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GLOBALLY ATTRACTING DOMAINS IN TWO-DIMENSIONAL REVERSIBLE CHEMICAL DYNAMICAL SYSTEMS

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Abstract. We investigate the differential equations of reversible chemical reactions containing two intermediates, and endowed with mass-action kinetics. The model of such a reaction is a dynamical system on the positive quadrant of the plane. We prove that the positive half trajectories of any dynamical system determined by a reaction of the above type are bounded and cannot tend to the origin. We show that if the dynamical system has no linear first integral, then there is a closed bounded globally attracting domain in the positive quadrant. Consequently, a dynamical system of this type has at least one stationary point in the positive quadrant.

1. Introduction

The investigation of some reversible chemical mechanisms shows that the reversibility stabilizes the system. We can mention two examples of this phenomenon. The first is the reversible Lotka-Volterra model for which Hering proved that this system has a globally asymptotically stable stationary point [6]. The second is the Lotka-Volterra-Autocatalator (LVA) model which is explosive (the trajectories tend to infinity) [3,4]. In [12] it was proved that the positive half trajectories of the reversible LVA model are bounded, and there is a sufficient condition for the existence of a globally asymptotically stable stationary point.

From the chemical point of view this behavior is expected for every reversible chemical mechanism endowed with mass-action kinetics. Namely,

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if a component X is produced in a reaction, then the backward reaction consumes X, and furthermore, due to mass-action kinetics the reaction rate of the backward reaction surpasses that of the forward reaction for enough large concentration of X, therefore, the time derivative of this concentration is negative provided that the concentration is large enough. Similarly, if the concentration of one of the components is very decreased then the derivative of the concentration function of this component is positive, because of the reversibility. Schnakenberg deals with this problem in [11, Chapter 7.6]. He states that the trajectories cannot tend to infinity, but he does not give a rigorous proof.

These qualitative ideas above suggest that the positive half trajectories of a reversible chemical system are contained in a closed bounded domain in the positive quadrant of the phase plane. This paper is devoted to prove this statement for the general case of two-dimensional systems.

In order to prove this statement we shall construct for every point of the positive quadrant of the phase plane a closed broken line, such that the positive half trajectory starting from the point cannot leave the domain bordered by the broken line. Moreover, we generally prove the existence of globally attracting sets bordered by closed broken lines. There is only one exception, the so-called conservative system (that is, for which there is a linear first integral); for it there is no globally attracting domain, but the boundedness of the positive half trajectories is true.

We shall prove our main result in three steps:

A. We construct a family $\mathcal{K}$ of simple (without self crossing) broken lines, such that every simple broken line $k \in \mathcal{K}$ has the following properties:

1. The broken line k has one endpoint on the axis x, and the other on the axis y. Therefore, the broken line divides the positive quadrant into two parts, one of which is bounded, the other is unbounded.

2. The trajectories cannot go through the broken line k from the bounded part into the unbounded part.

3. If $k_1 \in \mathcal{K}$ and P is a point in the unbounded part determined by k_1, then there exists a broken line $k_2 \in \mathcal{K}$ containing the point P, that is $P \in k_2$.

We shall refer to the broken lines of the family $\mathcal{K}$ as *outer broken lines*. Their properties show that every positive half trajectory is bounded.

B. In the bounded part defined by the outer broken lines we construct an other family $\mathcal{B}$ of simple broken lines, such that every simple broken line $b \in \mathcal{B}$ has the following properties:

1. The broken line b has one endpoint on the axis x, and the other on the axis y. Therefore the broken line divides the positive quadrant into two parts, one of which is bounded the other one is unbounded.

2. The trajectories cannot go through the broken line b from the unbounded part into the bounded part.

3. If $b_1 \in \mathcal{B}$ and P is a point in the bounded part determined by b_1, then there is a broken line $b_2 \in \mathcal{B}$ containing the point P, that is $P \in b_2$.

We shall refer to the broken lines of the family $\mathcal{B}$ as *inner broken lines*. Their properties show that the trajectories starting in the positive quadrant cannot tend to the origin.

C. In this step we construct a family $\mathcal{Z}$ of simple closed broken lines. Each member of this consists of four parts: one is chosen from the family $\mathcal{K}$, the other one from the family $\mathcal{B}$, and they are supplemented with a horizontal and a vertical segment. Every closed broken line $z \in \mathcal{Z}$ has the following properties:

1. The broken line z is contained in the open positive quadrant and it divides the open positive quadrant into two connected parts, one is bounded (the *interior*), the other one is unbounded (the *exterior*).

2. The trajectories cannot go through the broken line z from the interior into the exterior.

3. If $z_1 \in \mathcal{Z}$ and P is a point in its exterior, then there is a broken line $z_2 \in \mathcal{Z}$ containing the point P, that is $P \in z_2$.

For the nonconservative case a further property holds:

4. The interior of z is a globally attracting set.

Those properties imply that the system has at least one stationary point in the positive quadrant [9].

2. Two dimensional reversible systems

A reversible chemical reaction with the intermediates X and Y and r reaction steps has the following mechanism:

$$\alpha_i X + \beta_i Y \rightleftarrows \gamma_i X + \delta_i Y, \qquad i = 1, 2 \ldots, r. \tag{1}$$

This mechanism with mass-action kinetics [2,7] yields the kinetic equations:

$$(2)\qquad \begin{aligned} \dot{X} &= \sum_{i=1}^{r}(\gamma_i - \alpha_i)\left(k_i X^{\alpha_i} Y^{\beta_i} - l_i X^{\gamma_i} Y^{\delta_i}\right), \\ \dot{Y} &= \sum_{i=1}^{r}(\delta_i - \beta_i)\left(k_i X^{\alpha_i} Y^{\beta_i} - l_i X^{\gamma_i} Y^{\delta_i}\right), \end{aligned}$$

where k_i $(i = 1, 2, \dots, r)$ are the rate constants belonging to the forward reactions in (1), l_i are the rate constants of the reverse reactions. For simplicity, X and Y denote not only the intermediates, but also their concentrations.

After introducing the notations

$$(3)\qquad J_i(X,Y) = k_i X^{\alpha_i} Y^{\beta_i} - l_i X^{\gamma_i} Y^{\delta_i},$$

$$a_i = \gamma_i - \alpha_i, \qquad b_i = \delta_i - \beta_i$$

the kinetic equations take the form:

$$(4)\qquad \begin{aligned} \dot{X} &= \sum_{i=1}^{r} a_i J_i(X,Y), \\ \dot{Y} &= \sum_{i=1}^{r} b_i J_i(X,Y). \end{aligned}$$

We can assume without loss of generality that $a_i \geq 0$, and at least one of the numbers a_i and b_i is nonzero for every $i = 1, 2, \dots, r$.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x,y) = \left(\sum_{i=1}^{r} a_i J_i(x,y),\ \sum_{i=1}^{r} b_i J_i(x,y)\right)$$

the vector field defined by equation (4).

We call system (4) *conservative* if there exists a vector $n = (\mu, \nu) \in \mathbb{R}^2$ such that for every $x > 0$, $y > 0$ the scalar product of $F(x,y)$ and n is zero, that is

$$\langle F(x,y), n\rangle = \sum_{i=1}^{r}(a_i\mu + b_i\nu) J_i(x,y) = 0.$$

(If μ and ν denote the molar mass of the components X and Y, then this relation formulates the mass conservation.)

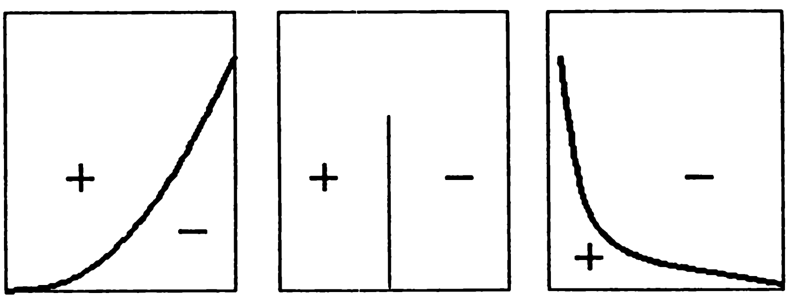
Note that the system is conservative if and only if it has a linear first integral. Indeed, for conservative system, the function $(x, y) \to \mu x + \nu y$ is a first integral.

The positive quadrant is denoted by

$$Q = \{(x, y) \in \mathbb{R}^2 : \ x > 0, \ y > 0\}.$$

The following curves will play an important role in the construction of the broken lines:

$$G_i = \{(x, y) \in Q : \ J_i(x, y) = 0\}.$$



$b_i < 0$ $\qquad$ $b_i = 0$ $\qquad$ $b_i > 0$

Figure 1a.

If $b_i \neq 0$, then the curve G_i is the graph of the function:

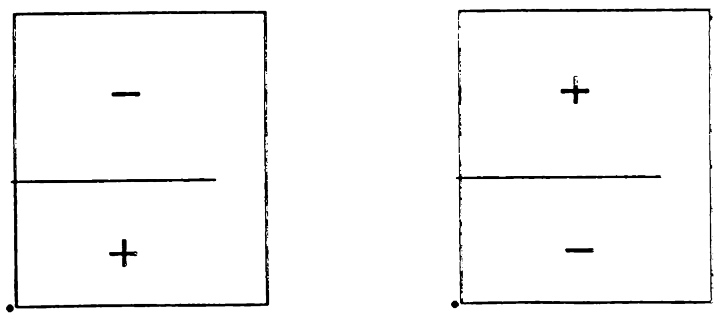
(5) $$x \mapsto x^{-a_i/b_i} c_i^{1/b_i},$$

while in the case $a_i \neq 0$ it is the graph of the function

(6) $$y \mapsto y^{-b_i/a_i} c_i^{1/a_i},$$

where $c_i = k_i/l_i$.

Fig. 1a illustrates the curves G_i in the case $a_i > 0$ for different values of b_i. For $a_i = 0$ these curves are shown in Fig. 1b. In the Figures we display the sign of the function J_i in the regions determined by the curves G_i.



$b_i < 0$ $b_i > 0$

Figure 1b.

3. The construction of the outer broken lines

Let us divide the index set $\{1, 2, ..., r\}$ into the following parts (we have r reaction steps):

$$I = \{1 \leq i \leq r : \quad a_i \neq 0, b_i > 0\}, \qquad L = \{1, 2, \ldots, r\} \setminus I.$$

Thus the set I contains the indices of those reaction steps, where in the forward reaction both X and Y is produced. We introduce further subsets of the set L:

$$L_1 = \{i \in L : \quad b_i \neq 0, \quad -a_i/b_i < 1 \text{ or } [-a_i/b_i = 1 \text{ and } c_i^{1/b_i} < 1]\},$$

$$L_2 = \{i \in L : \quad a_i \neq 0, \quad -b_i/a_i < 1 \text{ or } [-b_i/a_i = 1 \text{ and } c_i^{1/a_i} < 1]\}.$$

In the set L may be elements besides the elements of $L_1 \cup L_2$. The indices which are not contained in $L_1 \cup L_2$ are exactly those for which $-a_i/b_i = 1$ and $c_i^{1/b_i} = 1$, this condition is equivalent to the condition $-b_i/a_i = 1$ and $c_i^{1/a_i} = 1$. The graph of the functions belonging to these indices is the line $x = y$.

Let $\mathcal{F}$ be the set of the functions determined by the expression (5) belonging to the indices $i \in I \cup L_1$. Let $\mathcal{G}$ be the set of the functions determined by the expression (6) belonging to the indices $i \in I \cup L_2$. That is

$$\mathcal{F} = \left\{ f: \quad (0,\infty) \to \mathbb{R}: \quad f(x) = x^{-a_i/b_i} c_i^{1/b_i}, \quad i \in I \cup L_1 \right\},$$

$$\mathcal{G} = \left\{ g: \quad (0,\infty) \to \mathbb{R}: \quad g(y) = y^{-b_i/a_i} c_i^{1/a_i}, \quad i \in I \cup L_2 \right\}.$$

Since the sets $\mathcal{F}$ and $\mathcal{G}$ contain power functions the exponent of which is less than one, therefore there exists a positive number ρ with the following properties:

1. If $f \in \mathcal{F}$, resp. $g \in \mathcal{G}$ then $f(x) < x$, resp. $g(y) < y$ for every $x > \rho$, $y > \rho$.

2. If $f_1, f_2 \in \mathcal{F}$ and $f_1 \neq f_2$, resp. $g_1, g_2 \in \mathcal{G}$ and $g_1 \neq g_2$ then there cannot exist $x > \rho$, resp. $y > \rho$ for which $f_1(x) = f_2(x)$, resp. $g_1(y) = g_2(y)$.

Let ρ be a positive number having these properties and we consider the functions in the sets $\mathcal{F}$ and $\mathcal{G}$ defined on the interval $[\rho, \infty)$.

Note that the number of elements of the set $\mathcal{F}$ ($\mathcal{G}$) may be less than the number of elements of the set $I \cup L_1$ ($I \cup L_2$), because it can occur that some indices determine the same function.

Because of the second property above the functions in the set $\mathcal{F}$ and $\mathcal{G}$ can be ordered. Let the elements of the set $\mathcal{F}$ be (if $\mathcal{F} \neq \emptyset$) :

$$f_1 < f_2 < \ldots < f_M.$$

Let the elements of the set $\mathcal{G}$ be (if $\mathcal{G} \neq \emptyset$):

$$g_1 > g_2 > \ldots g_N.$$

In the case $\mathcal{F} \neq \emptyset$ let us introduce the functions:

$$\Phi_p(x) = \frac{f_p(x) + f_{p+1}(x)}{2}, \qquad x > \rho, \quad 1 \leq p \leq M,$$

where $f_{M+1}(x) = x$.

In the case $\mathcal{G} \neq \emptyset$ let us introduce the functions:

$$\gamma_s(y) = \frac{g_s(y) + g_{s-1}(y)}{2}, \qquad y > \rho, \quad 1 \leq s \leq N,$$

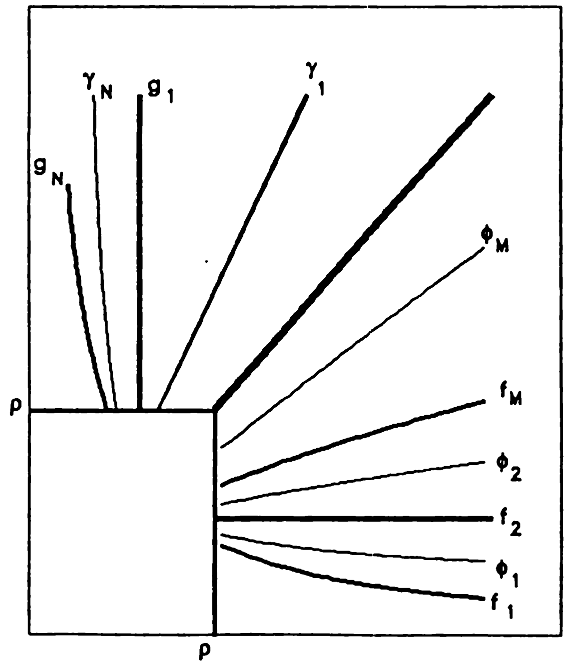


Figure 2.

where $g_0(y) = y$. It is obvious that

$$\Phi_1 < \Phi_2 < \ldots < \Phi_M \qquad \text{and} \qquad \gamma_1 > \gamma_2 > \ldots > \gamma_N .$$

Let

$$\Omega = Q \setminus (0,\rho) \times (0,\rho).$$

This region is divided into $M + N + 1$ parts by the graphs of functions Φ_p ($1 \le$ $\le p \le M$) and γ_s ($1 \le s \le N$). Let us denote these parts by $V_1, V_2, \ldots, V_{M+N+1}$ (Figure 3). Thus if $M \neq 0$, then for $1 \le p \le M$

$$V_p = \{(x,y) \in \Omega; \quad x > \rho, \quad \Phi_{p-1}(x) < y < \Phi_p(x)\},$$

where $\Phi_0(x) = 0$. If $N \neq 0$, then for $1 \le s \le N$

$$V_{M+1+s} = \{(x,y) \in \Omega : \quad y > \rho, \quad \gamma_s(y) > x > \gamma_{s+1}(y)\},$$

where $\gamma_{M+N+2}(y) = 0$. The set V_{M+1} is the complement of the union of sets V_p and V_{M+1+s} $(1 \le p \le M, 1 \le s \le N)$ defined above with respect to the set Ω.

Now we can define the outer broken lines. The endpoints are on the axis x and y and the breaking points on the graphs of the functions Φ_p and γ_s.

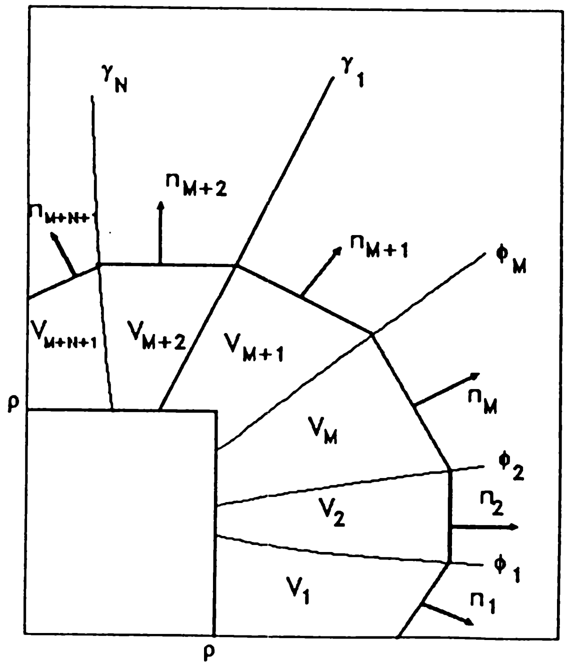


Figure 3.

We give a normal vector $n_p \in \mathbb{R}^2$ to every region V_p $(p = 1, 2, ..., M+N+1)$ in the following three cases. (This vector will be the outward (pointing to the unbounded region) normal vector of that segment of the broken line which is in the region V_p.)

1. If $M \neq 0$, then for every $1 \le p \le M$ there exists an index $i \in I \cup L_1$ such that the function $f_p \in \mathcal{F}$ belongs to the index i, that is

$$f_p(x) = x^{-a_i/b_i} c_i^{1/b_i}, \qquad \text{where} \quad i \in I \cup L_1.$$

Let

$$n_p = \operatorname{sgn} b_i (b_i, -a_i). \tag{7}$$

2. If $p = M + 1$, then let

$$n_{M+1} = (1, 1).$$

3. If $N \neq 0$, then for every $1 \leq s \leq N$ there exists an index $i \in I \cup L_2$ such that the function $g_s \in \mathcal{G}$ belongs to the index i, that is

$$g_s(y) = y^{-b_i/a_i} c_i^{1/a_i}, \qquad \text{where} \quad i \in I \cup L_2.$$

Let

$$n_{M+1+s} = (-b_i, a_i). \tag{8}$$

It is easy to verify the following: if we choose any point of the closure of the region V_p ($p = 1, 2, \ldots M + N + 1$) and draw the line with the normal vector n_p through this point, then this line has exactly two intersection points with the boundary of the region V_p. Therefore the segment of the line between the two intersection points is in the region V_p.

Lemma 1. *For every point of the set Ω there can be given a simple broken line with the following properties:*

(1) The point is in the broken line.

(2) The endpoints of the broken line are on the boundary of Ω.

(3) The breaking points are on the boundaries of the regions V_p.

(4) The normal vector of the segment of the broken line lying in V_p is n_p.

Proof. Let $P \in \Omega$ an arbitrary point. There exists a number p such that $P \in \overline{V_p}$. Let us consider the segment containing the point P with the normal vector n_p in the region V_p. The endpoints of this segment may be in the following positions:

1. on the boundary of V_p and V_{p+1};
2. on the boundary of V_p and V_{p-1};
3. on the line $x = \rho$ or $y = \rho$.

If one of the endpoints is in position 3., then this point is on the boundary of Ω, therefore this point is the endpoint of the broken line. If one of the endpoints is in position 1., then the broken line can be continued from this point in the region V_{p+1} with the normal vector n_{p+1}. If the endpoint is in position 2., then we can continue the broken line in the region V_{p-1}. We follow this procedure until the endpoints are in position 3. or in the regions V_1 or V_{M+N+1}, that is in the axes x and y. This completes the proof.

It is easy to see that among the broken lines constructed above there are broken lines with one endpoint in the axis x and the other on the axis y. Let us denote by $\mathcal{K}$ the set of these broken lines. The construction of the outer broken lines shows that they have the 1. and 3. properties listed in the introduction in step A. The only thing that remains to prove that they have the 2. property, too.

To prove this fact we show that the tangent vector of the trajectory $(F(x,y))$ and the outward normal vector of the broken line (n_p) cannot have an acute angle in any point of the broken line, that is, the scalar product of these vectors is not positive.

Let the coordinates of the vector n_p $(p = 1, 2, ..., M + N + 1)$ be (μ_p, ν_p). Thus the scalar product in question is

$$\langle F(x,y),\ n_p\rangle = \sum_{j=1}^{r}(a_j\mu_p + b_j\nu_p)J_j(x,y).$$

Lemma 2. *Let $1 \le p \le M + N + 1$, $(x,y) \in \overline{V_p}$ and $1 \le j \le r$. Then*

$$(a_j\mu_p + b_j\nu_p)J_j(x,y) \le 0.$$

Proof. We prove the lemma in three cases:
1. $1 \le p \le M$ (if $M \ne 0$),
2. $p = M + 1$,
3. $M + 2 \le p \le M + N + 1$ (if $N \ne 0$).

We show in all the three cases that the factors of the investigated product have different signs. The sign of the value $J_j(x,y)$ can be determined with the help of Figure 1.

1. In this case the coordinates of the normal vector are defined in expression (7), where i denotes the index which belongs to the function f_p. We investigate the following cases according to the value of j.

(i) $b_j = 0$. Since $p \le M$ and $(x,y) \in \overline{V_p}$ therefore $J_j(x,y) < 0$, hence using (7)

$$(a_j\mu_p + b_j\nu_p)J_j(x,y) \le 0.$$

(ii) $b_j \ne 0$ and $-a_j/b_j < -a_i/b_i$. Then we have to consider two cases according to the sign of b_j. If $b_j > 0$, then from (7)

$$a_j\mu_p + b_j\nu_p > 0,$$

and $J_j(x,y) < 0$ since $-a_j/b_j < -a_i/b_i$. If $b_j < 0$, then similarly

$$a_j\mu_p + b_j\nu_p < 0,$$

and $J_j(x,y) > 0$. Thus in both cases

$$(a_j\mu_p + b_j\nu_p)J_j(x,y) < 0.$$

(iii) $b_j \neq 0$ and $-a_j/b_j = -a_i/b_i$. Then from (7)

$$a_j\mu_p + b_j\nu_p = 0.$$

(iv) $b_j \neq 0$ and $-a_j/b_j > -a_i/b_i$. Then in the same way as in case (ii), we can distinguish two cases according to the sign of b_j and we get in both cases that $J_j(x,y)$ and $a_j\mu_p + b_j\nu_p$ have different signs, hence

$$(a_j\mu_p + b_j\nu_p)J_j(x,y) < 0.$$

Thus the first case is proved.

2. In this case $n_{M+1} = (1,1)$, therefore $a_j\mu_p + b_j\nu_p = a_j + b_j$. We distinguish the following cases according to the value of j:

(i) $b_j \geq 0$. Then $J_j(x,y) < 0$ and $a_j + b_j > 0$.

(ii) $b_j < 0$ and $-a_j/b_j < 1$. Then $J_j(x,y) > 0$ and $a_j + b_j < 0$.

(iii) $b_j < 0$ and $-a_j/b_j > 1$. Then $J_j(x,y) < 0$ and $a_j + b_j > 0$.

(iv) $b_j < 0$ and $-a_j/b_j = 1$. Then $a_j + b_j = 0$.

Thus in all the cases

$$(a_j\mu_p + b_j\nu_p)J_j(x,y) \leq 0.$$

3. In this case the normal vector is defined in (8). We have similar subcases as in case 1., therefore we do not give the proof.

Now we have proved that the positive half trajectories of system (4) are bounded, because it has been shown that for any point of the positive quadrant there exists an outer broken line $k \in \mathcal{K}$ such that the trajectory starting from the point cannot leave the bounded region determined by the broken line k.

4. The construction of the inner broken lines

Let us introduce the following index set:

$$H = \{1 \leq i \leq r : \quad a_i \neq 0, \quad b_i < 0\}.$$

We assume further that $H \neq \emptyset$. Namely, if $H = \emptyset$, then we do not need the inner broken lines, but we can go to Section 5 and construct the closed broken lines.

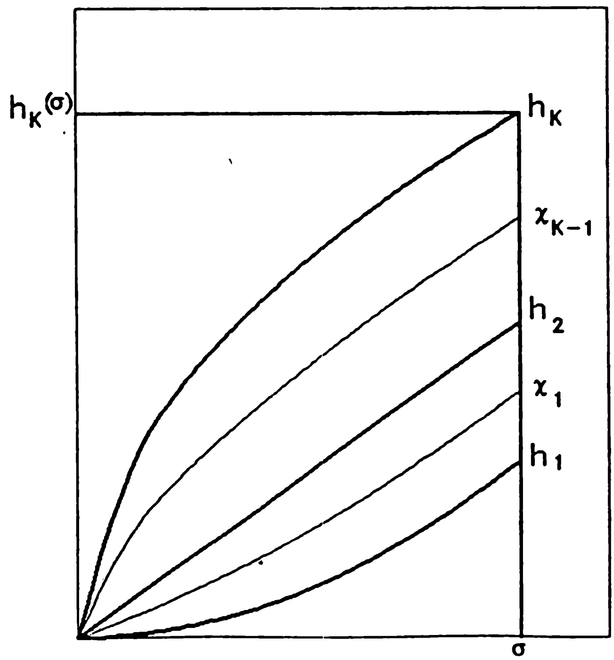


Figure 4.

Let $\mathcal{H}$ be the set of the functions determined by the expression (5) belonging to the indices $i \in H$, that is

$$\mathcal{H} = \left\{ h : [0, \infty) \to \mathbb{R} : \quad h(x) := x^{-a_i/b_i} c_i^{1/b_i}, \quad i \in H \right\}.$$

It is easy to see that there exists a positive number $\sigma < \rho$ with the following properties:

1. If $h_1, h_2 \in \mathcal{H}$, $h_1 \neq h_2$, then there is no $x \in (0, \sigma]$ such that $h_1(x) = h_2(x)$.

2. If $i \notin H$, then the graphs of the functions determined by the expressions (5) and (6) do not intersect the rectangle $[0, \sigma] \times [0, h_K(\sigma)]$, where h_K denotes the maximal element of $\mathcal{H}$.

From this time we consider the elements of $\mathcal{H}$ as functions defined in the interval $[0, \sigma]$. Because of the first property the functions in the set $\mathcal{H}$ can be ordered. Let the elements of the set $\mathcal{H}$ be

$$h_1 < h_2 < \ldots < h_K.$$

Note that the number of the elements of the set $\mathcal{H}$ may be less than the number of the elements of the set H, because it can occur that some indices determine the same function.

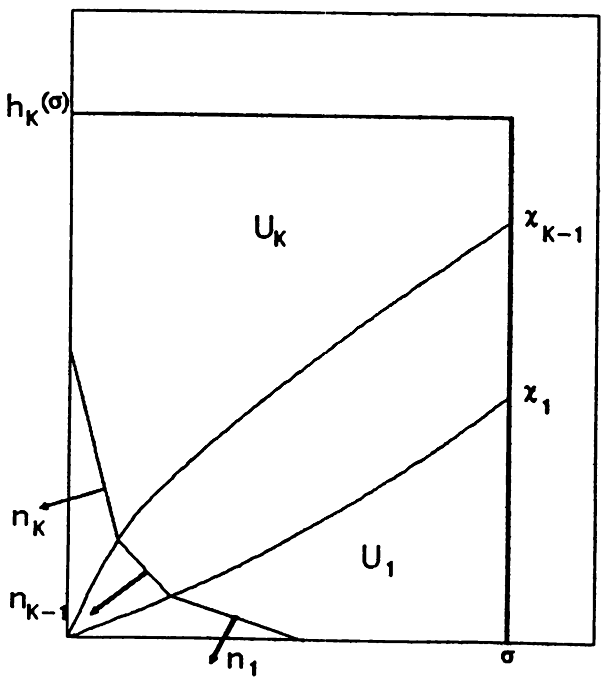


Figure 5.

Let us introduce the functions

$$\chi_p(x) = \frac{h_p(x) + h_{p+1}(x)}{2}, \qquad 0 < x < \sigma, \quad 1 \leq p \leq K - 1.$$

Let $U = [0, \sigma] \times [0, h_K(\sigma)]$. The graphs of the functions χ_p divide the rectangle U into K parts, let us denote these parts by $U_1, U_2, \ldots, U_K$ (Figure 5). That is

$$U_p = \{(x, y) \in U : \quad 0 < x < \sigma, \quad \chi_{p-1} < y < \chi_p(x)\}, \qquad 1 \leq p \leq K,$$

where $\chi_0(x) = 0$ and $\chi_K(x) = h_K(\sigma)$ for all $x \in [0, \sigma]$.

We shall give in every domain U_p $(p = 1, 2, ..., K)$ a normal vector $n_p \in \mathbb{R}^2$, this vector will be the outward normal vector of that segment of the inner broken line which lies in U_p. Let $p \in \{1, 2, ..., K\}$, let $i \in H$ be the index which belongs to the function $h_p \in \mathcal{H}$, that is

$$h_p(x) = x^{-a_i/b_i} c_i^{1/b_i}.$$

Let $n_p = (b_i, \ -a_i)$.

It is easy to verify the following: if we choose any point of the closure of the region U_p $(p = 1, 2, ..., K)$ and draw the line with the normal vector n_p through this point, then this line has exactly two intersection points with the boundary of the region U_p. Therefore the segment of the line between the two intersection points is in the region U_p.

Lemma 3. *For any point of the set U there can be given a simple broken line, with the following properties:*

(1) The point is in the broken line.

(2) The endpoints of the broken line are on the boundary of U.

(3) The breaking points are in the boundaries of the regions U_p.

(4) The normal vector of the segment of the broken line lying in U_p is n_p.

The proof is similar to that of Lemma 1.

It is easy to see that among the broken lines constructed above there are broken lines with one endpoint in the axis x and the other in the axis y. Let us denote the set of these broken lines by $\mathcal{B}$. The construction of the inner broken lines shows that they have the 1. and 3. properties listed in the introduction in step **B**. The only thing that remains to prove that they have the 2. property, too.

To prove this fact we show that the tangent vector of the trajectory $(F(x, y))$ and the outward normal vector of the broken line (n_p) cannot have an acute angle in any point of the broken line, that is the scalar product of these vectors is not positive.

Let the coordinates of the vector n_p $(p = 1, 2, ..., K)$ be (μ_p, ν_p). Thus the scalar product in question is:

$$\langle F(x, y), \ n_p \rangle = \sum_{j=1}^{r} (a_j \mu_p + b_j \nu_p) J_j(x, y).$$

Lemma 4. *Let $1 \leq p \leq K$, $(x, y) \in \overline{U_p}$, and $1 \leq j \leq r$. Then*

$$(a_j \mu_p + b_j \nu_p) J_j(x, y) \leq 0.$$

The proof is similar to that of Lemma 2.

Now we have proved that the trajectories starting in the positive quadrant cannot tend to the origin, more precisely the origin is not the ω-limit point of any trajectory in the positive quadrant.

5. The construction of the closed broken lines

In this section we distinguish two cases:

1. The set $\mathcal{H}$ defined in Section 4 is empty.
2. $\mathcal{H}$ is not empty.

1. In this case there are no inner broken lines, that is $\mathcal{B} = \emptyset$, therefore for an arbitrary outer broken line we shall give a horizontal and a vertical segment, which, together with the outer broken line, form the desired closed broken line.

We note that the case $\mathcal{F} \cup \mathcal{G} = \emptyset$ is trivial, namely, in this case $I = L_1 = L_2 = \emptyset$, therefore $a_i = -b_i$ $(1 \leq i \leq r)$ thus equation (4) reduces to $\dot{X} = -\dot{Y}$. Therefore we can assume that there exist outer broken lines, that is $\mathcal{K}$ is not empty.

Let $k \in \mathcal{K}$ be an outer broken line. Let x_0 be a positive number which is less than the first coordinate of the breaking points of k in the positive quadrant (Figure 6). Then the vertical line $x = x_0$ intersects that segment of the broken line k the left endpoint of which is in the axis y. Let us denote the second coordinate of this intersection point by y_1. Let y_0 be a positive number which is less than the second coordinate of the breaking points of k in the positive quadrant. Then the horizontal line $y = y_0$ intersects that segment of the broken line k the lower endpoint of which is in the axis x. Let us denote the first coordinate of this intersection point by x_1. Let us denote the segment between the points (x_0, y_0) and (x_0, y_1) by $\mathcal{A}$, and the segment between the points (x_0, y_0) and (x_1, y_0) by $\mathcal{C}$.

The part of the broken line k between the points (x_0, y_1) and (x_1, y_0) with the segments $\mathcal{A}$ and $\mathcal{C}$ form the desired closed broken line. The outward normal vectors belonging to the segments $\mathcal{A}$ and $\mathcal{C}$ are $n_A = (-1, 0)$ and $n_C = (0, -1)$. Let $(x, y) \in \mathcal{A}$. If $j \in \{1, 2, ..., r\}$ is an index such that $a_j \neq 0$, then $J_j(x, y) > 0$ because of the definition of $\mathcal{A}$. Therefore

$$\langle F(x,y),\ n_A \rangle = \sum_{j=1}^{r} -a_j J_j(x,y) \leq 0,$$

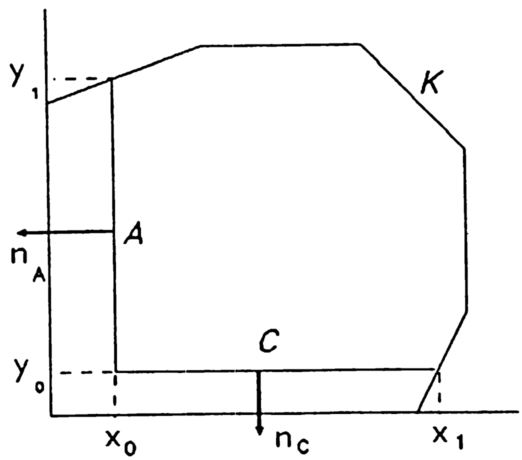


Figure 6.

which means that the trajectories cannot go to the left through the segment $\mathcal{A}$. Similarly in case $(x,y) \in \mathcal{C}$, if $j \in \{1,2,...,r\}$ is an index such that $b_j \neq 0$, then $b_j J_j(x,y) > 0$. Therefore

$$\langle F(x,y),\ n_C \rangle = \sum_{j=1}^{r} -b_j J_j(x,y) \leq 0,$$

which means that the trajectories cannot go down through the segment $\mathcal{C}$.

2. Now let us assume that $\mathcal{H} \neq \emptyset$. Let $k \in \mathcal{K}$ be an outer and $b \in \mathcal{B}$ be an inner broken line. Let x_0 be a positive number which is less than the first coordinate of the breaking points of k and b in the positive quadrant (Figure 7). Then the vertical line $x = x_0$ intersects those segments of the broken lines k and b the left endpoints of which are in the axis y. Let us denote the second coordinates of the intersection points by y_K and y_B. Let y_0 be a positive number which is less than the second coordinate of the breaking points of k and b in the positive quadrant. Then the horizontal line $y = y_0$ intersects those segments of the broken lines k and b the lower endpoints of which are in the axis x. Let us denote the first coordinate of these intersection points by x_K and x_B. Let us denote the segment between the points (x_0, y_B) and (x_0, y_K) by $\mathcal{D}$, and the segment between the points (x_B, y_0) and (x_K, y_0) by $\mathcal{E}$. The part of the broken line k between the points (x_0, y_K) and (x_K, y_0), and the part of the broken line b between the points (x_0, y_B) and (x_B, y_0) with the segments $\mathcal{D}$ and

$\mathcal{E}$ form the desired closed broken line. The outward normal vectors belonging to the segments $\mathcal{D}$ and $\mathcal{E}$ are $n_D = (-1, 0)$ and $n_E = (0, -1)$.

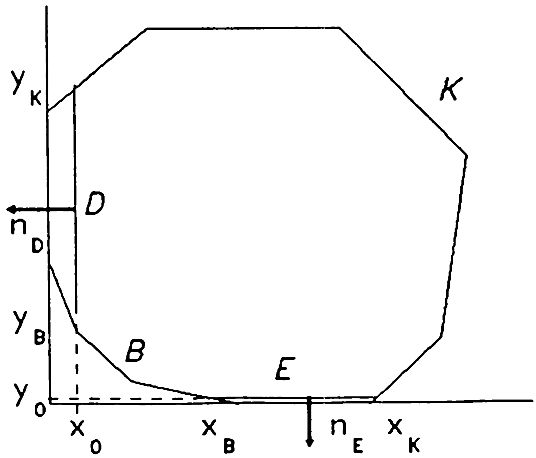


Figure 7.

Let $(x, y) \in \mathcal{D}$. If $j \in \{1, 2, ..., r\}$ is an index such that $a_j \neq 0$, then $J_j(x, y) > 0$ because of the definition of $\mathcal{D}$. Therefore

$$\langle F(x, y),\ n_D \rangle = \sum_{j=1}^{r} -a_j J_j(x, y) \leq 0,$$

which means that the trajectories cannot go to the left through the segment $\mathcal{D}$. Similarly, in the case $(x, y) \in \mathcal{E}$, if $j \in \{1, 2, ..., r\}$ is an index such that $b_j \neq 0$, then $b_j J_j(x, y) > 0$. Therefore

$$\langle F(x, y),\ n_E \rangle = \sum_{j=1}^{r} -b_j J_j(x, y) \leq 0,$$

which means that the trajectories cannot go down through the segment $\mathcal{E}$.

Let us denote by $\mathcal{Z}$ the set of all closed broken lines constructed above. We have proved about them the following

Lemma 5. *1. Every closed broken line $z \in \mathcal{Z}$ is contained in the open positive quadrant and it divides the open positive quadrant into two connected parts, one is bounded (the interior), the other one is unbounded (the exterior).*

2. The trajectories cannot go through any broken line $z \in \mathcal{Z}$ from the interior into the exterior.

3. If $z_1 \in \mathcal{Z}$ and P is a point in its exterior, then there is a broken line $z_2 \in \mathcal{Z}$ containing the point P, that is $P \in z_2$.

Theorem. *(1) Every positive half trajectory of system (4) starting in the positive quadrant is contained in a closed bounded domain which is in the positive quadrant.*

(2) Moreover, if system (4) is nonconservative, then there exist closed globally attracting domains in the positive quadrant.

Proof. (1) Let $P \in Q$. We shall distinguish three cases according to the position of the point P.

1. There exists an outer broken line containing the point P.
2. There exists an inner broken line containing the point P.
3. The point P is not contained in any outer and inner broken line.

We prove the statement of the theorem in these cases separately.

1. Let $k \in \mathcal{K}$ be an outer broken line containing the point P. Then using the method outlined in this section (both in the cases $\mathcal{H} = \emptyset$ and $\mathcal{H} \neq \emptyset$) we get a simple closed broken line $z \in \mathcal{Z}$ containing the point P. (In the construction the inner broken line (if there exists) is arbitrary.) The trajectories cannot leave the interior of the closed broken line z. Thus the trajectory starting from the point P cannot leave this bounded domain.

2. This case is similar to case 1, therefore we do not detail the proof.

3. Since there is no outer broken line containing the point P, if we choose an arbitrary outer broken line $k \in \mathcal{K}$, then P is in the bounded part determined by k (because of the 3. property of the outer broken lines). Similarly, if there are inner broken lines, that is $\mathcal{H} \neq \emptyset$, then for an arbitrary inner broken line $b \in \mathcal{B}$ the point P is in the unbounded domain determined by b. Hence using the method outlined in the beginning of this section we get a horizontal and a vertical segment such that the point P is in the inner part of the closed broken line formed by the broken lines k and b and the horizontal and the vertical segment. Therefore the trajectory starting from the point P cannot leave this bounded domain.

This completes the proof of the first part, now let us consider the second one.

Let $z \in \mathcal{Z}$ a closed broken line and let us denote by $n(x, y)$ the outward normal vector of the broken line in the point (x, y). The 2. statement of Lemma 5 means that

$$\langle F(x,y),\ n(x,y)\rangle \leq 0$$

in every point $(x,y) \in z$. Now we shall prove that if the system (4) is nonconservative, then this scalar product is negative, that is

$$(9) \qquad \langle F(x,y),\ n\rangle < 0.$$

Hence the statement of the theorem will be proved, namely, following the train of thought used in the Liapunov function technique let us fix any closed broken line $z \in \mathcal{Z}$. In the exterior of z there is no stationary point and no periodic orbit because of (9), but the positive half trajectories are bounded, therefore their ω-limit sets are in the interior of the broken line z. Hence they reach the interior in finite time [1,5,8,10]. Thus the interior of the closed broken line z is a bounded globally attracting set.

Now let us prove inequality (9). Let the normal vector $n = (\mu, \nu)$. Then

$$\langle F(x,y),\ n\rangle = \sum_{j=1}^{r}(a_j\mu + b_j\nu)J_j(x,y).$$

In Lemma 2 and Lemma 4 we showed that every term of the sum is non positive, therefore it is enough to prove that there is at least one negative term.

Since the system is nonconservative, therefore for the normal vector n there is an index $j \in \{1,2,...,r\}$ such that

$$a_j\mu + b_j\nu \neq 0.$$

We shall prove the following: if (x,y) is a point of that segment of the broken line the normal vector of which is n, then $J_j(x,y) \neq 0$, that is the j-th term of the sum is negative. In order to prove this, we show that if in the point (x,y) of the broken line $J_j(x,y) = 0$, then $a_j\mu + b_j\nu = 0$, where $n = (\mu,\nu)$ is the normal vector of the broken line in the point (x,y).

We have three cases.

1. The point (x,y) is in one of the inner broken lines. Then this point is in the graph of the function

$$h_p(x) = x^{-a_j/b_j}c_j^{1/b_j}$$

belonging to the index j and $n = n_p = (b_j, -a_j)$. Thus $a_j\mu + b_j\nu = 0$.

2. The point (x,y) is in one of the outer broken lines. Then this point is in the graph of the function $f_p \in \mathcal{F}$ or $g_S \in \mathcal{G}$ belonging to the index j, and the normal vector is determined by the formula (7) or (8), therefore $a_j\mu + b_j\nu = 0$.

3. The point (x,y) is in the vertical or horizontal segment of the broken line. If it is in the vertical one, then $a_j = 0$, because $J_j(x,y) = 0$. The

normal vector is $n = (-1, 0)$, hence $a_j \mu + b_j \nu = 0$. Similarly if the point is in the horizontal segment, then $b_j = 0$, the normal vector is $n = (0, -1)$, hence $a_j \mu + b_j \nu = 0$.

This completes the proof of the theorem.

Corollary. *The system (4) has at least one stationary point in the positive quadrant.*

Proof. If the system is conservative then there are infinitely many stationary points. If the system is nonconservative, then there is a bounded globally attracting domain, hence in this domain there is at least one stationary point because of the Poincaré-Bendixson theorem [1,5,8,10].

The statement of this corollary is already known in a more general case for reversible chemical systems [9].

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