \dots, m-1, & & & r &= 2, 3, \dots, m-2. \end{aligned}$$

(ii)  $n$  is even

$$C_{n,r} = C_{n-1,2r}, \quad n = 2, 4, \dots, m-1, \quad r = 1, 2, \dots, \frac{m-n}{2}.$$

Thus, using the Taylor expansion and (2.48), we have

$$(2.49) \quad |f_k^{(m)}(\varphi_{k,0}) - S_k^{(m)}| = |f_k^{(m)}(\varphi_{k,0}) - f_k^{(m)}| \leq \omega(D^m f; h),$$

and also from the Taylor expansion and (2.47)

$$(2.50) \quad |f_k^{(m-1)} - S_k^{(m-1)}| \leq$$

$$\leq \frac{h}{m!} \left| \sum_{L=1}^{m-1} C_{(m-1)L} f_k^{(m)}(\varphi_{kL}) - C_{m-1,1} S_k^{(m)} \right| \leq C_j h \omega(D^m f; h),$$

where

$$m!C_j = \sum_{L=1}^{m-1} C_{(m-1)L} = C_{m-1,1}.$$

$$(2.51) \quad \begin{aligned} & |f_k^{(m-2)} - S_k^{(m-2)}| \leq \\ & \leq \frac{h^2}{m!} \left| \sum_{L=1}^m C_{(m-2)L} f_k^{(m)}(\varphi_{kL}) - C_{m-2,1} S_k^{(m)} \right| \leq C_k h^2 \omega(D^m f; h), \end{aligned}$$

where

$$m!C_k = \sum_{L=1}^m C_{(m-2)L} = C_{m-2,1}.$$

To obtain the estimate  $|f_k^{(n)} - S_k^{(n)}|$  we proceed as follows:

(a) when  $n$  is odd, using the Taylor expansion and (2.44), we get

$$(2.52) \quad \begin{aligned} & |f_k^{(n)} - S_k^{(n)}| \leq \\ & \leq n! \left\{ \frac{C_{n,1}}{(n+1)!} |f_k^{(n+1)} - S_k^{(n+1)}| h + \frac{C_{n,2}}{(n+2)!} |f_k^{(n+2)} - S_k^{(n+2)}| h^2 + \right. \\ & \left. + \frac{C_{n,3}}{(n+3)!} |f_k^{(n+3)} - S_k^{(n+3)}| h^3 + \dots + \frac{C_{n,m-n}}{m!} |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}| h^{m-n} \right\} \leq \\ & \leq C_D h^{m-n} \omega(D^m f; h); \end{aligned}$$

(b) when  $n$  is even, using the Taylor expansion and (2.45), we have

$$(2.53) \quad \begin{aligned} & |f_k^{(n)} - S_k^{(n)}| \leq \\ & \leq n! \left\{ \frac{C_{n,1}}{(n+2)!} |f_k^{(n+2)} - S_k^{(n+2)}| h^2 + \frac{C_{n,2}}{(n+4)!} |f_k^{(n+4)} - S_k^{(n+4)}| h^4 + \right. \\ & \left. + \frac{C_{n,3}}{(n+6)!} |f_k^{(n+6)} - S_k^{(n+6)}| h^6 + \dots + \frac{1}{m!} C_{n, \frac{m-n}{2}} |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}| \right\} \leq \\ & \leq C_L h^{m-n} \omega(D^m f; h). \end{aligned}$$

Similarly, for the other estimates

$$(2.54) \quad |f_k^{(4)} - S_k^{(4)}| \leq \\ \leq 4! \left\{ \frac{5}{6!} |f_k^{(6)} - S_k^{(6)}| h^2 + \frac{21}{8!} |f_k^{(8)} - S_k^{(8)}| h^4 + \dots + \right. \\ \left. + \frac{1}{m!} C_{4, \frac{m-4}{2}} |f_k^{(m)}(\varphi_{4L}) - S_k^{(m)}| h^{m-4} \right\} \leq C_m h^{m-n} \omega(D^m f; h),$$

$$(2.55) \quad |f_k^{(3)} - S_k^{(3)}| \leq \\ \leq 3! \left\{ \frac{2}{4!} |f_k^{(4)} - S_k^{(4)}| h + \frac{5}{5!} |f_k^{(5)} - S_k^{(5)}| h^2 + \dots + \right. \\ \left. + \frac{1}{m!} C_{3, m-3} |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}| h^{m-3} \right\} \leq C_n h^{m-n} \omega(D^m f; h),$$

$$(2.56) \quad |f_k^{(2)} - S_k^{(2)}| \leq \\ \leq 2! \left\{ \frac{1}{4!} |f_k^{(4)} - S_k^{(4)}| h^2 + \frac{1}{6!} |f_k^{(6)} - S_k^{(6)}| h^4 + \dots + \right. \\ \left. + \frac{1}{m!} |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}| h^{m-2} \right\} \leq C_P h^{m-2} \omega(D^m f; h),$$

$$(2.57) \quad |f_k^{(1)} - S_k^{(1)}| \leq \\ \leq 1! \left\{ \frac{1}{2!} |f_k^{(2)} - S_k^{(2)}| h + \frac{1}{3!} |f_k^{(3)} - S_k^{(3)}| h^2 + \dots + \right. \\ \left. + \frac{1}{m!} |f_k^{(m)}(\varphi_{k1}) - S_k^{(m)}| h^{m-1} \right\} \leq C_I h^{m-1} \omega(D^m f; h).$$

By substituting (2.49)-(2.57) into (2.1) when  $m$  is even, we get

$$|f(x) - S_{k,m}(x)| \leq$$

$$\leq \left\{ C_I + \frac{1}{2!}C_P + \frac{1}{3!}C_n + \dots + \frac{1}{(m-1)!}C_j + \frac{1}{m!} \right\} h^m \omega(D^m f; h) = \\ = C_N h^m \omega(D^m f; h),$$

where

$$C_N = C_I + \frac{1}{2!}C_P + \frac{1}{3!}C_n + \dots + \frac{1}{(m-2)!}C_k + \frac{1}{(m-1)!}C_j + \frac{1}{m!},$$

$C_I, C_P, C_n, \dots, C_j$  are constants.

Thus  $S_{k,m}(x)$  is convergent to  $f(x)$ . Similarly, we can prove that  $S_{k,m}^{(j)}(x)$  is convergent to  $f^{(j)}(x)$ ,  $j = 1(1)m$ . So from cases 1 and 2 we complete the proof of Theorem 2.1.

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