

## **A REPRESENTATION THEOREM FOR THE OPERATOR SPACE $L(L^\infty(X); Z)$**

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*To Professor J. Balázs on his 75-th birthday*

### **1. Introduction**

It is well-known that the dual of  $L^p$  built on an abstract measure space is  $L^q$ , where  $1 \leq p < \infty$ ,  $p^{-1} + q^{-1} = 1$ , and that the dual of  $L^\infty$  is the space of finitely additive, absolutely continuous set functions, which have bounded variation.

A possible generalization is described in Benedek and Panzone [2], where the authors take successively the  $p_i$ -norm in the  $i$ -th variable ( $i = 1, \dots, n$ ;  $1 \leq p_i \leq \infty$ ) of a scalar-valued, measurable function defined on the Cartesian product of  $n$   $\sigma$ -finite measure spaces, and the mixed norm space  $L^{(p_1, \dots, p_n)}$  consists of functions for which these values are finite. It is shown in [2] that the dual of  $L^{(p_1, \dots, p_n)}$  is  $L^{(q_1, \dots, q_n)}$  ( $1 \leq p_i < \infty$ ,  $p_i^{-1} + q_i^{-1} = 1$ ).

Another generalization (that contains the previous one in a certain sense) is given in Diestel and Uhl [4], where the functions are defined on a finite measure space and take their values from a Banach-space  $X$ . The corresponding  $L^p$ -space  $L^p(X)$  is defined similarly to the scalar-valued case, but the absolute value is replaced by the  $X$ -norm. In [4] it is shown that the dual of  $L^p(X)$  is  $L^q(X^*)$ , whenever  $1 \leq p < \infty$ ,  $p^{-1} + q^{-1} = 1$  and  $X^*$  has the Radon-Nikodym property, that is every countably additive, absolutely continuous vector measure of bounded variation can be regarded as the indefinite integral of a Bochner-integrable function.

A third possibility for generalization is that instead of the functional space  $(L^p)^* = L(L^p; \mathbf{K})$  we model the linear operator space  $L(L^p; X)$ , where  $X$  is a Banach space and  $\mathbf{K}$  is the set of the real or complex numbers. Such a result for  $p = \infty$  is given in Diestel and Uhl [4] (cf. special cases 1) in this paper).

In this paper we shall deal with the representation of the operator space  $L(L_0^\infty(X); Z)$ , where  $X$  and  $Z$  are Banach spaces,  $L_0^\infty(X)$  is the closure of the subspace of simple functions in  $L^\infty(X)$ . This representation will be given using  $Y$ -valued, finitely additive, absolutely continuous and bounded vector measures, where  $Y$  is a Banach space, which is connected with  $X$  and  $Z$  by a bilinear operator satisfying the "bijection property". The paper contains a characterization of  $L_0^\infty(X)$ , and finally, two important special cases, too.

## 2. Definitions and preliminaries

Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$ ,  $(Z, \|\cdot\|_Z)$  be Banach spaces over the same (real or complex) scalar field  $\mathbf{K}$ , and  $B : X \times Y \rightarrow Z$  be a bounded bilinear operator, which satisfies the following "bijection property": the mapping  $Y \rightarrow L(X; Z)$ ,  $y \mapsto B(\cdot, y)$  is bijective, where  $L(X; Z)$  denotes the space of bounded linear operators from  $X$  to  $Z$ . We remark that the mapping  $y \mapsto B(\cdot, y)$  is injective if and only if  $B$  is nondegenerated, that is  $B(x, y) = 0$  ( $x \in X$ ) implies  $y = 0$  (cf. Halmos [5]). If it does not give rise to confusion, we denote  $B(x, y)$  simply by  $xy$  and call it as the "product" of  $x$  and  $y$ , and  $B$  as the "multiplication".

Let  $\Omega$  be a nonempty set,  $\mathcal{A} \subset 2^\Omega$  be a field. Denote by  $\mathcal{P}$  the set of  $\mathcal{A}$ -measurable finite partitions of  $\Omega$ , and by  $\{E_1, \dots, E_m\}$  or shortly  $\{E_i\}$ , the elements of  $\mathcal{P}$ . So  $E_i \in \mathcal{A}$ ,  $E_i \cap E_j = \emptyset$  if  $i \neq j$ ,  $E_1 \cup \dots \cup E_m = \Omega$ . A set function  $\mu : \mathcal{A} \rightarrow Y$  is called  $Y$ -valued vector measure if it is finitely additive. Define the semivariation of  $\mu$  by

$$\|\mu\| := \sup \left\{ \left\| \sum_{i=1}^m x_i \mu(E_i) \right\|_Z : \{E_i\} \in \mathcal{P}, x_i \in X, \|x_i\|_X \leq 1 \right\},$$

and the total variation of  $\mu$  by

$$|\mu| := \sup \left\{ \sum_{i=1}^m \|\mu(E_i)\|_Y : \{E_i\} \in \mathcal{P} \right\}$$

as in Bartle [1]. Remark that these variations are not necessarily finite and that they are norms in the vector spaces of vector measures of finite semi- and total variations, respectively.

### Proposition 1.

a)  $\|\mu\| \leq \|B\| \cdot |\mu|$ .

b) If  $X = \mathbf{K}$ ,  $Z = Y$ ,  $B(x, y) = x \cdot y$ , then

$$\|\mu\| = \sup \{ \|y^* \circ \mu\| : y^* \in Y^*, \|y^*\|_{Y^*} \leq 1 \}.$$

c) If  $X$  is reflexive,  $Y = X^*$ ,  $Z = \mathbf{K}$ ,  $B(x, y) = y(x)$ , then

$$\|\mu\| = |\mu|.$$

**Proof.** a) is obvious, for b) see Diestel and Uhl [4], Proposition 11. Proving c),  $\|\mu\| \leq |\mu|$  follows immediately from  $\|B\| \leq 1$  and a). To see the opposite inequality it is enough to show that

$$\left\{ \sum_i \|\mu(E_i)\|_{X^*} : \{E_i\} \in \mathcal{P} \right\} \subset \left\{ \left| \sum_i \mu(E_i)(x_i) \right| : \{E_i\} \in \mathcal{P}, x_i \in X, \|x_i\|_X \leq 1 \right\}.$$

Really, since  $X$  is reflexive, by the Hahn-Banach theorem there exists  $x_i \in X$  such that  $\|x_i\|_X = 1$  and  $\mu(E_i)(x_i) = \|\mu(E_i)\|_{X^*}$ .

In the remainder part of this paper  $(\Omega, \mathcal{A}, P)$  denotes an arbitrary probability measure space. A vector measure  $\mu$  is called absolutely continuous with respect to  $P$  (briefly  $\mu \ll P$ ) if  $P(E) = 0$  implies  $\mu(E) = 0$ . Denote by  $BA(\Omega, \mathcal{A}, P; Y)$  or simply by  $BA(Y)$  the vector space

$$BA(Y) := \{ \mu : \mathcal{A} \rightarrow Y : \mu \text{ is additive, } \|\mu\| < \infty, \mu \ll P \}.$$

Endowing this space with the norm  $\|\mu\|$ , it is a Banach-space.

Let

$$S(X) := S(\Omega, \mathcal{A}, P; X) := \left\{ \sum_{i=1}^m x_i \cdot 1_{E_i} : x_i \in X, \{E_i\} \in \mathcal{P} \right\}$$

be the vector space of the  $X$ -valued simple functions, where  $1_E$  denotes the characteristic function of  $E \in \mathcal{A}$ . We say that a function  $f : \Omega \rightarrow X$  is  $P$ -measurable if it is the  $P$ -almost everywhere limit of a sequence of simple functions, i.e. there exist  $f_n \in S(X)$  ( $n \in \mathbf{N}$ ) such that

$$P \left( \left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} \|f_n(\omega) - f(\omega)\|_X = 0 \right\} \right) = 1.$$

Let us introduce the vector space of all (equivalence classes of)  $X$ -valued,  $P$ -measurable, essentially bounded functions, the space  $L^\infty(\Omega, \mathcal{A}, P; X)$  or shortly  $L^\infty(X)$  (cf. Diestel and Uhl [4], IV.1.). This is a Banach space under the norm

$$\|f\|_{L^\infty(X)} := \operatorname{ess\,sup}_{\omega \in \Omega} \|f(\omega)\|_X.$$

Finally, let  $L_0^\infty(\Omega, \mathcal{A}, P; X)$ , or shortly  $L_0^\infty(X)$  be the closure of  $S(X)$  in  $L^\infty$ -norm.

**Proposition 2.** *Let  $X$  be a Banach space having Schauder-basis  $(e_j : j \in \mathbf{N})$ ,  $\|e_j\|_X = 1$ , further  $f \in L^\infty(X)$ . Then  $f \in L_0^\infty(X)$  if and only if*

$$(1) \quad \lim_{N \rightarrow \infty} \left\| f - \sum_{j=1}^{N-1} e_j^*(f(\cdot)) \cdot e_j \right\|_{L^\infty(X)} = 0.$$

(Here  $e_j^*$  ( $j \in \mathbf{N}$ ) denote the coordinate functionals.)

**Proof.** First let  $f = \sum_{i=1}^m x_i \cdot 1_{E_i} \in S(X)$ , and  $\varepsilon > 0$ . Then for every  $i = 1, \dots, m$  there exist  $N_i \in \mathbf{N}$  such that

$$\left\| \sum_{j=N}^{\infty} e_j^*(x_i) \cdot e_j \right\|_X < \varepsilon \quad (N \geq N_i).$$

For every  $N \geq \max\{N_1, \dots, N_m\} \geq N_i$  and for every  $\omega \in E_i \subset \Omega$  we have

$$\left\| \sum_{j=N}^{\infty} e_j^*(f(\omega)) \cdot e_j \right\|_X = \left\| \sum_{j=N}^{\infty} e_j^*(x_i) \cdot e_j \right\|_X < \varepsilon.$$

Taking ess sup it follows (1).

Now let  $f \in L_0^\infty(X)$ . Then there exists a sequence  $f_n \in S(X)$  ( $n \in \mathbf{N}$ ) with the property

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{L^\infty(X)} = 0.$$

Denoting by  $C$  the supremum of the norms of the partial sum operators in  $X$  (which is depending only on the chosen Schauder-basis), we deduce for any  $n, N \in \mathbf{N}$ ,  $N \geq 2$  and for any  $\omega \in \Omega$

$$\left\| f(\omega) - \sum_{j=1}^{N-1} e_j^*(f(\omega)) \cdot e_j \right\|_X \leq$$

$$\begin{aligned}
&\leq \left\| f(\omega) - f_n(\omega) - \sum_{j=1}^{N-1} e_j^*(f(\omega) - f_n(\omega)) \cdot e_j \right\|_X + \\
&\quad + \left\| f_n(\omega) - \sum_{j=1}^{N-1} e_j^*(f_n(\omega)) \cdot e_j \right\|_X \leq \\
&\leq (C+1) \cdot \|f(\omega) - f_n(\omega)\|_X + \left\| f_n(\omega) - \sum_{j=1}^{N-1} e_j^*(f_n(\omega)) \cdot e_j \right\|_X.
\end{aligned}$$

Taking ess sup we obtain

$$\left\| \sum_{j=N}^{\infty} e_j^*(f(\cdot)) \cdot e_j \right\|_{L^\infty(X)} \leq (C+1) \cdot \|f - f_n\|_{L^\infty(X)} + \left\| \sum_{j=N}^{\infty} e_j^*(f_n(\cdot)) \cdot e_j \right\|_{L^\infty(X)},$$

which shows (1).

Conversely, let  $\varepsilon > 0$  and  $f \in L^\infty(X)$  satisfying (1). Then there exists an  $N \in \mathbf{N}$ ,  $N \geq 2$  with the property

$$(2) \quad \left\| \sum_{j=N}^{\infty} e_j^*(f(\cdot)) \cdot e_j \right\|_{L^\infty(X)} < \varepsilon/2.$$

Since  $\|e_j^*\| \leq 2C$  ( $j \in \mathbf{N}$ ), it follows from the estimation

$$|e_j^*(f(\omega))| \leq \|e_j^*\| \cdot \|f(\omega)\| \leq 2C \cdot \|f(\omega)\|$$

that  $e_j^*(f(\cdot)) \in L^\infty(\mathbf{K})$ . Since  $S(\mathbf{K})$  is dense in  $L^\infty(\mathbf{K})$ , one can find for every  $j \in \{1, \dots, N-1\}$  a simple function  $\varphi_j \in S(\mathbf{K})$  such that

$$(3) \quad \|e_j^*(f(\cdot)) - \varphi_j\|_{L^\infty(\mathbf{K})} < \varepsilon/2(N-1).$$

The functions  $\varphi_j$  ( $j = 1, \dots, N-1$ ) can be assumed to be generated by the same partition  $\{E_1, \dots, E_m\}$ . It is easy to see that the function

$$g(\omega) := \sum_{j=1}^{N-1} \varphi_j(\omega) \cdot e_j \quad (\omega \in \Omega)$$

is in  $S(X)$ . The independence of  $e_1, \dots, e_{N-1}$  implies that for any  $\omega \in \Omega$  holds  $e_j^*(g(\omega)) = \varphi_j(\omega)$  and

$$\begin{aligned} & \|f(\omega) - g(\omega)\|_X \leq \\ & \leq \left\| \sum_{j=1}^{N-1} e_j^*(f(\omega) - g(\omega)) \cdot e_j \right\|_X + \left\| \sum_{j=N}^{\infty} e_j^*(f(\omega) - g(\omega)) \cdot e_j \right\|_X \leq \\ & \leq \sum_{j=1}^{N-1} |e_j^*(f(\omega)) - \varphi_j(\omega)| + \left\| \sum_{j=N}^{\infty} e_j^*(f(\omega)) \cdot e_j \right\|_X. \end{aligned}$$

Taking ess sup and using (2) and (3) we obtain  $\|f - g\|_{L^\infty(X)} < \varepsilon$ , which implies  $f \in L_0^\infty(X)$ .

This proposition makes us possible to give example for a function being in  $L^\infty(X)$ , but not in  $L_0^\infty(X)$ , provided that  $X$  has Schauder-basis  $(e_j : j \in \mathbf{N})$ ,  $\|e_j\|_X = 1$ . Suppose that  $\Omega$  can be written as union of a countably infinite set family  $\{E_j \in \mathcal{A} : j \in \mathbf{N}\}$ , where the sets  $E_j$  are disjoint, and  $P(E_j) > 0$ . (For example the intervals  $E_j := \left[1 - \frac{1}{j}, 1 - \frac{1}{j+1}\right)$  ( $j \in \mathbf{N}$ ) in  $[0, 1)$ .)

The function  $f := \sum_{j=1}^{\infty} e_j \cdot 1_{E_j}$  is obviously  $P$ -measurable, moreover, for every

$N \in \mathbf{N}$  holds that  $\left\| \sum_{j=N}^{\infty} e_j^*(f(\omega)) \cdot e_j \right\|_X = 1$ , if  $\omega \in \bigcup_{j=N}^{\infty} E_j$ , and 0 otherwise.

Since  $P\left(\bigcup_{j=N}^{\infty} E_j\right) > 0$ , we obtain that  $\text{ess sup}_{\omega \in \Omega} \left\| \sum_{j=N}^{\infty} e_j^*(f(\omega)) \cdot e_j \right\|_X = 1$ . This relation shows on one hand (for  $N = 1$ ) that  $f \in L^\infty(X)$ , and it implies on the other hand that  $f \notin L_0^\infty(X)$ , because (1) is not true.

Note that for finite dimensional  $X$  we have  $L_0^\infty(X) = L^\infty(X)$ .

### 3. Integration and model for the operator space

Following the way in Bartle [1] let us introduce the integral of an  $X$ -valued simple function  $f = \sum_{i=1}^m x_i \cdot 1_{E_i} \in S(X)$  with respect to the vector measure

$\mu \in BA(Y)$  using the "multiplication" defined by the bilinear operator  $B$ , as

$$I_\mu(f) := \int_{\Omega} f d\mu := \sum_{i=1}^m x_i \mu(E_i).$$

One can easily verify that  $I_\mu : S(X) \rightarrow Z$  is a well-defined linear operator (cf. [1]).

**Proposition 3.** *The operator  $I_\mu$  is bounded, and  $\|I_\mu\| = \|\mu\|$ .*

**Proof.** Define the following real number sets:

$$\begin{aligned} H_1 &:= \left\{ \left\| \sum_{i=1}^m x_i \mu(E_i) \right\|_Z : \{E_i\} \in \mathcal{P}, x_i \in X, \max_{i=1, \dots, m} \|x_i\|_X \leq 1 \right\}, \\ H_2 &:= \left\{ \left\| \sum_{i=1}^m x_i \mu(E_i) \right\|_Z : \{E_i\} \in \mathcal{P}, x_i \in X, \max_{i: P(E_i) \neq \emptyset} \|x_i\|_X \leq 1 \right\}, \\ H_3 &:= \left\{ \left\| \sum_{i=1}^m x_i \mu(E_i) \right\|_Z : \{E_i\} \in \mathcal{P}, x_i \in X, \max_{i: \mu(E_i) \neq 0} \|x_i\|_X \leq 1 \right\}. \end{aligned}$$

It is plausible that  $H_3 \subset H_1 \subset H_2$  and - using  $\mu \ll P$  - that  $H_2 \subset H_3$ . The mapping

$$(4) \quad \Phi : BA(Y) \rightarrow L(L_0^\infty(X); Z), \quad \Phi(\mu) := I_\mu$$

is a well-defined linear isometry, consequently it is an injection.

**Proposition 4.** *Every  $F \in L(L_0^\infty(X); Z)$  has the form  $I_\mu$  with a suitable  $\mu \in BA(Y)$ .*

**Proof.** For a given  $E \in \mathcal{A}$  the mapping  $x \mapsto F(x \cdot 1_E)$  is in  $L(X; Z)$ . Since our "multiplication" (i.e. the operator  $B$ ) satisfies the "bijection property", there exists a unique element  $\mu(E) \in Y$  such that

$$B(x, \mu(E)) = F(x \cdot 1_E) \quad (x \in X).$$

So, we have defined a set function  $\mu : \mathcal{A} \rightarrow Y$ . To see its additivity, after verifying  $B(x, \mu(E \cup F)) = B(x, \mu(E) + \mu(F))$  ( $E \cap F = \emptyset$ ,  $x \in X$ ) use that  $B$  is nondegenerated. To see  $\mu \ll P$  take any  $E \in \mathcal{A}$  with  $P(E) = 0$  and any  $x \in X$ . It follows from  $\|x \cdot 1_E\|_{L^\infty(X)} = 0$  that

$$B(x, \mu(E)) = F(x \cdot 1_E) = F(0) = 0 \quad (x \in X),$$

which implies  $\mu(E) = 0$ . Let us prove that  $\|\mu\| < \infty$ . Let  $\{E_i\} \in \mathcal{P}$ ,  $x_i \in X$ ,  $\|x_i\|_X \leq 1$ . Then

$$\left\| \sum_{i=1}^m x_i \mu(E_i) \right\|_Z = \left\| \sum_{i=1}^m F(x_i \cdot 1_{E_i}) \right\|_Z \leq \|F\| \cdot \left\| \sum_{i=1}^m x_i \cdot 1_{E_i} \right\|_{L^\infty(X)} \leq \|F\|,$$

which shows us the desired result. Now we have proved that  $\mu \in BA(Y)$ . Since

$$I_\mu \left( \sum_{i=1}^m x_i \cdot 1_{E_i} \right) = \sum_{i=1}^m x_i \mu(E_i) = \sum_{i=1}^m F(x_i \cdot 1_{E_i}) = F \left( \sum_{i=1}^m x_i \cdot 1_{E_i} \right),$$

the equality  $I_\mu = F$  holds on the dense subspace  $S(X)$ , consequently on  $L_0^\infty(X)$ , too.

We can summarize our results in the following

**Theorem 1.** *A model for the operator space  $L(L_0^\infty(X); Z)$  is  $BA(Y)$  by the isometric isomorphism (4).*

#### 4. Special cases

1) Let  $X := \mathbf{K}$ ,  $Y$  be an arbitrary Banach space over  $\mathbf{K}$ ,  $Z := Y$ ,  $B(x, y) := x \cdot y$  ( $x \in X$ ,  $y \in Y$ ).

$B$  is obviously nondegenerated, and for any  $f \in L(X; Z) = L(\mathbf{K}; Y)$  holds  $B(\cdot, f(1)) = f$ , therefore  $B$  satisfies the "bijection property". Moreover,  $\dim X = \dim \mathbf{K} = 1 < \infty$  implies  $L_0^\infty(\mathbf{K}) = L^\infty(\mathbf{K})$ . Applying Theorem 1, we obtain (cf. Diestel and Uhl [4]) as special case the following

**Theorem 2.** *There is a one-to-one linear and isometric correspondence between the Banach spaces  $BA(Y)$  and  $L(L^\infty(\mathbf{K}); Y)$  defined by*

$$\Phi : BA(Y) \rightarrow L(L^\infty(\mathbf{K}); Y), \quad \Phi(\mu) := I_\mu.$$

Remark that choosing  $Y = \mathbf{K}$ , we obtain the well-known duality theorem  $L^\infty(\mathbf{K})^* \cong BA(\mathbf{K})$ .

2) Let  $X$  be an arbitrary Banach space over  $\mathbf{K}$ ,  $Y := X^*$ ,  $Z := \mathbf{K}$ ,  $B(x, y) := y(x)$  ( $x \in X$ ,  $y \in Y$ ).



In this case  $B(\cdot, y) = y(\cdot)$ , that is the mapping  $y \mapsto B(\cdot, y)$  is the identity of  $Y$ , therefore  $B$  satisfies the "bijection property". Applying Theorem 1, we obtain as special case the following

**Theorem 3.** *The dual of  $L_0^\infty(X)$  is  $BA(X^*)$  and the mapping*

$$\Phi : BA(X^*) \rightarrow L_0^\infty(X)^*, \quad \Phi(\mu) := I_\mu$$

*is an isometric isomorphism.*

Remark that for reflexive  $X$  the semivariation can be exchanged by the total variation (see Proposition 1). This special case is applied in Csörgő [3] and Weisz [6] for  $X = \ell^2$ .

Note that in case  $Y = \mathbf{K}$ ,  $Z = X$ ,  $B(x, y) := y \cdot x$  the "bijection property" is not satisfied.

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