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LACUNARY INTERPOLATION ON SOME NON–UNIFORMLY DISTRIBUTED NODES ON THE UNIT CIRCLE

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To Professor János Balázs on his 75th birthday

1. Introduction

The problem of Hermite-Birkhoff interpolation on real nodes has a well developed theory which is embodied in the well-known book by G.G. Lorentz et al. [4]. R.A. Lorentz and G.G. Lorentz [5] have extended the theory to multivariate interpolation also. But on the subject of interpolation on nodes on the unit circle, almost all the results deal with the case of equidistributed nodes which, except for rotation, are equivalent to some roots of unity. This observation was first made by Shen Xiecheng [1]. Following this remark, Xie Siquing [3] has shown that the problem of $(0, \dots, r-2, r)$ interpolation for any integer $r \geq 2$ is regular on the nodes which are obtained by projecting the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ vertically on the unit circle on putting $x = (z+z^{-1})/2$. His method comprises in finding a differential equation for $\omega_n(z) = z^n P_n^{(\alpha,\beta)}((z^2+1)/(2z))$ which is a polynomial of degree $2n$ and whose differential equation also turns out to be of the second degree. More recently Chui et al. [2] worked out the case of Lagrange interpolation on nodes which are perturbed slightly in a small neighbourhood of roots of unity. But they do not consider the case of lacunary interpolation. Fabrykowski et al. [7] considered the problem of lacunary interpolation on cube roots of unity with two Hermitian sequences and two non-zero entries in the third row.

Our object here is, in a sense, simpler but the apparent simplicity entails its own difficulties. We propose to study the regularity of the problem of $(0, m)$ interpolation on the zeros of $(z^{2n}+1)(z^2-1)$ and also on the zeros of $(z^{2n}+1)(z^n-1)$. It is clear that the nodes are not uniformly distributed any more in these cases because of the extra nodes at $+1$ and -1. It is known that on the roots of unity and (in particular) on the zeros of $z^{2n}+1$, the problem of $(0, m_1, \ldots, m_q)$ interpolation is regular [6] where $0 < m_1 < \ldots < m_q$ are distinct integers and $m_j \le jn$.

2. $(0, m)$ interpolation on zeros of $(z^{2n}+1)(z^2-1)$ $(m \ge 1$ integer)

We shall prove

Theorem 1. *The problem of* $(0, m)$ *interpolation on zeros of* $(z^{2n}+1)(z^2-1)$ *is regular for any integer* $m \ge 1$ *and* $m < 2n+2$.

Proof. For $m = 1$, we know that it is Hermite interpolation so we shall take $m \ge 2$. For $m > 2n+2$, the problem is known to be not regular. Here the data is $4n+4$ and so it is enough to show that if a polynomial $Q(z)$ is of degree $\le 4n+3$ satisfies

$$Q(z_k) = 0, \quad Q^{(m)}(z_k) = 0, \text{ where } z_k^{2n}+1 = 0 \quad k = 1\ldots, 2n, \tag{2.1}$$

$$Q(\pm 1) = 0, \quad Q^{(m)}(\pm 1) = 0, \tag{2.2}$$

then $Q(z)$ is identically zero.

We write

$$Q(z) = P_0(z) + z^{2n}P_1(z) + z^{4n}P_2(z),$$

where $P_0(z),\ P_1(z) \in \pi_{2n-1}$ and $P_2(z) \in \pi_3$. Then $Q(z) \in \pi_{4n+3}$ and from $Q(z_k) = 0$ we have

$$\sum_{\nu=0}^{2n-1} (a_{0,\nu} - a_{1,\nu} + a_{2,\nu}) z_k^\nu = 0, \quad k = 1, \ldots, 2n. \tag{2.2a}$$

Here we have set $P_j(z) = \sum_{\nu=0}^{2n-1} a_{j,\nu} z^\nu,\ j = 0, 1, 2$. Since $P_2(z) \in \pi_3$, we have

$$a_{2,\nu} = 0, \quad \nu \ge 4.$$

Thus from (2.2a) we have the identity

$$\sum_{\nu=0}^{2n-1} (a_{0,\nu} - a_{1,\nu} + a_{2,\nu}) z^\nu = 0,$$

which yields

$$\begin{cases} a_{0,\nu} - a_{1,\nu} + a_{2,\nu} = 0, & \nu = 0,1,2,3, \\ a_{0,\nu} - a_{1,\nu} = 0, & \nu = 4,5,\dots 2n-1. \end{cases} \tag{2.3}$$

Similarly, from $Q^{(m)}(z_k) = 0$, on setting $(a)_m := a(a-1)\dots(a-m+1)$, we get
(2.4)

$$\begin{cases} (\nu)_m a_{0,\nu} - (2n+\nu)_m a_{1,\nu} + (4n+\nu)_m a_{2,\nu} = 0, & \nu = 0,1,2,3 \\ (\nu)_m a_{0,\nu} - (2n+\nu)_m a_{1,\nu} = 0, & \nu = 4,\dots,2n-1. \end{cases}$$

From the second system of equations in (2.3) and (2.4), we see that

$$a_{0,\nu} = a_{1,\nu} = 0, \quad \nu = 4,\dots,2n-1.$$

Since we also require the conditions (2.2), we get the following four conditions:

$$\sum_{\nu=0}^{3} (a_{0,\nu} + a_{1,\nu} + a_{2,\nu}) = 0, \tag{2.5}$$

$$\sum_{\nu=0}^{3} (a_{0,\nu} + a_{1,\nu} + a_{2,\nu})(-1)^{\nu} = 0, \tag{2.6}$$

$$\sum_{\nu=0}^{3} [(\nu)_m a_{0,\nu} + (2n+\nu)_m a_{1,\nu} + (4n+\nu)_m a_{2,\nu}] = 0, \tag{2.7}$$

$$\sum_{\nu=0}^{3} [(\nu)_m a_{0,\nu} + (2n+\nu)_m a_{1,\nu} + (4n+\nu)_m a_{2,\nu}](-1)^{\nu} = 0. \tag{2.8}$$

Adding and subtracting we get from (2.5), (2.6) and (2.7), (2.8)

$$\sum_{\nu=0}^{1} [a_{0,2\nu} + a_{1,2\nu} + a_{2,2\nu}] = 0, \tag{2.9}$$

$$\sum_{\nu=0}^{1}[(2\nu)_m a_{0,2\nu} + (2n+2\nu)_m a_{1,2\nu} + (4n+2\nu)_m a_{2,2\nu}] = 0, \tag{2.10}$$

$$\sum_{\nu=0}^{1}[a_{0,2\nu+1} + a_{1,2\nu+1} + a_{2,2\nu+1}] = 0, \tag{2.11}$$

$$\sum_{\nu=0}^{1}[(2\nu+1)_m a_{0,2\nu+1} + (2n+2\nu+1)_m a_{1,2\nu+1} + (4n+2\nu+1)_m a_{2,2\nu+1}] = 0. \tag{2.12}$$

From (2.3), we get $a_{0,0} + a_{2,0} = a_{1,0}$ and $a_{0,2} + a_{2,2} = a_{1,2}$ so that from (2.9) we get

$$a_{1,0} + a_{1,2} = 0.$$

Similarly from (2.4) and (2.10), we have

$$(2n)_m a_{10} + (2n+2)_m a_{1,2} = 0.$$

The last two equations give $a_{10} = a_{12} = 0$. This yields

$$a_{0,0} + a_{2,0} = 0 \quad \text{and} \quad a_{0,2} + a_{2,2} = 0,$$

and from (2.4) for $\nu = 0$ and 2, we get

$$(0)_m a_{0,0} + (4n)_m a_{2,0} = 0 \quad \text{and} \quad (2)_m a_{0,2} + (4n+2)_m a_{2,2} = 0.$$

Combining these equations suitably proves that

$$a_{0,2\nu} = 0, \quad a_{2,2\nu} = 0, \quad \nu = 0, 1.$$

Similarly we get

$$a_{0,2\nu+1} = a_{1,2\nu+1} = a_{2,2\nu+1} = 0, \quad \nu = 0, 1.$$

Hence $Q(z) \equiv 0$. This completes the proof.

Remark. It is clear from the foregoing proof that the theorem is also valid when the nodes are zeros of $(z^{pn}+1)(z^2-1)$, $p > 2$.

3. Zeros of $(z^{2n}+1)(z^n-1)$

We shall now consider the problem of $(0,m)$ interpolation on the zeros of $(z^{2n}+1)(z^n-1)$. We shall prove

Theorem 2. *The problem of $(0,m)$ interpolation on the zeros of $(z^{2n}+ +1)(z^n-1)$ is regular for $m<3n$.*

Proof. As in Theorem 1, we set

$$Q(z)=P_0(z)+z^{2n}P_1(z)+z^{4n}P_2(z),$$

where $P_i(z)=\sum\limits_{\nu=0}^{2n-1}a_{i,v}z^\nu,\ i=0,1,2$, since in this case $Q(z)\in\pi_{6n-1}$. From $Q(z_k)=0$, where $z_k^{2n}=-1$, we get

$$\begin{aligned}&P_0(z_k)-P_1(z_k)+P_2(z_k)=0,\\&P_0^{(m)}(z_k)+(z^{2n}P_1)_{z_k}^{(m)}+(z^{4n}P_2)_{z_k}^{(m)}=0.\end{aligned}$$

From these we get as in Section 2,

$$\sum_{\nu=0}^{2n-1}(a_{0,\nu}-a_{1,\nu}+a_{2,\nu})z^\nu=0, \tag{3.1}$$

$$\sum_{\nu=0}^{2n-1}[(\nu)_m a_{0,\nu}-(\nu+2n)_m a_{1,\nu}+(\nu+4n)_m a_{2,\nu}]z^\nu=0. \tag{3.2}$$

If we denote $\xi_1,\dots,\xi_n$ to be the zeros of z^n-1, then $Q(\xi_k)=0$ and $Q^{(m)}(\xi_k)= =0$ $(k=1,\dots,n)$ yield the following conditions:

$$\sum_{\nu=0}^{n-1}[(a_{0,\nu}+a_{0,n+\nu})+(a_{1,\nu}+a_{1,n+\nu})+(a_{2,\nu}+a_{2,n+\nu})]z^\nu=0, \tag{3.3}$$

(3.4)

$$\sum_{\nu=0}^{n-1}[\{(\nu)_m a_{0,\nu}+(n+\nu)_m a_{0,n+\nu}\}+\{(2n+\nu)_m a_{1,\nu}+(3n+\nu)_m a_{1,n+\nu}\}+ \\ +\{(4n+\nu)_m a_{2,\nu}+(5n+\nu)_m a_{2,n+\nu}\}]z^\nu=0.$$

From (3.1)-(3.4) we get the following system of equations:

$$(3.5)\qquad \begin{cases} a_{0,\nu} - a_{1,\nu} + a_{2,\nu} = 0 \\ (\nu)_m a_{0,\nu} - (\nu + 2n)_m a_{1,\nu} + (\nu + 4n)_m a_{2,\nu} = 0, \\ \qquad (\nu = 0, 1, \dots, 2n-1) \end{cases}$$

$$(3.6)\qquad \begin{cases} (a_{0,\nu} + a_{0,n+\nu}) + (a_{1,\nu} + a_{1,n+\nu}) + (a_{2,\nu} + a_{2,n+\nu}) = 0, \\ ((\nu)_m a_{0,\nu} + (n+\nu)_m a_{0,n+\nu}) + ((2n+\nu)_m a_{1,\nu} + (3n+\nu)_m a_{1,n+\nu}) + \\ \quad +((4n+\nu)_m a_{2,\nu} + (5n+\nu)_m a_{2,n+\nu}) = 0 \\ \qquad (\nu = 0, 1, \dots, n-1). \end{cases}$$

Writing the first equation (3.5) for ν and for $n+\nu$ and adding, and on using the first equation in (3.6) we get

$$a_{1,\nu} + a_{1,n+\nu} = 0.$$

Similarly from the second equation in (3.5) for ν and $n+\nu$, and the second equation in (3.6) we get

$$(2n+\nu)_m a_{1,\nu} + (3n+\nu)_m a_{1,n+\nu} = 0,$$

so that $a_{1,\nu} = a_{1,n+\nu} = 0, \quad \nu = 0, 1, \dots, n-1$. This leads to the equations

$$\begin{array}{ll} a_{0,\nu} + a_{2,\nu} = 0, & a_{0,n+\nu} + a_{2,n+\nu} = 0, \\ (\nu)_m a_{0,\nu} + (\nu+4n)_m a_{2,\nu} = 0, & (\nu+n)_m a_{0,n+\nu} + (\nu+5n)_m a_{2,n+\nu} = 0, \end{array}$$

whence we get $a_{0,\nu} = a_{2,\nu} = 0, (\nu = 0, 1, \dots, 2n-1)$. This completes the proof.

Remark. The above method shows that the problem of $(0,m)$ interpolation is also regular on the nodes given by the zeros of $(z^{pn}+1)(z^n-1)$.

4. Regularity of $(0, 1, \dots, r-2, r)$ on zeros of $(z^{2n}+1)(z^2-1)$

When $r = 2$, this is a special case of Theorem 1. We believe that a more general result is true, i.e. $(0, 1, \dots r, r+m)$ is also regular on the zeros of $(z^{2n}+1)(z^2-1)$ for $m < (r+1)(2n+2)$. But our method here does not apply here. Here we shall prove

Theorem 3. *The problem of* $(0,1,\dots,r-2,r)$ *is regular on the zeros of* $(z^{2n}+1)(z^2-1)$.

Proof. Since the number of data is $r(2n+2)$, we set

$$Q(z)=(z^{2n}+1)^{r-1}(z^2-1)^{r-1}P(z),\qquad P(z)\in\pi_{2n+1}. \tag{4.1}$$

Then it is clear that

$$\begin{cases} Q^{(\nu)}(z_k)=0, \quad \nu=0,1\dots r-2, \quad z_k^{2n}=-1 \quad (k=1,\dots 2n),\\ Q^{(\nu)}(-1)=Q^{(\nu)}(+1)=0, \quad \nu=0,1,\dots,r-2. \end{cases} \tag{4.2}$$

We shall show that if we require

$$Q^{(r)}(z_k)=0,\ Q^{(r)}(-1)=Q^{(r)}(1)=0 \qquad (k=1,\dots,2n),$$

then $P(z)=0$.

From $Q^{(r)}(+1)=0$ and $Q^{(r)}(-1)=0$ we get from (4.1)

$$\begin{cases} 2P'(1)+(r-1)(2n+1)P(1)=0,\\ 2P'(-1)-(r-1)(2n+1)P(-1)=0. \end{cases} \tag{4.3}$$

If we set $P(z)=\sum\limits_{j=0}^{2n+1}a_jz^j$, then (4.3) is equivalent to

$$\begin{cases} \sum\limits_{\nu=0}^{2n+1}\big(2\nu+(r-1)(2n+1)\big)a_\nu=0,\\ \sum\limits_{\nu=0}^{2n+1}\big(2\nu+(r-1)(2n+1)\big)a_\nu(-1)^\nu=0. \end{cases}$$

From these we at once obtain

$$\begin{cases} \sum\limits_{\nu=0}^{n}\{4\nu+(r-1)(2n+1)\}a_{2\nu}=0,\\ \sum\limits_{\nu=0}^{n}\{2(2\nu+1)+(r-1)(2n+1)\}a_{2\nu+1}=0. \end{cases} \tag{4.4}$$

From $Q^{(r)}(z_k)=0, k=1,\dots,2n$, we get

$$z_k^2[2z_kP'(z_k)+(r-1)(2n+3)P(z_k)]=2z_kP'(z_k)+(r-1)(2n-1)P(z_k).$$

Using the fact that $z_k^{2n} = -1$, we obtain from the above, after some simplification, a polynomial of degree $\le 2n-1$ vanishing at the zeros of $z^{2n}+1$. Thus we have the identity

(4.5)
$$\sum_{\nu=2}^{2n-1} \{2(\nu-2)+(r-1)(2n+3)\}a_{\nu-2}z^{\nu} -$$
$$-\sum_{\nu=0}^{3}\{2(\nu+2n-2)+(r-1)(2n+3)\}a_{2n+\nu-2}z^{\nu} =$$
$$= \sum_{\nu=0}^{2n-1} \{2\nu+(r-1)(2n-1)\}a_{\nu}z^{\nu} - \sum_{\nu=0}^{1}\{2(\nu+2n)+(r-1)(2n-1)\}a_{\nu}z^{\nu}.$$

Comparing like powers of z on both sides gives the following system of equations

$$\{2(\nu-2)+(r-1)(2n+3)\}a_{\nu-2} = \{2\nu+(r-1)(2n-1)\}a_{\nu}$$

$$(\nu = 4, 5, \dots, 2n-1), \tag{4.6}$$

and for $\nu = 0, 1, 2, 3$, we have

$$\{2(2n-2)+(r-1)(2n+3)\}a_{2n-2} = 4na_0, \tag{4.7}$$

$$\{2(2n-1)+(r-1)(2n+3)\}a_{2n-1} = 4na_1, \tag{4.8}$$

$$-\{4n+(r-1)(2n+3)\}a_{2n}+(r-1)(2n+3)a_0 = \{4+(r-1)(2n-1)\}a_2, \tag{4.9}$$

$$\begin{aligned} -\{2(2n+1)+(r-1)(2n+3)\}a_{2n+1} + \{2+(r-1)(2n+3)\}a_1 = \\ = \{6+(r-1)(2n-1)\}a_3. \end{aligned} \tag{4.10}$$

From (4.6) we can write $a_{2\nu-2}$ in terms of $a_{2\nu}$ and obtain

$$a_{2\nu-2} = A_{\nu-1}a_{2\nu}, \quad \text{where} \quad A_{\nu-1} = \frac{4\nu+(r-1)(2n-1)}{4(\nu-1)+(r-1)(2n+3)}, \tag{4.11}$$

where $A_{\nu-1} < 1$ for $r \ge 2$ $(\nu = 2, \dots, n-1)$. If we set

$$A_{n-1} := \frac{4n}{4(n-1)+(r-1)(2n+3)} < 1,$$

then from (4.7) we have $a_{2n-2} = A_{n-1}a_0$. From (4.4) and (4.11) we now obtain

$$(r-1)(2n+1)a_0 + \sum_{\nu=1}^{n-1}\{4\nu+(r-1)(2n+1)\}a_{2\nu} + \{4n+(r-1)(2n+1)\}a_{2n} = 0,$$

which can be written as

$$A_n a_0 + \{4n + (r-1)(2n+1)\}a_{2n} = 0, \tag{4.12}$$

where

$$0 < A_n = (r-1)(2n+1) + \sum_{\nu=1}^{n-1}\{4\nu + (r-1)(2n+1)\}\prod_{\mu=\nu}^{n-1} A_\mu.$$

From (4.9) we have

$$(r-1)(2n+3)a_0 - \{4 + (r-1)(2n-1)\}a_2 - \{4n + (r-1)(2n+3)\}a_{2n} = 0$$

which gives

$$B_n a_0 - \{4n + (r-1)(2n+3)\}a_{2n} = 0, \tag{4.13}$$

where

$$B_n = (r-1)(2n+3) - (4 + (r-1)(2n-1))\prod_{\mu=1}^{n-1} A_\mu.$$

Since $0 < A_\mu < 1$, we see that

$$\begin{aligned} B_n &> (r-1)(2n+3) - (4 + (r-1)(2n-1)) > \\ &> 4(r-2) \geq 0 \quad \text{if} \quad r \geq 2. \end{aligned}$$

The determinant of the two homogeneous equations (4.12) and (4.13) is clearly non-zero. Hence $a_0 = a_{2n} = 0$ which shows from (4.11) that all the a_j's are zero. This completes the proof.

5. Zeros of $(z^{2n}+1)(z^n-1)$

We shall now prove

Theorem 4. *The problem of* $(0,1,\dots,r-2,r)$ *interpolation is regular on the zeros of* $(z^{2n}+1)(z^n-1)$.

Proof. Here the number of data is $3nr$ and so we set

$$Q(z) = (z^{2n}+1)^{r-1}(z^n-1)^{r-1}P(z), \quad P(z) \in \pi_{3n-1}.$$

It is enough to show that if $Q^{(r)}(z_k)=0,\ k=0,1,\dots 2n-1$, where z_k is a zero of $(z^{2n}+1)(z^n-1)$, then Q is identically zero.

Let ξ_k^n $(k=0,1\dots n-1)$ be a primitive root of unity. Then $Q^{(r)}(\xi_k)=0$, $(k=0,1,\dots n-1)$ yields

$$2\xi_k P'(\xi_k) + (r-1)(3n-1)P(\xi_k) = 0, \quad k=0,1,\dots,n-1. \tag{5.1}$$

If we set $P(z) = \sum_{j=0}^{3n-1} a_j z^j$, then from (5.1) we get

$$2\sum_{j=0}^{3n-1} j a_j \xi_k^j + (r-1)(3n-1)\sum_{j=0}^{3n-1} a_j \xi_k^j = 0.$$

Rearranging the terms on the left and keeping in mind that $\xi_k^n=1$, we obtain a polynomial in ξ_k of degree $\le n-1$ which vanishes at the n-th roots of unity. So we have the identity

$$2\left[\sum_{j=0}^{n-1} j a_j z^j + \sum_{j=0}^{n-1}(j+n)a_{j+n}z^j + \sum_{j=0}^{n-1}(j+2n)a_{j+2n}z^j\right]+$$

$$+(r-1)(3n-1)\left[\sum_{j=0}^{n-1} a_j z^j + \sum_{j=0}^{n-1} a_{j+n}z^j + \sum_{j=0}^{n-1} a_{j+2n}z^j\right] = 0.$$

This yields the following equation in the a_j's:

$$\begin{aligned} \{2j+(r-1)(3n-1)\}a_j + \{2(j+n)+(r-1)(3n-1)\}a_{j+n} &+ \\ + \{2(j+2n)+(r-1)(3n-1)\}a_{j+2n} &= 0 \\ (j=0,1,\dots,n-1).& \end{aligned} \tag{5.2}$$

If we denote by η_k the zeros of $z^{2n}+1$, then from $Q^{(r)}(\eta_k)=0$ $(k=1,\dots,2n)$, we get

$$\left[\left(\frac{z^{2n}+1}{z-\eta_k}\right)^{r-1}(z^n-1)^{r-1}P(z)\right]'_{z=\eta_k} = 0.$$

Simple calculations then give the relation

$$2\eta_k(\eta_k^n - 1)P'(\eta_k) + (r-1)[(4n-1)\eta_k^n - (2n-1)]P(\eta_k) = 0$$

$$(k = 1, \ldots, 2n).$$

Since $P(z) = \sum\limits_{j=0}^{3n-1} a_j z^j$, we get, as in the earlier case, the following two equations

$$\begin{aligned} &\{2j + (r-1)(2n-1)\}a_j + \{2(j+n) + (r-1)(4n-1)\}a_{j+n} - \\ &\qquad - \{2(j+2n) + (r-1)(2n-1)\}a_{j+2n} = 0 \\ &\qquad (j = 0, 1, \ldots, n-1), \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\{2(j-n) + (r-1)(4n-1)\}a_{j-n} - \{2j + (r-1)(2n-1)\}a_j - \\ &\qquad - \{2(j+n) + (r-1)(4n-1)\}a_{j+n} = 0 \\ &\qquad (j = n, \ldots, 2n-1). \end{aligned} \tag{5.5}$$

Replacing j by $j+n$ in (5.5), we get

$$\begin{aligned} &\{2j + (r-1)(4n-1)\}a_j - \{2(j+n) + (r-1)(2n-1)\}a_{j+n} - \\ &\qquad - \{2(j+2n) + (r-1)(4n-1)\}a_{j+2n} = 0 \\ &\qquad (j = 0, 1, \ldots, n-1). \end{aligned} \tag{5.6}$$

From (5.2), (5.4) and (5.6), we see that the determinant of this system of homogeneous equations is given by $\Delta(j)$, where

$$\Delta(j) := \begin{vmatrix} a & a+2n & a+4n \\ b & c+2n & -(b+4n) \\ c & -(b+2n) & -(c+4n) \end{vmatrix}$$

$$j = 0, 1, \ldots, n-1,$$

where

$$a := 2j + (r-1)(3n-1), \quad b := 2j + (r-1)(2n-1), \quad c := 2j + (r-1)(4n-1).$$

Expanding $\Delta(j)$ in terms of the elements of the last column we see that

$$\Delta(j) = -(a+4n)[b(b+2n)+c(c+2n)] + (b+4n)[-a(b+2n)-c(a+2n)]- \\ -(c+4n)[a(c+2n)-b(a+2n)].$$

Since $a(c+2n)-b(a+2n) = 2n(r-1)(b+2n) > 0$, it follows that $\Delta(j) < 0$ for $j = 0, 1, \ldots, n-1$. This completes the proof.

Remark. The above method can be used to show that the problem is regular also on zeros of $(z^{pn}+1)(z^n-1)$.

6. Another approach to $(0, m)$ interpolation

We shall use a different method to prove the regularity of $(0, m)$ interpolations on the zeros of $(z^{2n}+1)(z^2-1)$. This method has been used in [6] for regularity of $(0, m_1, \ldots, m_q)$ interpolation on the n-th roots of unity. We observe that if $z_k^{2n} = -1$, then

$$z_k^m [D^m(z^{2n}P_1)]_{z_k} = -\sum_{\nu=0}^{m} \binom{m}{\nu} (2n)_{m-\nu} z_k^{\nu} (D^{\nu}P_1)_{z_k},$$

so that we may set

$$G_{1,2np}(D) := (-1)^p \sum_{\nu=0}^{m} \binom{m}{\nu} (2np)_{m-\nu} z^{\nu} D^{\nu}.$$

It is easy to check that $G_{1,2np}(D)z^s = (2np+s)_m z^s$.

Now we set $Q(z) = P_0(z) + z^{2n}P_1(z) + z^{4n}P_2(z)$, when $P_0(z), P_1(z) \in \pi_{2n-1}$ while $P_2(z) \in \pi_3$. From $Q(z_k) = 0$ and $Q^{(m)}(z_k) = 0$, we can derive

$$\text{(6.1)} \qquad \begin{cases} P_0(z) - P_1(z) = -P_2(z), \\ (z^m D^m)P_0(z) - G_{1,2n}(D)P_1(z) = -G_{1,2n}(D)P_2. \end{cases}$$

Solving these differential equations, we obtain $P_0(z)$ and $P_1(z)$. Thus we have

$$P_0(z) = \frac{G_{1,4n}(D) - G_{1,2n}(D)}{G_{1,2n}(D) - z^m D^m} P_2(z),$$

$$P_1(z) = \frac{G_{1,4n}(D) - z^m D^m}{G_{1,2n}(D) - z^m D^m} P_2(z).$$

If we put $P_2(z) = \sum_{\nu=0}^{3} a_{2,\nu} z^\nu$, then

$$\text{(6.2)} \qquad \begin{cases} P_0(z) = \sum_{\nu=0}^{3} a_{2,\nu} \dfrac{(\nu+4n)_m - (\nu+2n)_m}{(\nu+2n)_m - (\nu)_m} z^\nu, \\ P_1(z) = \sum_{\nu=0}^{3} a_{2,\nu} \dfrac{(\nu+4n)_m - (\nu)_m}{(\nu+2n)_m - (\nu)_m}. \end{cases}$$

From $Q(\pm 1) = 0$ and $Q^{(m)}(\pm 1) = 0$, we get

$$\text{(6.3)} \qquad P_0(1) + P_1(1) + P_2(1) = 0, \quad P_0(-1) + P_1(-1) + P_2(-1) = 0$$

and

$$\text{(6.4)} \qquad \left[z^m D^m P_0(z) + G_{1,2n}(D) P_1(z) + G_{1,2n}(D) P_2(z)\right]_{z=\pm 1} = 0.$$

Using (6.2) in (6.3) and (6.4), we can get equations containing only the coefficients of $P_2(z)$. By elementary calculations we then get

$$\sum_{\nu=0}^{1} a_{2,2\nu} (2\nu+2n)_m \frac{(2\nu+4n)_m - (2\nu)_m}{(2\nu+2n)_m - (2\nu)_m} = 0,$$

$$\sum_{\nu=0}^{1} a_{2,2\nu} \frac{(2\nu+4n)_m - (2\nu)_m}{(2\nu+2n)_m - (2\nu)_m} = 0.$$

Since the determinant of this system is $(2\nu+2m)_m - (2n)_m \neq 0$ we see that $a_{2,0} = a_{2,2} = 0$. Similarly we get $a_{2,1} = a_{2,3} = 0$. This shows that $P_2(z) \equiv 0$ and hence $P_0(z), P_1(z)$ are also zero. This completes the proof.

The method used above to prove regularity can be used to find the fundamental polynomials of interpolation in the case of Theorem 2. We shall return to this aspect later. Another question which arises naturally is whether the results of Theorems 1 and 2 can be extended to $(0, m_1, m_2)$ interpolation. We have been able to prove that $(0, m_1, m_2)$ interpolation on the zeros of $(z^{2n}+1)(z^2-1)$ is regular but this required tedious computation with determinants. We propose to return to these questions later.

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