Annales Univ. Sci. Budapest., Sect. Comp. 16 (2004) 31-44

INEQUALITIES FOR STOPPED RANDOM WALKS

N.L. Bassily (Cairo, Egypt)

Dedicated to Professor János Balázs on his 75-th birthday

Abstract. Consider the stopped partial sums of i.i.d. random variables $Y_1, Y_2, \ldots$ with the stopping time ν. Gut and Janson (1986) have investigated the effect of the conditions $E(|S_\nu|^p) < +\infty$ and $E(\nu^p) < +\infty$ and proved that in this case $E(|Y_1|^p) < +\infty$. Also, they have shown that if $E(|S_\nu|^p) < +\infty$ and $E(|Y_1|^p) < +\infty$ then necessarily $E(\nu^p) < < +\infty$ provided that $E(Y_1) \neq 0$. Here $p \geq 1$ is a power. We consider the quantities $A_1^{(p)} = \sup_{n\geq 1} E(|S_{\nu\wedge n}|^p)$, and $A_2^{(p)} = E\left(\sup_{n\geq 1} |S_{\nu\wedge n}|^p\right)$.

In this paper two other systems of conditions are investigated. Namely, if $A_1^{(p)}$ is finite and if $E(Y_1) \neq 0$, then necessarily $E(\nu^p) < +\infty$. Also, we prove that if $E(|Y_1|^p)$ and $E(\nu^p)$ are finite then the same holds for $A_2^{(p)}$. Moreover, the case when $a = E(Y_1) = 0$ is also treated. Some extensions and improvements of results obtained by Chow and Teicher (1978) and of Klass (1988) concerning $A_2^{(p)}$ are obtained.

1. Introduction and summary

Let $Y_1, Y_2, \ldots$ be i.i.d. random variables. Consider the σ-fields $\mathcal{F}_n = = \sigma(Y_1, \ldots, Y_n)$, $n \geq 1$. Let also ν be a stopping time adapted to the sequence $\{\mathcal{F}_n\}_{n=1}^\infty$ of σ-fields. Consider the generalized random walk $S_0 = 0$, $S_n = = Y_1 + \ldots + Y_n$, $n = 1, 2, \ldots$, and the corresponding stopped random walks $S_0 = 0$, $S_{\nu\wedge n}$, $n = 1, 2, \ldots$.

If ν is finite with probability 1 then the limit $\lim_{n\to\infty} S_{\nu\wedge n} = S_\nu$ exists and is finite a.s. except on the event $\{\nu = +\infty\}$, where S_ν is not defined. We define $S_\nu = 0$ on this event which has probability zero by assumption. Therefore S_ν has the form

$$S_\nu = \sum_{i=1}^{\infty} S_i I(\nu = i) = \sum_{i=1}^{\infty} Y_i(+\infty > \nu \geq i),$$

where $I(A)$ stands for the indicator of event A.

The sequence $S_{\nu\wedge n}$ is of the form

$$S_{\nu\wedge n} = \sum_{i=1}^{n} S_i I(\nu = i) + S_n I(\nu > n) = \sum_{i=1}^{n} Y_i I(+\infty > \nu \geq i).$$

Throughout this paper we suppose that $P(\nu < +\infty) = 1$.

Gut and Janson proved the following assertions. Let $p \geq 1$.

a) If $E(|S_\nu|^p) < +\infty$, $E(|Y_1|^p) < +\infty$, then so is $E(\nu^p) < \infty$, provided that $a = E(Y_1) \neq 0$ (generalization of Blackwell's theorem (1953)).

b) If $E(|S_\nu|^p) < +\infty$ and $E(\nu^p) < +\infty$ then necessarily $E(|Y_1|^p) < +\infty$. If $E(|S_\nu|^p) = +\infty$, $E(Y_1) = 0$ and $E(\nu) < +\infty$ then $E(|Y_1|^p) < +\infty$.

In the light of results of the present paper the following three conditions are equivalent:

1) $E(|Y_1|^p) < +\infty, \quad E(\nu^p) < +\infty \quad \text{and} \quad E(Y_1) \neq 0;$

2) $E(|Y_1|^p) < +\infty, \quad E(|S_\nu|^p) < +\infty \quad \text{and} \quad E(Y_1) \neq 0;$

3) $E(|S_\nu|^p) < +\infty, \quad E(\nu^p) < +\infty.$

Any of them implies that $A_2^{(p)} < +\infty$ and yet more, namely we have

$$A_3^{(p)} = E\left(\left(\sum_{i=1}^{\infty} |Y_i| I(+\infty > \nu \geq i)\right)^p\right) < \infty.$$

2. Standard extensions of Gut and Janson results

In this direction we prove the following result.

Theorem 1. *For $p \geq 1$ we have*

$$(i) \qquad A_1^{(p)} < +\infty, \quad E(Y_1) = a \neq 0 \implies E(\nu^p) < +\infty,$$

$$(ii) \qquad E(\nu^p) < +\infty, \quad E(|Y_1|^p) < +\infty \implies A_2^{(p)} < +\infty.$$

Proof. *Consider first the case $p = 1$.*

Suppose that $A_1^{(1)} = \sup_{n \geq 1} E(|S_{\nu \wedge n}|) < +\infty$ (which implies trivially that $E(|Y_1|) < \infty$). Then for every n we have

$$|E(S_{\nu \wedge n})| \leq E(|S_{\nu \wedge n}|) \leq A_1^{(1)} < +\infty.$$

Since $E(S_{\nu \wedge n}) = aE(\nu \wedge n)$ it follows that

$$|a|E(\nu \wedge n) \leq A_1^{(1)} < +\infty.$$

Letting $n \to +\infty$ and noting that $a \neq 0$ we get $|a|E(\nu) \leq A_1^{(1)} < +\infty$, which implies the finiteness of $E(\nu)$.

Now suppose that $E(|Y_1|) < +\infty$ and $E(\nu) < +\infty$. Since

$$\sup_{n \geq 1} |S_{\nu \wedge n}| \leq \sum_{i=1}^{\infty} |Y_i| I(\nu \geq i),$$

and using Wald's identity it follows that

$$A_2^{(1)} = E\left(\sup_{n \geq 1} |S_{\nu \wedge n}|\right) \leq E(|Y_1|)E(\nu) < +\infty.$$

Remarks. (1) The conditions $A_2^{(1)} < +\infty$ and $A_1^{(1)} < +\infty$ are equivalent provided that $a \neq 0$.

(2) Note that $E(|S_\nu|) < +\infty$ if $A_2^{(1)} < +\infty$. The finiteness of $E(S_\nu)$ does not imply in general neither that of $E(Y_1)$ nor that of $E(\nu)$. Counterexamples can be constructed in an obvious way (cf. Gut and Janson (1986)).

For $1 < p \leq 2$ we have:

Suppose $A_1^{(p)} < +\infty$. Then trivially $E(|Y_1|^p) < +\infty$. If, in addition $a = E(Y_1) > 0$, then by what we have proved, we have $E(\nu) < +\infty$. Consider the Doob decomposition of the submartingale $(S_{\nu\wedge n}, \mathcal{F}_n)$. Then the corresponding martingale is

$$(S_{\nu\wedge n} - a(\nu\wedge n), \mathcal{F}_n).$$

By the Burkholder-Davis-Gundy inequality (cf. Burkholder (1973))

$$E\left(\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu\wedge n)|^p\right) \leq C_p E\left(\sum_{i=1}^{\infty}(Y_i - a)^2 I(+\infty > \nu \geq i)\right)^{p/2},$$

where $C_p > 0$ is a constant depending only on p. Since $p/2 \leq 1$ and as an application to Wald's identity it follows that

$$E\left(\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu\wedge n)|^p\right) \leq C_p E\left(|Y_1 - a|^p\right) E(\nu) < +\infty.$$

Here we used the independence of $Y_i - a$ and $I(\nu \geq i)$. Now, by the C_p-inequality,

$$a^p E\left((\nu\wedge n)^p\right) \leq 2^{p-1}\left\{E\left(\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu\wedge n)|^p\right) + E\left(|S_{\nu\wedge n}|^p\right)\right\},$$

and letting $n \to +\infty$ we get

$$a^p E\left(\nu^p\right) \leq 2^{p-1}\{C_p E(|Y_1 - a|^p)E(\nu) + \sup_{n\geq 1} E(|S_{\nu\wedge n}|^p)\}.$$

Since $a > 0$ we can see that $E(\nu^p) < +\infty$. We proceed similarly when $a < 0$. Suppose now that $E(|Y_1|^p) < +\infty$ and $E(\nu^p) < +\infty$. Then again by the C_p-inequality

$$\sup_{n\geq 1} |S_{\nu\wedge n}|^p \leq 2^{p-1}\left\{\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu\wedge n)|^p + a^p\nu^p\right\}.$$

By what we have proved above, we have

$$A_2^{(p)} =$$

$$= E\left(\sup_{n\geq 1} |S_{\nu\wedge n}|^p\right) \leq 2^{p-1}\left\{E\left(\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu\wedge n)|^p\right) + a^p E(\nu^p)\right\} < +\infty.$$

The obtained estimation holds also in case $a = 0$. This proves the assertion by noting that in case $a < 0$ we consider the stopped random walk $(-S_{\nu\wedge n}, \mathcal{F}_n)$.

Finally, for $p \geq 2$, we have:

If $A_1^{(p)} < +\infty$, then trivially $E(|Y_1|^p) < +\infty$. Suppose in addition that $a = E(Y_1) \neq 0$. It follows that for $1 \leq r \leq 2$ $E(|Y_1|^r)$ and $A_1^{(r)}$ are finite. Let k be the smallest positive integer for which $1 \leq p2^{-k} \leq 2$ and denote $r = p2^{-k}$. Then, by what we proved, it follows that $E(\nu^{p2^{-k}}) < +\infty$. Since $p2^{-k+1} \geq 2$, it follows from Theorem 1 of Bassily, Ishak and Mogyoródi (1987) that

$$E\left(\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu\wedge n)|^{p2^{-k+1}}\right) < +\infty,$$

since $E(|Y_1|^{p2^{-k+1}}) < +\infty$ and $E(\nu^{p2^{-k}}) < +\infty$ provided that $p2^{-k+1} \leq p$. From this by the C_p-inequality we have

$$a^{p2^{-k+1}} E(\nu^{p2^{-k+1}}) \leq$$

$$\leq 2^{p2^{-k+1}-1}\left\{E\left(\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu\wedge n)|^{p2^{-k+1}}\right) + \sup_{n\geq 1}\left(|S_{\nu\wedge n}|^{p2^{-k+1}}\right)\right\} < +\infty.$$

This implies the finiteness of $E(\nu^{p2^{-k+1}})$. If $p2^{-k+1} \leq p$ and since $E(\nu^{p2^{-k+1}})$ and $E(|Y_1|^{p2^{-k+2}})$ being finite, we have

$$E\left(\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu\wedge n)|^{p2^{-k+2}}\right) < +\infty.$$

From this by the same manner we deduce the finiteness of

$$E\left(\nu^{p2^{-k+2}}\right).$$

Following this procedure step by step we finally arrive at the finiteness of $E(\nu^p)$.

Now suppose that $E(|Y_1|^p)$ and $E(\nu^p)$ are finite. Again, by the C_p-inequality we have

$$A_2^{(p)} = E\left(\sup_{n\geq 1} |S_{\nu\wedge n}|^p\right) \leq 2^{p-1}\left\{E\left(\sup_{n\geq 1} |S_{\nu\wedge n} - a(\nu\wedge n)|^p\right) + |a|^p E(\nu^p)\right\}$$

and the first term on the right hand side is also finite since $E(|Y_1|^p)$ and $E(\nu^{p/2})$ are finite, cf. Theorem 1 of Bassily, Ishak and Mogyoródi (1987). This means that $A_2^{(p)} < +\infty$. This proves the assertion.

Now let us introduce the quantity $A_3^{(p)} = E\left(\left(\sum_{i=1}^{\infty} |Y_i| I(+\infty > \nu \geq i)\right)^p\right)$. The following result is also true.

Theorem 2. *Let $p \geq 1$. Then the condition*

$$A_2^{(p)} < +\infty \quad \textit{whenever} \quad a = E(Y_1) \neq 0 \implies A_3^{(p)} < +\infty.$$

Conversely, if this last holds, then $A_2^{(p)} < +\infty$.

Thus, all the results that are true for $A_2^{(p)}$ are automatically true for $A_3^{(p)}$ and vice versa.

3. Convergence in L_1 of the stopped random walk

In this section we consider the convergence in L_1 of the stopped random walk $\{S_{\nu \wedge n}\}$. We shall see that the same conditions are necessary to ensure that $\sup_{n \geq 1} |S_{\nu \wedge n}|$ belong to L_1.

Theorem 3. *Let $\{S_{\nu \wedge n}\}$ be a stopped random walk and suppose that $E(S_\nu)$ is finite. If $S_{\nu \wedge n}$ converges in L_1 (to S_ν) then necessarily $a = E(Y_1)$ is finite. Moreover, $A_1^{(1)} = \sup_{n \geq 1} E(|S_{\nu \wedge n}|) < +\infty$. If, in addition, $a \neq 0$, then $E(\nu)$ is also finite.*

Remark. The finiteness of $E(S_\nu)$ does not imply in general neither that of $a = E(Y_1)$ nor that of $E(\nu)$ (cf. Gut and Janson (1986)).

Proof. If $\lim_{n \to +\infty} E(|S_\nu - S_{\nu \wedge n}|) = 0$, then there exists an index, say n_0, such that for $n \geq n_0$ we have $E(|S_\nu - S_{\nu \wedge n}|) < +\infty$, which implies

$$E(|S_{\nu \wedge n}|) \leq E(|S_\nu - S_{\nu \wedge n}|) + E(|S_\nu|) < +\infty,$$

since by assumption $E(|S_\nu|) < +\infty$. From this

$$E(|S_{\nu \wedge (n+1)} - S_{\nu \wedge n}|) = E(|Y_{n+1} I(\nu \geq n+1)|) < +\infty.$$

Y_{n+1} and $I(\nu \geq n+1)$ being independent we can see that

$$E(|Y_{n+1}|) P(\nu \geq n+1) < +\infty.$$

If $P(\nu \geq n+1) > 0$ this means that $E(|Y_1|) = E(|Y_{n+1}|) < +\infty$. If $P(\nu \geq n+ +1) = 0$ then ν is bounded a.s. and $P(\nu \leq k) = 1$, where k is the largest positive integer such that $P(\nu = k) > 0$. Now let us show that $Y_k I(\nu = k) = Y_k I(\nu \geq k)$ is of finite expectation. In fact, the random vector

$$\left(\sum_{i=1}^{k-1} Y_i I(\nu \geq i), \quad I(\nu = k) \right)$$

and the random variable Y_k are independent. From it follows that

$$E(S_\nu) = E\left(\sum_{i=1}^{k-1} Y_i(\nu \geq i) + Y_k I(\nu = k) \right) =$$

$$= E\left(\sum_{i=1}^{k-1} Y_i I(\nu \geq i) + Y_k I(\nu \geq k) \right) = \int\limits_R \int\limits_{R^2} (x + yz) dQ(y) dQ'(x,z)$$

is finite, where Q denotes the distribution of the Y_i's, whilst Q' is the joint distribution of the vector

$$\left(\sum_{i=1}^{k-1} Y_i I(\nu \geq i), \quad I(\nu = k) \right).$$

Since $E(|S_\nu|) < +\infty$ by Fubini's theorem we have for Q'-almost all (x,z) the integral

$$\int\limits_R (x + yz) dQ(y) = x + z \int\limits_R y dQ(y)$$

is finite. Since $P(\nu = k) > 0$, z can be taken to be different from 0. This means that $\int\limits_R y dQ(y) = E(Y_k) = E(Y_1) = a$ is finite.

Now let us show that $A_1^{(1)} < +\infty$. In fact, the random variables $Z_n = = \sum\limits_{i=1}^{n} S_i I(\nu = i)$ converge in L_1 to S_ν. Further

$$\left| S_\nu - \sum_{i=1}^{n} S_i I(\nu = i) \right| = \sum_{i=n+1}^{\infty} |S_i| I(\nu = i)$$

and the right hand side tends decreasingly to 0 as $n \to +\infty$. By the monotone convergence theorem and by the integrability of $|S_\nu|$ it follows that

$$E\left(\left|S_\nu - \sum_{i=1}^{n} S_i I(\nu = i)\right|\right) \downarrow 0$$

as $n \to +\infty$. Since by our assumption $S_{\nu\wedge n} \to S_\nu$ in L_1, we deduce the limit relation

$$\lim_{n\to+\infty} E(|S_n| I(\nu > n)) = 0,$$

and we have

$$|S_n| I(\nu > n) = |S_{\nu\wedge n}| - |S_\nu| I(\nu \leq n).$$

Integrating both sides of this relatiion and noting that

$$E(|S_\nu| I(\nu \leq n)) \to E(|S_\nu|),$$

we get

$$\lim_{n\to+\infty} E(|S_{\nu\wedge n}|) = E(|S_\nu|) < +\infty.$$

Each term of the sequence is finite since

$$E(S_{\nu\wedge n}) = aE(\nu \wedge n).$$

This means that

$$A_1^{(1)} = \sup_{n\geq 1} E(|S_{\nu\wedge n}|) < +\infty.$$

Finally, if $a = E(Y_1) \neq 0$ then by Theorem 1 the last relation implies that $E(\nu)$ is finite.

This proves the assertion.

4. The case $a = E(Y_1) = 0$

It is of interest to give an upper estimate for the random variable $\sup_{n\geq 1} |S_{\nu\wedge n}|^p$, where $a = E(Y_1) = 0$. Below we generalize a result of Chow, Robbins and Siegmund (1971) and of Chow and Teicher (1978) by obtaining an estimate for $A_2^{(p)}$ in terms of some moments of Y_1 and of ν, where p is a power such that $1 \le p \le 2$. Our method of proof is different and leads to explicit upper bound (cf. Ishak (1992)).

Theorem 4. *Let $a = E(Y_1) = 0$ and $1 \le p \le q \le 2$. If $E(|Y_1|^q)$ and $E(\nu^{p/q})$ are finite then*

$$A_2^{(p)} \le C_p (p/2)^{-p/2} E(|Y_1|^q) E(\nu^{p/q}),$$

where C_p is a constant depending only on p.

Proof. By using the Burkholder-Davis-Gundy inequality we have

$$A_2^{(p)} \le C_p E\left\{\left[\sum_{i=1}^{\infty} Y_i^2 I(\nu \ge i)\right]^{p/2}\right\} = C_p E\left\{\left[\sum_{i=1}^{\infty} Y_i^2 I(\nu \ge i)\right]^{q/2}\right\}^{p/q},$$

where $C_p > 0$ is a constant depending only on p, and here $q/2 \le 1$ and $p/q \le 1$. The following is true

$$E\left\{\sum_{i=1}^{\infty} Y_i^2 I(\nu \ge i)\right\}^{p/2} \le E\left\{\sum_{i=1}^{\infty} |Y_i|^q I(\nu \ge i)\right\}^{p/q}.$$

By using the concavity lemma (cf. Burkholder (1973) and Mogyoródi (1981)) we have

$$E\left\{\sum_{i=1}^{\infty} Y_i^2 I(\nu \ge i)\right\}^{p/2} \le (p/2)^{-p/2} E\left\{\sum_{i=1}^{\infty} E\left[|Y_i|^q I(\nu \ge i) \mid \mathcal{F}_{i-1}\right]\right\}^{p/q} =$$

$$= (p/2)^{-p/2} E\left\{[E(|Y_i|^q]^{p/q} \cdot \left[\sum_{i=1}^{\infty} I(\nu \ge i)\right]^{p/q}\right\} =$$

$$= (p/2)^{-p/2} [E(|Y_1|^q)]^{p/q} \cdot E(\nu^{p/q}).$$

Here we have used the fact that Y_i and $\mathcal{F}_{i-1}$ are independent and $I(\nu \ge i)$ is $\mathcal{F}_{i-1}$-measurable. This proves our assertion.

As an easy consequence we formulate now the following

Corollary 1. *Let $a = 0$ and $1 \le p \le q \le 2$. If $E(|Y_1|^q)$ and $E(\nu^{p/q})$ are finite, then*

$$E(|S_\nu|^p) \le C_p (p/2)^{-p/2} E(|Y_1|^q) E(\nu^{p/q}).$$

Proof. By the preceding theorem the conditions imply that

$$A_1^{(p)} < +\infty.$$

Consequently, $S_{\nu\wedge n} \to S_\nu$ in L_p and a.s. From this

$$E\left(\lim_{n\to+\infty} |S_{\nu\wedge n}|^p\right) = E(|S_\nu|^p) \leq$$

$$\leq E\left(\sup_{n\geq 1} |S_{\nu\wedge n}|^p\right) \leq C_p(p/2)^{-p/2} E(|Y_1|^q)E(\nu^{p/q}).$$

This proves the corollary.

On the basis of the preceding theorem the result of Chow and Teicher (1978) follows as a special case.

Corollary 2. *If for some p, where $1 \leq p \leq 2$, $E(|Y_1|^p)$ and $E(\nu^{1/p})$ are finite, then $A_2^{(p)} < +\infty$.*

Now we consider the case $p \geq 2$. It was proved by Burkholder (1973) and Mogyoródi (1977) that

$$c_p\left\{E(s^p) + \sum_{i=1}^{\infty} E(|d_i|^p)\right\} \leq E(X^{*p}) \leq C_p\left\{E(s^p) + \sum_{i=1}^{\infty} E(|d_i|^p)\right\}, \tag{1}$$

where c_p and C_p are positive constants depending only on p, and

$$s = s(X) = \left(\sum_{i=1}^{\infty} E(d_i^2|\mathcal{F}_{i-1}|)\right)^{\frac{1}{2}}$$

is the so-called conditional quadratic variation of a square integrable martingale $(X_n, \mathcal{F}_n)$ with difference sequence $d_o = 0$, $d_i = X_i - X_{i-1}$, $i \geq 1$ and with $X^* = \sup_{n\geq 1} |X_n|$ is the corresponding maximal function.

To provide additional perspective we formulate the following results without proof specialized to the present situation with the martingale differences $d_0 = 0 = S_0$ and $d_i = Y_i I(+\infty > \nu \geq i)$. These results are immediate consequences of conditional version of Rosenthal's inequality (1).

Theorem 5. *For $p \geq 2$ the moment $A_2^{(p)} < +\infty$ if and only if $E(|Y_1|^p)$ and $E(\nu^{p/2})$ are finite. In this case we have the inequality*

$$c_p\left\{\sigma^p E(\nu^{p/2}) + E(|Y_1|^p)E(\nu)\right\} \leq A_2^{(p)} \leq C_p\left\{\sigma^p E(\nu^{p/2}) + E(|Y_1|^p)E(\nu)\right\}.$$

Here $c_p > 0$ and $C_p > 0$ are constants depending only on p and σ^2 denotes the variance of Y_1.

Further, if $E(|Y_1|^p)$ and $E(\nu^{p/2})$ are finite then

$$c_p\left(\frac{p}{p-1}\right)^{-p}[\sigma^2 E(\nu^{p/2})+E(|Y_1|^p)E(\nu)]\le$$

$$\le E(|S_\nu|^p)\le C_p[\sigma^p E(\nu^{p/2})+E(|Y_1|^p)\cdot E(\nu)].$$

The results of this theorem show that the random variable S_ν belongs to the Hardy space H_p if and only if $E(|Y_1|^p)$ and $E(\nu^{p/2})$ are finite.

Recently, M.Klass (1988) proved that in case $E(Y_1)=0$ and for a power $p\ge 1$ we have $A_2^{(p)}\le CE(a_\nu^*)$, which by uniform integrability implies the validity of Wald's equation. Here $a_n^*=E\left(\max\limits_{1\le k\le n}|S_k|^p\right)$ and $C>0$ is a constant depending only on p.

This result was refined and sharpened in the form of two-sided inequality, cf. Bassily (1991). For the sake of completeness of this work and to provide an additional perspective concerning $A_2^{(p)}$ we formulate this result with a very compact proof.

Theorem 6. *For $p\ge 1$ we have*

$$cE(|S_\nu|^p)\le A_2^{(p)}\le CE(|S_\nu|^p),$$

where in the case $1\le p\le 2$ we also suppose that $\sigma^2=E(Y_1^2)$ is finite (and $=1$). Here $c>0$ and $C>0$ are two constants depending only on the distribution of Y_1 and on p and are independent of the choice of ν.

Proof. The idea of the proof is mainly based on the following inequalities:

a) If $p\ge 2$ then

$$c_p\left[\sigma^p E(\nu^{p/2})+E(|Y_1|^p)E(\nu)\right]\le A_2^{(p)}\le C_p\left[\sigma^2 E(\nu^{p/2})+E(|Y_1|^p)E(\nu)\right],$$

where $c_p>0$ and C_p are constants depending only on p. It is clear that the left and right-hand sides are finite if and only if so are $E(|Y_1|^p)$ and $E(\nu^{p/2})$, cf. Bassily, Ishak and Mogyoródi (1987). This inequality can be written in the following form

$$c_p\sigma^p E(\nu^{p/2})\le A_2^{(p)}\le 2C_p E(|Y_1|^p)E(\nu^{p/2}). \tag{2}$$

b) For $1 \le p \le 2$ Burkholder and Gundy (1970) have proved the following two-sided inequality: if $\sigma^2 = 1$ then

$$c_{p,d} E(\nu^{p/2}) \le A_2^{(p)} \le C_p E(\nu^{p/2}), \tag{3}$$

where $C_p > 0$ is a constant depending only on p whilst $c_{p,d} > 0$ is such a constant which depends not only on p, but also on $d = E(|Y_1|)$.

c) By using the Marcinkiewicz-Zygmund inequality ((1937) and (1938)) one can easily prove that for $p \ge 2$ supposing $E(|Y_1|^p) < +\infty$ for every $n = 1, 2, 3, \ldots$ we have

$$c_p^{(1)} \sigma^p n^{p/2} \le E(|S_n|^p) \le C_p^{(1)} E(|Y_1|^p) n^{p/2}, \tag{4}$$

where $c_p^{(1)} > 0$ and $C_p^{(1)} > 0$ are constants depending only on p.

If $1 \le p \le 2$ and $\sigma^2 = E(Y_1^2)$ is finite then

$$E(|S_n|^p) \le C_p^{(1)} \sigma^p n^{p/2}, \qquad n = 1, 2, \ldots \tag{5}$$

From (4) and (5) we have for all $p \ge 1$ that

$$E(|S_\nu|^p) \le K E(\nu^{p/2}) \tag{6}$$

holds with some $K > 0$ depending only on p and the distribution of Y_1, and is independent of the choice of ν. Applying (6) to the left-hand side of (2) in case $p \ge 2$ and to the left-hand side of (3) in case $1 \le p \le 2$ we have with some constant $c > 0$ that

$$cE(|S_\nu|^p) \le A_2^{(p)}, \qquad p \ge 1. \tag{$*$}$$

For $p \ge 2$ by using the inequality (4) it is clear that

$$E(\nu^{p/2}) \le K_1 E(|S_\nu|^p) \tag{7}$$

holds with the $K_1 = \dfrac{1}{c_p^{(1)}} \sigma^{-p}$ positive constant. Applying (7) to the right-hand side of (2) we have with a positive constant $K_2 = 2C_p \dfrac{1}{c_p^{(1)}} \sigma^{-p} E(|Y_1|^p)$, which does not depend on the choice of ν, that

$$A_2^{(p)} \le K_2 E(|S_\nu|^p), \qquad p \ge 2. \tag{8}$$

Also, for $1 \leq p \leq 2$, by using the Burkholder-Davis-Gundy and the Marcinkiewicz-Zygmund inequality one can prove that

$$A_2^{(p)} \leq 4 \frac{C_p}{c_p^{(1)}} E(|S_\nu|^p), \qquad 1 \leq p \leq 2. \tag{9}$$

Making use of (8) and (9), we have with $C > 0$ that

$$A_2^{(p)} \leq C E(|S_\nu|^p), \qquad p \geq 1. \tag{$**$}$$

Now, employing (*) and (**), we have for $p \geq 1$ that

$$cE(|S_\nu|^p) \leq A_2^{(p)} \leq CE(|S_\nu|^p).$$

Here $c > 0$ and $C > 0$ are two contants depending only on p and on the distribution of Y_1 and are independent of the choice of ν provided that $E(|Y_1|^p) < +\infty$ if $p \geq 2$, and $\sigma^2 < +\infty$ (and $= 1$) if $1 \leq p \leq 2$. This proves our assertion.

References

[1] **Bassily N.L., Ishak S. and Mogyoródi J.,** On Wald-type inequalities, *Annales Univ. Sci. Bud. Sect. Comp.,* **8** (1987), 5-24.

[2] **Bassily N.L.,** On an inequality of M.J.Klass, *Annales Univ. Sci. Bud. Sect. Comp.,* **11** (1991), 53-62.

[3] **Blackwell D.,** Extension of a renewal theorem, *Pacific J.Math.,* **3** (1953), 315-320.

[4] **Burkholder D.L.,** Distribution function inequalities for martingales, *Annals of Probab.,* **1** (1973), 19-42.

[5] **Burkholder D.L. and Gundy R.F.,** Extrapolation and interpolation of quasilinear operators on martingales, *Acta Math.,* **124** (1970), 249-304.

[6] **Chow Y.S., Robbins H. and Siegmund D.,** *Great expectations: The theory of optimal stopping,* Houghton Mifflin Co., Boston, 1971.

[7] **Chow Y.S. and Teicher H.,** *Probability theory, independence, interchangebility, martingales,* Springer, 1978.

[8] **Gut A. and Janson S.,** Converse results for existence of moments and uniform integrability for stopped random walks, *Annals of Probab.,* **14** (1986), 1296-1317.

[8] **Ishak S.,** Remark on a theorem of Burkholder and Gundy, *J. of Theor. Probab.,* **5** (1992), 153-158.

[9] **Klass M.J.,** A best possible improvement of Wald's equation, *Annals of Probab.,* **16** (1988), 840-853.

[10] **Marcinkiewicz J. and Zygmund A.,** Sur les fonctions indépendantes, *Fundamenta Math.,* (1937), 60-90.

[11] **Marcinkiewicz J. and Zygmund A.,** Quelques théorèmes sur les fonctions indépendantes, *Studia Math.,* **7** (1938), 104-120.

[12] **Mogyoródi J.,** On an inequality of H.R. Rosenthal, *Periodica Math. Hung.,* **8** (1977), 275-279.

[13] **Mogyoródi J.,** A concave function inequality for martingales, *Annales Univ. Sci. Bud. Sect. Math.,* **24** (1981), 265-271.

N.L. Bassily
Department of Mathematics
Faculty of Science
Ain Shams University
Cairo, Egypt