Annales Univ. Sci. Budapest., Sect. Comp. 16 (2004) 13-30

IDENTICAL CLASSES OF MULTIPLIERS FOR WALSH SERIES

S. Baron (Ramat Gan, Israel)
F. Schipp (Budapest, Hungary)

Dedicated to Professor János Balázs on his 75th birthday

1. Introduction

We shall denote the set of non-negative integers by $\mathbf{N}$, the set of real numbers by $\mathbf{R}$, and the set of dyadic rationals in the unit interval $\mathbf{I} = [0, 1)$ by $\mathbf{Q}$. In particular, each element of $\mathbf{Q}$ has the form $m/2^n$ for some $m, n \in \mathbf{N}$ with $0 \le m < 2^n$. Sequences are indexed by $\mathbf{N}$ and are written in the form $a = (a_n, n \in \mathbf{N})$. Furhermore, if the limits of a sum are not indicated, then it is taken with respect to $\mathbf{N}$.

By dyadic interval in $\mathbf{I}$ we mean intervals of the form $[m/2^n, (m+1)/2^n)$ for some $m, n \in \mathbf{N}$ with $0 \le m < 2^n$. Given $n \in \mathbf{N}$ and $x \in \mathbf{I}$ let $I_n(x)$ denote the dyadic interval of length 2^{-n} that contains x.

For each $n \in \mathbf{N}$ let $\mathcal{A}^n$ represent the σ-algebra generated by the dyadic intervals of length 2^{-n}. Thus every element of $\mathcal{A}^n$ is a finite union of intervals of the form $[k/2^n, (k+1)/2^n)$, where $k \in \mathbf{N}$ and $k < 2^n$. The collection of $\mathcal{A}^n$-measurable real functions on $\mathbf{I}$ will be denoted by $L(\mathcal{A}^n)$ and the set of dyadic step functions by $\mathcal{P}$, i.e.

$$\mathcal{P} := \bigcup_n L(\mathcal{A}^n)$$

(see [14], p.75). The set of *W-continuous* functions, i.e. the closure of dyadic step functions in the supremum norm

$$\|f\| := \sup_{t \in \mathbf{I}} |f(t)|$$

The research of second author was supported by the Hungarian Scientific Research Funds No. T 020457.

is denoted by $C_W := C_W(\mathbf{I})$. Let $C := C(\mathbf{I})$ denote the Banach space equivalent to $C[0,1]$, whose elements are received by restricting the functions in $C[0,1]$ to $\mathbf{I}$. The space C is a vector subspace of C_W. For details and another characterization of C_W see [14], p.9 and p.50. The Banach space of functions g of bounded variation on $\mathbf{I}$ with $g(0) = 0$ and with the norm $\|g\|_{BV} :=\mathrm{var}_{\mathbf{I}}(g)$ is denoted by $BV := BV(\mathbf{I})$ (cf. [11], p.135, [10], p.62).

We denote by $|Y|$ the Lebesgue measure of the measurable subset Y in $\mathbf{I}$. The $L^p := L^p(\mathbf{I})$ quasi-norm or norm $(0 < p \leq \infty)$ of a function $f \in L^p$ will be denoted by $\|f\|_p$. For any subspace $X \subseteq L^1$ let X_0 be the set of functions $f \in X$ whose integral is zero, i.e.

$$X_0 := \{f \in X : \int\limits_{\mathbf{I}} f(t)dt = 0\}.$$

The conditional expectation of a function $f \in L^1$ with respect to $\mathcal{A}^n$ is denoted by $E_n f$ and can be given in the form (see [14], pp.29 and 75)

$$(E_n f)(x) = \frac{1}{|I_n(x)|} \int\limits_{I_n(x)} f(t)\,dt \qquad (x \in \mathbf{I}, n \in \mathbf{N}).$$

For the definition of dyadic Hardy spaces H^p we shall use the *dyadic maximal function* which is defined by

$$f^*(t) := \sup_n |(E_n f)(t)| \quad (f \in L^1, t \in \mathbf{I}).$$

Let $p \geq 1$ and H^p the set of functions $f \in L^1$ for which

$$\|f\|_{H^p} := \|f^*\|_p < \infty.$$

The space H^1 can be identified with a proper subspace of L^1, but $H^p = L^p$ for $1 < p < \infty$ (see [14], pp.104 and 138).

It is well known that for $1 \leq p < \infty$ the dual of L^p is L^q, where $1/p+1/q = = 1$. The dual of H^1 is the space of functions of *bounded mean oscillation*, in notation BMO (see [14], p.107). A function $f \in L_0^2$ is said to belong to BMO if

$$\|f\|_{BMO} := \sup_n \|(E_n(f - E_n f)^2)^{1/2}\|_\infty < \infty.$$

The subspace of functions in BMO satisfying

$$\lim_n \|(E_n(f - E_n f)^2)^{1/2}\|_\infty = 0$$

is denoted by VMO, and it is called the space of functions of *vanishing mean oscillation.* The space VMO is the closure of the set $\mathcal{P}$ of dyadic step functions in BMO-norm and the dual of VMO is H_0^1 (see [14], p.114, Theorem 10).

Denote by $w = (w_n, n \in \mathbf{N})$ *the Walsh system in Paley's ordering.* The values of the Walsh functions at $x \in \mathbf{I}$ can be expressed by the bits of x in the binary expansion $x = \sum x_k 2^{-(k+1)}$, where each $x_k = 0$ or 1. If $x \in \mathbf{I} \setminus \mathbf{Q}$, then this expansion is uniquely determined. By the dyadic expansion of an $x \in \mathbf{Q}$, we shall always mean the one which terminates in 0's.

There is a direct connection between dyadic expansions and Walsh functions. Namely,

$$w_n(x) = (-1)^{[n,x]},$$

where

$$[n, x] := \sum n_k x_k \pmod 2 \quad (n \in \mathbf{N}, x \in \mathbf{I}),$$

and $n_k \in \{0, 1\}$ are the binary coefficients of $n \in \mathbf{N}$, i.e. $n = \sum n_k 2^k$. This implies that the Walsh functions behave almost like characters with respect to dyadic addition. Namely, for almost every $x, y \in \mathbf{I}$ we have

$$w_n(x \dot{+} y) = w_n(x)\, w_n(y) \quad (n \in \mathbf{N}), \tag{1.1}$$

where the dyadic sum of x and y is defined (cf. [14], p.10) by

$$x \dot{+} y = \sum |x_k - y_k|\, 2^{-(k+1)}.$$

Using the concept of dyadic addition we can introduce for an arbitrary function f defined on $\mathbf{I}$ the *dyadic translation operator* τ_y (cf. [14], p.13) by

$$(\tau_y f)(x) := f(x \dot{+} y) \quad (x, y \in \mathbf{I}).$$

The *dyadic convolution* of $f, g \in L^1$ is defined by

$$(f * g)(x) := \int_{\mathbf{I}} f(x \dot{+} y)\, g(y)\, dy \qquad (x \in \mathbf{I}). \tag{1.2}$$

The Banach subspaces of L^1 mentioned before are homogeneous with respect to the dyadic translation. A Banach space $X \subseteq L^1$ with the norm $\|\cdot\|_X$ is called a *homogeneous Banach space* if the set $\mathcal{P}$ of dyadic step functions is dense in X, $\|f\|_1 \le \|f\|_X$ for all $f \in X$ and its norm is translation invariant, i.e.

$$\text{if } f \in X \text{ and } y \in \mathbf{I} \quad \text{then} \quad \tau_y f \in X \quad \text{and} \quad \|\tau_y f\|_X = \|f\|_X.$$

For $1 \leq p < \infty$ the L^p and H^p spaces, the spaces C_W and VMO are homogeneous Banach spaces (see [14], pp.155, 9 and 107).

The Orlicz spaces L_Φ with the complementary Young function Ψ (see, for example, [3], p.291) satisfying $\Psi(1) \leq 1$ are also homogeneous Banach spaces. Indeed, since Φ satisfies $\triangle_2$-condition, we have that $\mathcal{P}$ is dense in L_Φ (see [20], p.86, Example 15). The inequality $\|f\|_1 \leq \|f\|_\Phi$ follows from the definition of the norms. At the end translation invariance of the integral yields that L_Φ is a homogeneous Banach space.

Furthermore the spaces L^∞, L_Ψ and BMO are duals of homogeneous Banach spaces (see [20], pp.138 and 150, and [14], p.114).

2. Walsh coefficients and Walsh series

Walsh series are closely connected to quasi-measures. Denote by $\Re$ the algebra of sets generated by the dyadic intervals in $\mathbf{I}$. By a quasi-measure we shall mean a real-valued set-function which is finitely additive on $\Re$. The restriction of every finite Borel-measure on $\mathbf{I}$ to $\Re$ is a quasi-measure, but not conversely (see [14], p.30).

We shall denote the collection of quasi-measures on $\Re$ by QM. Let VM be the set of quasi-measures with finite total variation, and let BM be the set of finite Borel-measures on $\mathbf{I}$. Moreover, denote by AM the set of absolutely continuous measures in BM. We recall (cf. [14], p.266) that the map $f \mapsto \nu^f$ defined by

$$\nu^f(J) := \int_J f(t)\,dt \qquad (J \in \Re) \tag{2.1}$$

is a 1-1 transformation from L^1 onto AM. Moreover, if $\|\nu\|$ denotes the total variation of $\nu \in VM$ then $\|\nu^f\| = \|f\|_1$, i.e. the map in (2.1) is isometric.

For any $\nu \in QM$ and $f \in \mathcal{P}$ the map

$$f \to \int_{\mathbf{I}} f(t)\,d\nu(t)$$

is a linear functional on $\mathcal{P}$. Moreover every linear functional on $\mathcal{P}$ is of this form .

If $\nu \in QM$ then the *Walsh-Fourier-Stieltjes coefficients* (shortly *Walsh coefficients*) of ν are defined by

$$\hat{\nu}(n) := \int_{\mathbf{I}} w_n(t)\, d\nu(t) \qquad (n \in \mathbf{N}). \tag{2.2}$$

Since each Walsh function is constant on sufficiently small dyadic intervals, this definition makes sense. It turns out (in contrary with the trigonometric case) that the map $\nu \mapsto \hat{\nu}$ is a 1-1 function from QM onto the space of sequences

$$\mathbf{s} := \{x : x = (x_n, n \in \mathbf{N}), x_n \in \mathbf{R}\}$$

(see [14], p.30).

For $f \in L^1$ the n-th Walsh-Fourier coefficient of f will be denoted by

$$\hat{f}(n) := \int_{\mathbf{I}} f(t)\, w_n(t)\, dt \qquad (n \in \mathbf{N}). \tag{2.3}$$

From (2.1), (2.2) and (2.3) it follows that

$$\widehat{\nu^f} = \hat{f} \qquad (f \in L^1). \tag{2.4}$$

The dyadic convolution of $f, g \in L^1$ satisfies (see [14], p.25)

$$(\widehat{f * g})(n) = \hat{f}(n) \cdot \hat{g}(n) \quad (n \in \mathbf{N}).$$

The convolution (1.2) can be extended for $\nu \in QM$ and $f \in \mathcal{P}$ by

$$(f * \nu)(x) := \int_{\mathbf{I}} f(x \dot{+} t)\, d\nu(t) \quad (x \in \mathbf{I}). \tag{2.5}$$

The Walsh series of $\nu \in QM$ is defined by

$$S\nu := \sum \hat{\nu}(n)\, w_n, \tag{2.6}$$

and the set of Walsh series will be denoted by $\mathcal{W}$. The Walsh series of $f \in L^1$ is defined as

$$Sf := S\nu^f. \tag{2.7}$$

By (2.6), (2.4) and (2.7) we have

$$Sf = \sum \hat{f}(n)\, w_n.$$

The partial sums of $S\nu$ are defined (cf. [14], p.27) by

$$S_0\nu = 0, \quad S_n\nu = \sum_{k=0}^{n-1} \hat{\nu}(k)\, w_k \quad (n \in \mathbf{N} \setminus \{0\}). \tag{2.8}$$

We shall investigate linear subspaces $U, V, \ldots$ of the space of quasi-measures QM and denote the corresponding space of the Walsh coefficients by $\hat{U}, \hat{V}, \ldots$, i.e.

$$\hat{U} := \{\hat{\nu} : \ \nu \in U\}.$$

Since L^1 by (2.1) can be identified with the subspace AM of QM, subspaces of L^1 can also be identified in this way.

A sequence $\lambda = (\lambda_n, n \in \mathbf{N})$ is called a *multiplier of the class* (U, V) if for every $\nu \in U$ we have

$$(\lambda_n \hat{\nu}(n), \quad n \in \mathbf{N}) \in \hat{V},$$

that is $T\nu \in V$ for every $\nu \in U$, where the multiplier operator $T : U \to V$ generated by the multiplier λ is given by

$$\widehat{T\nu}(n) := \lambda_n \hat{\nu}(n) \quad (n \in \mathbf{N}).$$

The aim of this paper is to find relations between several classes of multipliers, using the notions of complementary spaces and summability factors.

Let $A = (\alpha_{nk})$ be a triangular matrix summability method which maps series into sequences. The A-means of the Walsh series $S\nu$ are denoted by $\sigma_n\nu$, i.e.

$$\sigma_n\nu := \sum_{k=0}^{n} \alpha_{nk}\, \hat{\nu}(k)\, w_k \quad (n \in \mathbf{N}). \tag{2.9}$$

The operator σ_n is a linear map from QM to the set $\mathcal{W}$. The corresponding operator from $\mathbf{s}$ to $\mathbf{s}$ will also be denoted by σ_n, i.e.

$$\sigma_n\hat{\nu} := (\alpha_{nk}\hat{\nu}(k), \quad k \in \mathbf{N}) \quad (n \in \mathbf{N}).$$

By the definition (2.2) of Walsh coefficients, (2.9) and by (1.1) we have

$$(\sigma_n \nu)(x) = \sum_{k=0}^{n} \alpha_{nk}\, w_k(x) \int_{\mathbf{I}} w_k(t)\, d\nu(t) =$$
$$= \int_{\mathbf{I}} \sum_{k=0}^{n} \alpha_{nk}\, w_k(x) w_k(t)\, d\nu(t) =$$
$$= \int_{\mathbf{I}} \sum_{k=0}^{n} \alpha_{nk}\, w_k(x \dot{+} t)\, d\nu(t).$$

Denote K_n the kernels of the summability method A. In this case the kernels can be expressed by a one-variable function, i.e.

$$K_n := \sum_{k=0}^{n} \alpha_{nk} w_k.$$

Thus $\sigma_n \nu$ can be written in the form

$$(\sigma_n \nu)(x) = \int_{\mathbf{I}} K_n(x \dot{+} t)\, d\nu(t)$$

and by (2.5) we have

$$\sigma_n \nu = K_n * \nu \quad (n \in \mathbf{N}).$$

If $\nu \in AM$, i.e. $\nu = \nu^f$ for some $f \in L^1$ then analogously to (2.7) we introduce the notation

$$\sigma_n f := \sigma_n \nu^f$$

and by (2.4), (2.3) and (1.2) we obtain

$$\sigma_n f = K_n * f \quad (n \in \mathbf{N}).$$

The numbers

$$L_n := \|K_n\|_1 \quad (n \in \mathbf{N})$$

are called the *Lebesgue constants* of the summability method A.

Many of the methods A satisfy the conditions

(2.10) $$\sup_{n,k} |\alpha_{nk}| < \infty, \quad \lim_n \alpha_{nk} = 1$$

and

$$L_n = O(1). \tag{2.11}$$

We give the following two examples.

If for $2^j \le n < 2^{j+1}$ we set $\alpha_{nk} = 1$ if $k \le 2^j$ and $\alpha_{nk} = 0$ if $k > 2^j$, then $\sigma_n \nu = S_{2^j}\nu$ and the conditions (2.10) and (2.11) are satisfied (see also [14], p.7 and p.48). Thus results with respect to the subsequences of partial sums $(S_{2^j}\nu, j \in \mathbf{N})$ can be obtained in this way.

For the Cesàro method, i.e. if

$$\alpha_{nk} = A_{n-k}^{\alpha}/A_n^{\alpha} \quad \text{with} \quad A_n^{\alpha} := \binom{n+\alpha}{n}$$

the conditions (2.10) are satisfied. The claim (2.11) is known for $\alpha > 0$. For $\alpha = 1$ see [4], p.396, Lemma 2 and 3, and [14], p.46. For $0 < \alpha < 1$ see [19], p.966. The case $\alpha > 1$ follows from a result of Tandori (see [15], p.89; cf. [1], p.66 and 175). For $\alpha = 0$ condition (2.11) is not satisfied (see [12], p.104, and [14], p.34).

3. Some topological properties of A-complementary spaces

Let $\mathbf{x} \subset \mathbf{s}$ be a normed sequence space. The space $\mathbf{x}$ is called a *BK-space*, if $\mathbf{x}$ is complete and the coordinate wise convergence of a sequence follows from its convergence in norm (cf. [21], p.466). This implies that the coordinate functionals

$$\ell_k(x) := x_k \quad (x = (x_n, \quad n \in \mathbf{N}) \in \mathbf{s}, \quad k \in \mathbf{N}) \tag{3.1}$$

are bounded for every $k \in \mathbf{N}$.

Denote $A = (\alpha_{nk})$ a triangular matrix summability method, satisfying (2.10). The collection of sequences $y = (y_n, n \in \mathbf{N}) \in \mathbf{s}$ for which the series

$$\sum_n x_n y_n$$

is A-summable for all $x \in \mathbf{x}$, is a vector subspace of $\mathbf{s}$. This space is called the *A-complementary space* of $\mathbf{x}$ and will be denoted by $(\mathbf{x} \to A)$. Thus $y \in (\mathbf{x} \to A)$ if and only if the sequence $(F_n(x,y), n \in \mathbf{N})$ converges for all $x \in \mathbf{x}$, where

$$F_n(x,y) := \sum_{m=0}^{n} \alpha_{nm} x_m y_m \quad (n \in \mathbf{N}). \tag{3.2}$$

The space $(\mathbf{x} \to A)$ is not empty. Moreover in view of (2.10) the set

$$\mathbf{s}_0 := \{x = (x_n, n \in \mathbf{N}) : \exists N_x \in \mathbf{N} \quad \text{such that} \quad x_n = 0 \quad \forall n \geq N_x\}$$

generated by the unite sequences

$$e^{(k)} := (\delta_{kn}, n \in \mathbf{N}) \quad (k \in \mathbf{N})$$

is a subspace of $(\mathbf{x} \to A)$.

We prove

Theorem 3.1. *Let $\mathbf{s}_0 \subset \mathbf{x} \subset \mathbf{s}$ be a BK-space and suppose that A satisfies* (2.10)*. Then $(\mathbf{x} \to A)$ is also a BK-space with the norm*

$$\|y\|_{(\mathbf{x}\to A)} := \sup_n \sup_{\|x\|_{\mathbf{x}}\leq 1} |F_n(x,y)|. \tag{3.3}$$

Proof. The set $(\mathbf{x} \to A)$ is a vector space. We prove, that if $y \in (\mathbf{x} \to A)$, then $\|y\|_{(\mathbf{x}\to A)} < \infty$ in (3.3). In fact, let for $x \in \mathbf{x}$

$$G_n x := F_n(x,y) \quad (x \in \mathbf{x}) \tag{3.4}$$

and by (3.1)

$$G_n x = \sum_{m=0}^{n} \alpha_{nm} y_m \ell_m(x).$$

Since $\mathbf{x}$ is a BK-space we have that $(G_n, n \in \mathbf{N})$ is a sequence of continuous linear functionals on $\mathbf{x}$ (compare [21], p.471, Satz 4.4a). If $y \in (\mathbf{x} \to A)$, then according to the definition of the space $(\mathbf{x} \to A)$, the sequence $(G_n, n \in \mathbf{N})$ converges for each $x \in \mathbf{x}$, and by the Banach-Steinhaus theorem (see [9], p.41, [13], p.63, [10], p.204) we obtain

$$\|G_n\| = O(1) \quad (n \in \mathbf{N})$$

and therefore (3.3) and (3.4) implies

$$\|y\|_{(\mathbf{x}\to A)} = \sup_n \|G_n\| < \infty.$$

It is clear that (3.3) satisfies the axioms of a norm. In fact, let $\|y\|_{(\mathbf{x}\to A)} = 0$. If there exists a number k with $y_k \neq 0$, then by (2.10) and (3.2) we obtain $|F_n(e^{(k)}, y)| = |\alpha_{nk} y_k| > 0$ for large n. Therefore, $\|y\|_{(\mathbf{x}\to A)} > 0$.

Let us prove that $(\mathbf{x} \to A)$ is a complete space. Let $(y^{(i)}, i \in \mathbf{N})$ be a Cauchy sequence in $(\mathbf{x} \to A)$, where

$$y^{(i)} = (y_k^{(i)}, \quad k \in \mathbf{N}) \in \mathbf{s}.$$

Then by (3.2) and (3.3) for any $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that for any $i \geq N$ and any natural number j the inequality

$$\|y^{(i)} - y^{(i+j)}\|_{(\mathbf{x} \to A)} = \sup_n \sup_{\|x\|_{\mathbf{x}} \leq 1} |F_n(x, y^{(i)} - y^{(i+j)})| < \varepsilon \tag{3.5}$$

holds. If in (3.5) we put $x = e^{(k)}/\|e^{(k)}\|_{\mathbf{x}}$, then by (3.3) we obtain

$$|\alpha_{nk}(y_k^{(i)} - y_k^{(i+j)})|/\|e^{(k)}\|_{\mathbf{x}} \leq \|y^{(i)} - y^{(i+j)}\|_{(\mathbf{x} \to A)} < \varepsilon \tag{3.6}$$

for $i \geq N$ and any j. Therefore, by (2.10) from (3.6) we get

$$|y_k^{(i)} - y_k^{(i+j)}| \leq \|y^{(i)} - y^{(i+j)}\|_{(\mathbf{x} \to A)} \|e^{(k)}\|_{\mathbf{x}} \leq \varepsilon \|e^{(k)}\|_{\mathbf{x}}.$$

Consequently there exists a $y \in \mathbf{s}$ with

$$y_k^{(i)} \to y_k \quad \text{as} \quad i \to \infty$$

for all $k \in \mathbf{N}$. Recall that $F_n(x, y^{(i)} - y^{(i+j)})$ is a finite sum for any fixed n, and the inequality (3.5) is valid for any j. Hence taking the limit as $j \to \infty$ we have

$$\sup_{\|x\|_{\mathbf{x}} \leq 1} |F_n(x, y^{(i)} - y)| \leq \varepsilon \tag{3.7}$$

for every $i \geq N$ independently on n. Taking sup over n in (3.7), by (3.3) we obtain

$$\|y^{(i)} - y\|_{(\mathbf{x} \to A)} \leq \varepsilon \tag{3.8}$$

for all $i \geq N$. From (3.8) it follows, that $y \in (\mathbf{x} \to A)$. Indeed, let $x' \in \mathbf{x}$ with $\|x'\|_{\mathbf{x}} \leq 1$ and $i \geq N$. Then by (3.2) and (3.7) we have for any $n \in \mathbf{N}$ that

$$|F_n(x', y^{(i)}) - F_n(x', y)| = |F_n(x', y^{(i)} - y)| \leq$$

$$\leq \sup\{|F_n(x, y^{(i)} - y)| : \|x\|_{\mathbf{x}} \leq 1\} \leq \varepsilon.$$

Since $y^{(i)} \in (\mathbf{x} \to A)$, therefore $(F_n(x', y^{(i)}), n \in \mathbf{N})$ converges for every $i \in \mathbf{N}$. Consequently, $(F_n(x', y), n \in \mathbf{N})$ is also convergent, i.e. $y \in (\mathbf{x} \to A)$ by definition.

It remains now to prove the coordinate-wise convergence in the Banach space $(\mathbf{x} \to A)$. In fact, if we take $\tilde{x} := e^{(k)}/\|e^{(k)}\|_{\mathbf{x}}$, we obtain from (3.2) and (3.3) that

$$|y_k^{(i)} - y_k||\alpha_{nk}| = |F_n(\tilde{x}, y^{(i)} - y)| \, \|e^{(k)}\|_{\mathbf{x}} \le \|y^{(i)} - y\|_{(\mathbf{x}\to A)} \|e^{(k)}\|_{\mathbf{x}},$$

whence by (2.10)

$$|y_k^{(i)} - y_k| \le \|y^{(i)} - y\|_{(\mathbf{x}\to A)} \|e^{(k)}\|_{\mathbf{x}} \tag{3.9}$$

and (3.9) yields that $y_k^{(i)} \to y_k$ as $i \to \infty$. Consequently $(\mathbf{x} \to A)$ is a BK-space.

An immediate consequence of the above Theorem is

Corollary 3.2. *If the space* $\mathbf{x}$ *of sequences is a* BK*-space,* A *satisfies* (2.10) *and* $\mathbf{s}_0$ *is dense in* $\mathbf{x}$*, then* $y \in (\mathbf{x} \to A)$ *if and only if*

$$\|y\|_{(\mathbf{x}\to A)} < \infty. \tag{3.10}$$

Proof. The fact that $y \in (\mathbf{x} \to A)$ implies (3.10) is shown in the proof of Theorem 3.1. We shall prove the converse. Assume that (3.10) holds. Then for the sequence $(G_n, n \in \mathbf{N})$ of continuous linear functionals, defined by (3.4) we have $\|G_n\| = O(1)$. By the Banach–Steinhaus theorem it remains to prove that the sequence $(G_n x, n \in \mathbf{N})$ converges on a dense set in $\mathbf{x}$. If $x = e^{(k)}$ for some $k \in \mathbf{N}$, then

$$F_n(e^{(k)}, y) = \alpha_{nk} y_k \to y_k \quad \text{as} \quad n \to \infty$$

in view of (2.10). Since the linear hull of $\{e^{(k)} : k \in \mathbf{N}\}$ is $\mathbf{s}_0$ and $\mathbf{s}_0$ is dense in $\mathbf{x}$, it follows that $(G_n x, n \in \mathbf{N})$ converges for any $x \in \mathbf{x}$, that is $y \in (\mathbf{x} \to A)$.

If the space X of quasi–measures ν is a Banach space, then the space $\mathbf{x} = \hat{X}$ of Walsh coefficients $x = \hat{\nu}$ is also a Banach space with the norm $\|x\|_{\mathbf{x}}$, defined by

$$\|\hat{\nu}\|_{\hat{X}} := \|\nu\|_X. \tag{3.11}$$

The Walsh polynomials in QM correspond to the set $\mathbf{s}_0$ by this map, i.e. $\hat{\mathcal{P}} = \mathbf{s}_0$. Let $X \subset QM$ be a Banach space. Set $\mathbf{x} = \hat{X}$ and denote $(\mathbf{x} \to A)$ the complementary space of $\mathbf{x}$. The space $Y \subset QM$ of quasi measures satisfying

$$\hat{Y} = (\mathbf{x} \to A)$$

is called the *comlementary space of* X and will be denoted by $(X \to A)$ (cf. [3], p.299). Hence

$$(X \to A) = \{\mu : \mu \in QM, \quad (F_n(\hat{\nu}, \hat{\mu}), n \in \mathbf{N}) \in \mathbf{c} \quad \forall \nu \in X\},$$

where $\mathbf{c}$ is the space of all convergent sequences. By Theorem 3.1 the norm of $\mu \in (X \to A)$ is defined by

$$\|\mu\|_{(X\to A)} := \sup_n \sup_{\|\nu\|_X \le 1} |F_n(\hat{\nu}, \hat{\mu})|, \tag{3.12}$$

where the functionals F_n are introduced in (3.2), that is

$$F_n(\hat{\nu}, \hat{\mu}) = \sum_{m=0}^{n} \alpha_{nm}\, \hat{\nu}(m)\, \hat{\mu}(m).$$

We will suppose that $X \subset QM$ is a Banach space and X contains the Walsh polynomials and consequently $\mathbf{s}_0 \subset \hat{X}$. Applying Theorem 3.1 to special subsets of QM we get

Corollary 3.3. *If X is any of the spaces $C, C_W, L^p\,(p \ge 1), L_\Phi, L_\Psi, H^1$, BMO, VMO or BV and the summability method A satisfies condition* (2.10), *then the complementary space $(X \to A)$ is a BK-space with the norm* (3.12).

Proof. Let X one of these spaces. Then X is a Banach space. Moreover we have by (2.2) and (3.11) that

$$|\hat{\mu}(k)| \le \|\mu\|_{L^1} \le \|\mu\|_X = \|\hat{\mu}\|_{\hat{X}}.$$

Consequently, $\hat{X}$ is a BK-space. Now by Theorem 3.1 the A-complementary spaces $(X \to A)$ are BK-spaces with the norm (3.12). In the case if $X = BV$ we get immediately

$$|\hat{\mu}(k)| \le \|\mu\|_{BV} \tag{3.13}$$

for all $k \in \mathbf{N}$.

The analogue of these results for trigonometric series can be found in [5], Satz 2.1 and Satz 2.3, [6], p.373, [7], Satz 3', [17], Theorem 12 and Theorem 13.

4. Identical classes of multipliers

We shall need the following definition (see [9], p.34, [16], p.202).

Let X be a normed space with the dual space denoted by X'. A closed subset $\Gamma \subset X'$ is called a *norm-determining manifold for* X if for any $x \in X$ we have

$$\sup\{|\varphi(x)| : \|\varphi\|_{X'} \le 1, \quad \varphi \in \Gamma\} = \|x\|_X.$$

For example, the dual X' itself is a norm determining manifold for X. This follows from the Theorem of sufficiently many continuous linear functionals (see [16], p.186, Theorem 4.3-B, [10], p.117, Theorem 2).

In our investigations we consider sequence spaces $\mathbf{y} \subset \mathbf{s}$ such that the subspace $\mathbf{s}_0$ is dense in $\mathbf{y}$. In this case the linear functionals on $\mathbf{y}$ are of the form $\varphi(x) = \langle x, y\rangle$, where

$$\langle x, y\rangle := \sum_k x_k y_k \quad (x \in \mathbf{s}_0).$$

The uniquely determined sequence $y = (y_k, k \in \mathbf{N})$ is called the *generating sequence of the functional* φ.

Let $\mathbf{u}, \mathbf{v} \subset \mathbf{s}$. The sequence $\lambda = (\lambda_n, n \in \mathbf{N})$ is called a *multiplier of the class* $(\mathbf{u}, \mathbf{v})$ if for every sequence $x = (x_n, n \in \mathbf{N}) \in \mathbf{u}$ we have

$$\lambda x = (\lambda_n x_n, n \in \mathbf{N}) \in \mathbf{v}.$$

We prove the following

Theorem 4.1. *Let the summability method* A *satisfy* (2.10)*. Let* $\mathbf{x}$ *and* $\mathbf{y}'$ *be* BK*-spaces of sequences and suppose that the set* $\mathbf{s}_0$ *is dense in* $\mathbf{x}$ *and* $\mathbf{s}_0 \subset \mathbf{y}$*. Let* Γ *and* Λ *be norm–determining manifolds for* $\mathbf{y}$*. Then*

$$(\Gamma, (\mathbf{x} \to A)) = (\Lambda, (\mathbf{x} \to A)). \tag{4.1}$$

Proof. By the definition of multiplier and Corollary 3.2 it follows that λ is a multiplier of the class $(\Gamma, (\mathbf{x} \to A))$ if and only if the operator

$$T : \Gamma \to (\mathbf{x} \to A)$$

defined by

$$(Ty)_n := \lambda_n y_n \quad (n \in \mathbf{N})$$

is bounded, i.e.

$$\|Ty\|_{(\mathbf{x}\to A)} < \infty \quad \forall y \in \Gamma$$

and this is satisfied if $\|T\| < \infty$. From (3.3) we have

$$\|Ty\|_{(\mathbf{x}\to A)} = \sup\{|F_n(x, Ty)| : x \in \mathbf{x}, \quad \|x\|_{\mathbf{x}} \leq 1, \quad n \in \mathbf{N}\}. \tag{4.2}$$

Observe that

$$F_n(x, Ty) = F_n(Tx, y) \quad (x, y \in \mathbf{s}), \tag{4.3}$$

i.e. the operator T is self-adjoint with respect the bilinear functional F_n (cf. [16], p.214).

We show that the supremum of the norms in (4.2) with respect to $y \in \Gamma$ is the same for any norm-determining manifold of $\mathbf{y}$. To this end we introduce the multiplier operators T_n defined by

$$(T_n x)_k := \alpha_{nk}\lambda_k x_k \quad (k, n \in \mathbf{N},\ x \in \mathbf{s}).$$

Then

$$F_n(x, Ty) = \langle T_n x, y\rangle \quad (n \in \mathbf{N},\ x, y \in \mathbf{s}).$$

Since the map $u \to \langle u, y\rangle$ is a linear functional on $\mathbf{y}$, therefore for any norm-determining manifold $\Gamma \subseteq \mathbf{y}'$ we have

$$\sup\{|\langle u, y\rangle| : y \in \Gamma, \quad \|y\|_{\mathbf{y}'} \leq 1\} = \|u\|_{\mathbf{y}}.$$

Applying this for $u = T_n x$ we get

$$\sup\{|F_n(x, Ty)| : y \in \Gamma, \quad \|y\|_{\mathbf{y}'} \leq 1\} = \|T_n x\|_{\mathbf{y}}.$$

Consequently taking the supremum with respect to x we obtain

$$\sup\{|F_n(x, Ty)| : y \in \Gamma, \quad \|y\|_{\mathbf{y}'} \leq 1, \quad x \in \mathbf{x}, \quad \|x\|_{\mathbf{x}} \leq 1\} =$$

$$= \sup\{\|T_n x\|_{\mathbf{y}} : x \in \mathbf{x}, \quad \|x\|_{\mathbf{x}} \leq 1\} = \|T_n\|_{\mathcal{L}(\mathbf{x},\mathbf{y})}.$$

This and (4.2) implies that the norm of T can be expressed by

$$\|T\| = \sup\{\|Ty\|_{(\mathbf{x}\to A)} : y \in \Gamma, \quad \|y\|_{\mathbf{y}'} \leq 1\} = \sup_n \|T_n\|_{\mathcal{L}(\mathbf{x},\mathbf{y})},$$

and the right hand side is the same for any norm-determining manifold for $\mathbf{y}$, in particular also for Λ.

If $\mathcal{P}$ is dense in X, then $\hat{\mathcal{P}}$ is dense in $\hat{X}$, since the convergence in norm implies the weak convergece. Therefore applying Theorem 4.1 for Walsh coefficients we obtain the following Corollaries (cf. [8], p.141, [18], p.92).

Corollary 4.2. *Let the summability method A satisfy* (2.10). *Let $\hat{X}$ and $\hat{Y}'$ be BK-spaces of Walsh coefficients and suppose that the set $\mathcal{P}$ of dyadic step functions is dense in X. Let Γ and Λ be norm–determining manifolds for Y. Then*

$$(\Gamma, (X \to A)) = (\Lambda, (X \to A)).$$

Corollary 4.3. *If A satisfies* (2.10) *and X is a homogeneous Banach space, then*

$$(C_W, (X \to A)) = (L^\infty, (X \to A)).$$

Proof. Since $(L^1)' = L^\infty$, then L^∞ is a norm–determining manifold for the Banach space L^1. Now we have to prove that the Banach space C_W is also a norm–determining manifold for L^1. In fact, for each $f \in C_W \subset L^\infty$ there exists a functional $\varphi \in (L^1)'$ such that

$$\varphi(x) = \int_{\mathbf{I}} x(t)f(t)dt$$

for all $x \in L^1$ (see [20], p.142, [10], p.173-174), whence $\sigma \leq \|x\|_{L^1}$, where

$$\sigma = \sup\{|\varphi(x)| : \|\varphi\|_{L^\infty} \leq 1, \quad f \in C_W\}.$$

On the other hand for each $x \in L^1$ there exists a sequence of functions $f_n \in C_W$ such that

$$\|f_n\|_{L^\infty} \leq 1, \quad \lim_n f_n(t) = \operatorname{sgn} x(t) \quad (\text{a.e. in} \quad t \in \mathbf{I}),$$

and consequently

$$\sigma \geq |\varphi_n(x)| := \left| \int_{\mathbf{I}} x(t)f_n(t)dt \right| \to \int_{\mathbf{I}} |x(t)|dt = \|x\|_{L^1},$$

i.e. we obtain $\sigma = \|x\|_{L^1}$.

Now the set $\mathcal{P}$ is dense in each homogeneous Banach space, then it remains to use Corollary 4.2 for the case $Y = L^1, \Gamma = C_W$ and $\Lambda = L^\infty$.

Corollary 4.4. *If A satisfies* (2.10) *and X is a homogeneous Banach space, then*

$$(L^1, (X \to A)) = (BV, (X \to A)).$$

Proof. Since $C' = BV$ (see [11], p.135, [10], p.182), then BV is a norm–determining manifold for the Banach space C. But the Banach space L^1 is also a norm–determining manifold for C (cf. [2], p.217). Since $\mathcal{P}$ is dense in X, then it remains to use Corollary 4.2 for the case $Y = C$, $\Gamma = L^1$ and $\Lambda = BV$. We remark that $\widehat{BV}$ is by (3.13) also a BK–space.

From the Corollaries 4.3 and 4.4 with the help of Theorem 4.1 i) from [3] we obtain the next corollary, since we can suppose that A satisfies (2.10) and (2.11).

Corollary 4.5. *If X is a homogeneous Banach space, then the following identities between classes of multipliers hold:*

$$1) \quad (C_W, X') = (L^\infty, X'),$$

$$2) \quad (L^1, X') = (BV, X').$$

Theorem 4.6. *For any summability method A and arbitrary spaces X and Y, the inclusion*

$$(X, Y) \subseteq ((Y \to A), (X \to A))$$

holds.

Proof. Let λ be a multiplier of the class (X, Y), that is $\widehat{T\mu} \in \hat{Y}$ for any $\mu \in X$. Let $\nu \in (Y \to A)$. Since

$$(Y \to A) = \{\nu \in QM : (F_n(\hat{\pi}, \hat{\nu}), \quad n \in \mathbf{N}) \in \mathbf{c} \quad \forall \pi \in Y\}$$

and $\pi = T\mu \in Y$, it follows that $< \hat{\nu}, \widehat{T\mu} >$ is A-summable for any $\mu \in X$. By (4.3)

$$< \hat{\nu}, \widehat{T\mu} > = < \widehat{T\nu}, \hat{\mu} >$$

and we obtain that $< \widehat{T\nu}, \hat{\mu} >$ is A–summable for any $\mu \in X$. This is $\widehat{T\nu} \in$ $\in (X \to A)$.

Corollary 4.7. *For any homogeneous Banach spaces X and Y the following identities between classes of multipliers hold:*

$$1) \quad (X, Y) = (Y', X'),$$

$$2) \quad (X, Y') = (Y, X'),$$

$$3) \quad (X', Y) = (Y', X).$$

Proof. From Theorem 4.6 using Theorems 4.1 i) and 4.1 ii) from [3] we obtain, for example, if A satisfies (2.10) and (2.11), that

$$\begin{aligned}(X', Y) &\subseteq ((Y \to A), (X' \to A)) = (Y', X) \subseteq \\ &\subseteq ((X \to A), ((Y' \to A)) = \\ &= (X', Y).\end{aligned}$$

In the Corollaries 4.5 and 4.7 the spaces X' and Y' are L_Ψ if $\Psi(1) \leq 1, L^p$ if $p > 1, L^\infty, BMO$ and others.

References

[1] **Alexits G.,** *Konvergenzprobleme der Orthogonalreihen,* Akadémiai Kiadó, Budapest, 1960.

[2] **Baron S.,** Complementary spaces and multipliers of double Fourier series, *Ark. Mat.,* **28** (1990), 201-219.

[3] **Baron S. and Schipp F.,** On complementary spaces and multipliers for Walsh series, *Acta Sci. Math. Szeged,* **57** (1993), 289-303.

[4] **Fine N.J.,** On the Walsh functions, *Trans. Amer. Math. Soc.,* **65** (1949), 372-414.

[5] **Goes G.,** BK-Räume und Matrixtransformationen für Fourierkoeffizienten, *Math. Z.,* **70** (1959), 345-371.

[6] **Goes G.,** Komplementäre Fourierkoeffizientenräume und Multiplikatoren, *Math. Ann.,* **137** (1959), 371-384.

[7] **Goes G.,** Identische Multiplikatorenklassen und C_k–Basen in C_k–komplementäre Fourierkoeffizientenräumen, *Math. Nachr.,* **21** (1960), 150-159.

[8] **Goes G.,** Charakterisierung von Fourierkoeffizienten mit einem Summierbarkeitsfaktorentheorem und Multiplikatoren, *Studia Math.,* **19** (1960), 133-148.

[9] **Hille E. and Phillips R.S.,** *Functional Analysis and Semigroups,* Amer. Math. Soc. Coll. Publ., Providence, 1957.

[10] **Kantorowitsch L.W. und Akilow G.P.,** *Funktionalanalysis in normierten Räumen,* Akad. Verlag, Berlin, 1964.

[11] **Ljusternik L.A. und Sobolew W.I.,** *Elemente der Funktionalanalysis,* 5. Auflage, Akad. Verlag, Berlin, 1976.

[12] **Móricz F. and Schipp F.,** On the integrability and L^1–convergence of Walsh series with coefficients of bounded variation, *J.Math. Anal. Appl.,* **146** (1990), 99-109.

[13] **Riesz F. and Sz.-Nagy B.,** *Lectures on Functional Analysis,* Dover, New York, 1955.

[14] **Schipp F., Wade W.R., Simon P. and Pál J.,** *Walsh Series, an Introduction to Dyadic Harmonic Analysis,* Adam Hilger, Bristol-New York, 1990.

[15] **Tandori K.,** Über die Cesàrosche Summierbarkeit der orthogonalen Reihen, *Acta Sci. Math. Szeged,* **14** (1951/52), 85-95.

[16] **Taylor E. A.,** *Introduction to Functional Analysis,* Wiley, 1967.

[17] **Tõnnov M.,** T-complementary spaces of Fourier coefficients, *Tartu Ülik. Toimetised,* **192** (1966), 65-81. (in Russian)

[18] **Tõnnov M.,** Summability factors, Fourier coefficients and multipliers, *Tartu Ülik. Toimetised,* **192** (1966), 82-97. (in Russian)

[19] **Yano S.,** On approximation by Walsh functions, *Proc. Amer. Math. Soc.,* **2** (1951), 962-967.

[20] **Zaanen A.C.,** *Linear Analysis, Measure and Integral, Banach and Hilbert Spaces, Linear Integral Equations,* North-Holland, 1964.

[21] **Zeller K.,** Allgemeine Eigenschaften von Limitierungsverfahren, *Math. Z.,* **53** (1951), 463-487.

S. Baron
Department of Mathematics and Computer Science
Bar–Ilan University
Ramat Gan, 52900, Israel

F. Schipp
Department of Numerical Analysis
Eötvös Loránd University
VIII. Múzeum krt. 6-8.
H-1088 Budapest, Hungary