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EXISTENCE AND UNIQUENESS THEOREMS FOR A CLASS OF NONLINEAR VOLTERRA INTEGRO–DIFFERENTIAL EQUATIONS

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Abstract. In this paper existence and uniqueness theorems for first order and m-th order Nonlinear Volterra Integro-differential Equations (abbreviated NVIDE) are presented in a Banach space using the contraction mapping principle. Note that these theorems are also valid for the linear VIDE.

Introduction

Applying a numerical method to a VIDE, linear or nonlinear, one usually assumes that the problem has solution, thus the introduced here existence and uniqueness theorems can be applied directly to the given NVIDE to see whether it has unique solution or not.

At first we prove an existence and uniqueness theorem for NVIDE

$$x'(t) = f(t, x(t)) + \int_a^t K(t, \tau, x(\tau))d\tau,$$

having the initial condition $x(a) =: c$, it has a unique solution in a Banach space equipped with a generalized Bielecki's type norm $\|x\| :=\max\limits_t e^{-r(t)}|x(t)|$, $t \in [a, b]$ and $r(t)$ is an auxiliary function (explained in the theorem).

In Theorem 2 we discuss a generalization for the next form of NVIDE of order m

$$x^{(m)}(t) = f\left(t,\ \{x^{(j)}(t)\}_{j=0}^{m-1}, IKX\right)$$

having the i.c. $x^{(j)}(a) =: c_j$ for $j = 0, 1, \ldots, m-1$; such that $IKX := := \int_a^t K\left(t, \tau, \{x^{(j)}(\tau)\}_{j=0}^{m-1}\right)$, where

$$\{x^{(j)}(t)\}_{j=0}^{m-1} = x(t), x'(t), x''(t), \ldots, x^{(m-1)}(t)$$

and $x^{(0)}(t) := x(t)$, IKX means the operator $I := \int_a^t (\ )d(\ )$ applied to the kernel $KX := K\left(t, \tau, \{x^{(j)}(t)\}_{j=0}^{m-1}\right)$.

Definition 1. A continuous function $f(t, X)$ defined in the $n+1$-dimensional region $D:\ a \le t \le b,\ |x_i| < \infty,\ i = 1, 2, \ldots, n$, satisfies a Lipschitz condition on X in D for a Lipschitz coefficient $\ell > 0$ if

$$|f(t, X) - f(t, Y)| \le \ell |X - Y| := \ell \sum_{i=1}^{n} |x_i - y_i|$$

for every (t, X) and (t, Y) in D.

1. First order NVIDE in additive form

Theorem 1. *The first order nonlinear Volterra integro-differential equation*

$$x'(t) = f(t, x(t)) + \int_a^t K(t, \tau, x(\tau))d\tau \tag{1.1}$$

with initial condition $x(a) =: c$ and satisfying

1. $K(t, \tau, u)$ is continuous and satisfies $|K(t, \tau, u)| \le M_1$ for every (t, τ, u) in

$$B1:\quad a \le t \le b, \quad a \le \tau \le b, \quad |u - c| \le T < \infty, \tag{1.2}$$

moreover K satisfies the Lipschitz condition

$$|K(t, \tau, u_1) - K(t, \tau, u_2)| \le \ell_1 |u_1 - u_2| \tag{1.3}$$

for every (t, τ, u_1) and (t, τ, u_2) in $B1$;

2. $f(t,u)$ is continuous in

$$(1.4)\qquad B2: \quad a \le t \le b, \quad |u-c| \le T < \infty$$

and satisfies $|f(t,u)| \le M_2$ for every $t,u)$ in it, as well as f is Lipschitzian in $B2$, i.e.

$$(1.5)\qquad |f(t,u_1) - f(t,u_2)| \le \ell_2 |u_1 - u_2|$$

for every $t,u_1)$ and (t,u_2) in $B2$;

has a unique solution in $E \subset D$ (the Banach space of all functions $x(t) \in \in C'[a,b]$ equipped with the norm

$$(1.6)\qquad \|x\| := \max_t e^{-r(t)} |x(t)|, \quad for \quad t \in [a,b],$$

where $r(t) = \nu \mathcal{L}(t-a)$, $\mathcal{L} := \max(\ell_1, \ell_2, 1)$, and the finite number $\nu \ge 2$) such that

$$(1.7)\qquad E := \{x(t) \in C'(\Delta): \ |x(t)-c| \le T, \quad for \ \ \Delta := |t-a| \le \delta\}$$

with $\delta = \min((b-a), T/M)$, where $M := M_3\left(1 + \frac{b-a}{2}\right)$ and $M_3 = = \max(M_1, M_2)$.

Proof. Integrate both sides of (1.1) from a to t to show that it is equivalent to the integral equation

$$(1.8)\qquad x(t) = c + \int_a^t f(\tau, x(\tau))d\tau + \int_a^t \int_a^\tau K(\tau, \mu, x(\mu))d\mu d\tau.$$

In order to have a fixed point problem choose the r.h.s. of (1.8) to be $Q(x)t$ (a nonlinear operator), then consider the following difference

$$|Q(x)t - c| \le \int_a^t |f(\tau, x(\tau))|d\tau + \int_a^t \int_a^\tau |K(\tau, \mu, x(\mu))|d\mu d\tau \le$$

$$(1.9)\qquad \le M_2(t-a) + M_1 \frac{(t-a)^2}{2} \le M_3(t-a)\left(1 + \frac{t-a}{2}\right) \le$$

$$M_3(t-a)\left(1+\frac{b-a}{2}\right) \le M\delta \le T,$$

which means that for every $(t, \tau, x(\tau))$ in $B1$ and for every $(t, x(t))$ in $B2$ with $a \le \tau \le t \le a+\delta$ the operator $Q: \ E \to E$.

Next, to show that Q is conraction, consider the difference

$$|Q(x) - Q(y)|(t) \le \tag{1.10}$$

$$\le \int_a^t |f(\tau, x(\tau)) - f(\tau, y(\tau))| d\tau + \int_a^t \int_a^\tau |K(\tau, \mu, x(\mu)) - K(\tau, \mu, y(\mu))| d\mu d\tau.$$

Make use of the Lipschitz conditions (1.3) and (1.5) in (1.10) as well as $\mathcal{L} :=$ $:= \max(\ell_1, \ell_2, 1)$ to get the following inequality

$$|Q(x) - Q(y)|(t) \le \mathcal{L} \int_a^t |x(\tau) - y(\tau)| d\tau + \mathcal{L} \int_a^t \int_a^\tau |x(\mu) - y(\mu)| d\mu d\tau. \tag{1.11}$$

Multiply the r.h.s. of (1.11) by $e^{-\nu\mathcal{L}(t-a)}e^{\nu\mathcal{L}(t-a)}$ to get

$$|Q(x) - Q(y)|(t) \le \mathcal{L} \int_a^t |x(\tau) - y(\tau)| e^{-\nu\mathcal{L}(\tau-a)} e^{\nu\mathcal{L}(\tau-a)} d\tau +$$

$$+\mathcal{L} \int_a^t \int_a^\tau |x(\mu) - y(\mu)| e^{-\nu\mathcal{L}(\mu-a)} e^{\nu\mathcal{L}(\mu-a)} d\mu d\tau. \tag{1.12}$$

Now take the maximum in the terms of r.h.s. as follows

$$|Q(x) - Q(y)|(t) \le$$

$$\le \mathcal{L} \int_a^t \left(\max_\tau |x(\tau) - y(\tau)| e^{-\nu\mathcal{L}(\tau-a)}\right) e^{\nu\mathcal{L}(\tau-a)} d\tau + \tag{1.13}$$

$$+\mathcal{L} \int_a^t \int_a^\tau \left(\max_\mu |x(\mu) - y(\mu)| e^{-\nu\mathcal{L}(\mu-a)}\right) e^{\nu\mathcal{L}(\mu-a)} d\mu d\tau.$$

According to (1.6), (1.13) is equivalent to

$$(1.14)\qquad |Q(x)-Q(y)|(t) \le \mathcal{L}\|x-y\| \int_a^t \left(e^{\nu\mathcal{L}(\tau-a)} + \int_a^\tau e^{\nu\mathcal{L}(\mu-a)} d\mu \right) d\tau.$$

(1.14) implies that

$$(1.15)\qquad |Q(x)-Q(y)|(t) \le \|x-y\| \left(\frac{1+\nu}{\nu^2}(e^{\nu\mathcal{L}(t-a)}-1) - \frac{t-a}{\nu} \right),$$

where $\frac{1}{\nu\mathcal{L}} \le \frac{1}{\nu}$, since $\mathcal{L} \ge 1$ and $\nu \ge 2$ is used.

Multiply both sides of (1.15) by $e^{-\nu\mathcal{L}(t-a)}$ to have

$$e^{-\nu\mathcal{L}(t-a)}|Q(x)-Q(y)|(t) \le$$

$$(1.16)\qquad \begin{aligned} &\le \|x-y\| \left(\frac{1+\nu}{\nu^2}(1-e^{-\nu\mathcal{L}(t-a)}) - \frac{t-a}{\nu} e^{-\nu\mathcal{L}(t-a)} \right) \le \\ &\le \|x-y\| \left(\frac{1+\nu}{\nu^2}(1-\min_t e^{-\nu\mathcal{L}(t-a)}) - \min_t \frac{t-a}{\nu} e^{-\nu\mathcal{L}(t-a)} \right) \le \\ &\le \frac{1+\nu}{\nu^2}\left(1-e^{-\nu\mathcal{L}(b-a)}\right)\|x-y\|. \end{aligned}$$

The most r.h.s. of (1.16) is now constant, i.e. independent of t, thus (1.16) is valid for every $t \in [a,b]$; whence for the maximum of its l.h.s.

$$(1.17)\qquad \max_t e^{-\nu\mathcal{L}(t-a)}|Q(x)-Q(y)|(t) \le \frac{1+\nu}{\nu^2}\left(1-e^{-\nu\mathcal{L}(b-a)}\right)\|x-y\|$$

which, according to (1.6), gives

$$(1.18)\qquad \|Q(x)-Q(y)\| \le \frac{1+\nu}{\nu^2}\left(1-e^{-\nu\mathcal{L}(b-a)}\right)\|x-y\|.$$

It is clear that $0 < \frac{1+\nu}{\nu^2}\left(1-e^{-\nu\mathcal{L}(b-a)}\right) < 1$ for any finite $\nu \ge 2$ and $\mathcal{L} \ge 1$, it can be considered as the contraction coefficient of $Q(x)t$. Hence Q is a contraction operator and Banach's fixed point theorem is applicable to guarantee the existence of a unique solution for the problem (1.1) in E.

2. m-th order NVIDE in general form

Theorem 2. *Let us consider the m-th order NVIDE*

$$x^{(m)}(t) = f\left(t, \{x^{(j)}(t)\}_{j=0}^{m-1}, IKX\right), \tag{2.1}$$

having initial conditions $x^{(j)}(a) =: c_j$ *for* $j = 0, 1, \ldots, m-1$; *where* $x^{(0)}(a) = x(a) =: c_0$ *and*

$$IKX := \int_a^t K\left(t, \tau, \{x^{(j)}(\tau)\}_{j=0}^{m-1}\right). \tag{2.2}$$

Let the following conditions be posed:

1. the kernel $K\left(t, \tau, \{u_j\}_{j=0}^{m-1}\right)$ *of (2.2) is continuous for every* t, τ *and* u_j, $j = 0, 1, 2, \ldots, m-1$ *in B1 given by*

$$B1: \quad a \le t \le b, \qquad a \le \tau \le b, \qquad |u_j - c_j| \le T_j < \infty \tag{2.3}$$

for $j = 0, 1, 2, \ldots, m-1$; *as well as* K *is bounded there by M1, i.e.*

$$|KX| \le \left|K\left(t, \tau, \{u_j\}_{j=0}^{m-1}\right)\right| \le M1. \tag{2.4}$$

2. K satisfies the following Lipschitz condition on w_j, $j = 0, 1, 2, \ldots, m-1$ *in B1 for a Lipschitz coefficient* $\ell_1 > 0$

$$\left|K\left(t, \tau, \{w_j\}_{j=0}^{m-1}\right) - K\left(t, \tau, \{v_j\}_{j=0}^{m-1}\right)\right| \le \ell_1 \sum_{j=0}^{m-1} |w_j - v_j| \tag{2.5}$$

for every $\left(t, \tau, \{w_j\}_{j=0}^{m-1}\right)$ *and* $\left(t, \tau, \{v_j\}_{j=0}^{m-1}\right)$ *in B1.*

3. $f\left(t, \{u_j\}_{j=0}^{m}\right)$ *is continuous and bounded by M2, i.e.* $\left|f\left(t, \{u_j\}_{j=0}^{m}\right)\right| \le M2$ *in B2 given by*

$$B2: \quad a \le t \le b, \qquad |u_j - c_j| \le T_j < \infty, \qquad |u_m| \le T_m < \infty \tag{2.6}$$

for $j = 0, 1, 2, \ldots, m-1$ *and is Lipschitzian there for an* $\ell_2 > 0$, *i.e.*

$$\begin{aligned} &\left|f\left(t, \{w_j\}_{j=0}^{m-1}, IZ1\right) - f\left(t, \{v_j\}_{j=0}^{m-1}, IZ2\right)\right| \le \\ &\le \ell_2 \left(\sum_{j=0}^{m-1} |w_j - v_j| + \ell_1 I\left(\sum_{j=0}^{m-1} |w_j - v_j|\right)\right) \end{aligned} \tag{2.7}$$

for every $\left(t,\{w_j\}_{j=0}^{m-1}, IZ1\right)$ *and* $\left(t,\{v_j\}_{j=0}^{m-1}, IZ2\right)$ *in B2 given by (2.6); in (2.7) we used the inequality*

$$|IZ1 - IZ2| \le I|Z1 - Z2| \le \ell_1 I \left(\sum_{j=0}^{m-1} |w_j - v_j|\right), \tag{2.8}$$

where I *is as defined before,*

$$Z1 := K\left(t,\tau,\{w_j\}_{j=0}^{m-1}\right) \quad and \quad Z2 := K\left(t,\tau,\{v_j\}_{j=0}^{m-1}\right).$$

If the NVIDE (2.1) satisfies the abovementioned conditions, then it has a unique solution $x(t) \in C^{(m)}(\Delta)$ *in* $E \subset D$, *where*

$$E := \left\{x(t) \in C^{(m)}(\Delta): \quad |x^{(j)}(t) - c_j| \le T\right\} \tag{2.9}$$

for $j = 0, 1, 2, \ldots, m-1$ *and* $\Delta := |t - a| \le \delta$, *where*
(2.10)

$$\delta := \min\left((b-a), \frac{T}{M}\right), \quad T =: \min(T_0, T_1, \ldots, T_m),$$

$$M := M3 \sum_{j=1}^{m}\left(\frac{(b-a)^{j-1}}{j!}\right), \quad M3 = \max(|c_j|, M_1, M_2), \quad j = 1, \ldots, m-1.$$

D is the Banach space of all functions $x(t) \in C^{(m)}[a,b]$ *equipped with the norm*

$$||x|| := \max_t e^{-r(t)} \sum_{j=0}^{m-1} |x^{(j)}(t)|, \qquad t \in [a,b], \tag{2.11}$$

$r(t) = \nu\mathcal{L}(t-a)$ *for a finite* $\nu \ge 2$ *and* $\mathcal{L} = \max(\ell_1, \ell_2, 1)$, *noting that* $x^{(0)}(t) = = x(t)$.

Proof. Integrating both sides of (2.1) from a to t m times one can show that it is equivalent to the integral equation

$$x(t) = \sum_{j=0}^{m-1} \frac{(t-a)^j}{j!} + \tag{2.12}$$

$$+ \int_a^t \int_a^{\mu_1} \int_a^{\mu_2} \cdots \int_a^{\mu_{m-1}} f\left(\mu_m, \{x^{(j)}(\mu_m)\}_{j=0}^{m-1}, \ IKX\right) d\mu_m d\mu_{m-1} \ldots d\mu_1.$$

To form a fixed point problem denote the r.h.s. of (2.12) by $Q(x)t$, a nonlinear operator. First we have to show that $Q : E \to E$ (E is the space given by (2.9)).

For any $\left(\mu_m, \{x^{(j)}(\mu_m)\}_{j=0}^{m-1},\ IKX\right)$ in $B2$ given by (2.6) for any $a \le t \le$ $\le a+\delta$ it is clear that

$$|Q(x)t - c_0| \le \sum_{j=1}^{m-1} |c_j| \frac{(t-a)^j}{j!} + \tag{2.13}$$

$$+ \int_a^t \int_a^{\mu_1} \int_a^{\mu_2} \cdots \int_a^{\mu_{m-1}} \left| f\left(\mu_m, \{x^{(j)}(\mu_m)\}_{j=0}^{m-1},\ IKX\right) \right| d\mu_m d\mu_{m-1} \ldots d\mu_1 \le$$

$$\le \sum_{j=1}^{m-1} |c_j| \frac{(t-a)^j}{j!} + M2 \frac{(t-a)^m}{m!} \le M3(t-a) \sum_{j=1}^{m} \frac{(t-a)^{j-1}}{j!} \le$$

$$\le M3(t-a) \sum_{j=1}^{m} \frac{(b-a)^{j-1}}{j!} \le M\delta \le T.$$

(2.2) and (2.4) for any $a \le t \le a+\delta$ imply

$$|IKX| \le \int_a^t \left| K\left(t, \tau, \{x^{(j)}(\tau)\}_{j=0}^{m-1}\right) \right| d\tau \le M1(t-a) \le M\delta \le T, \tag{2.14}$$

thus from (2.13) and (2.14) we conclude that $Q : E \to E$.

Next (to prove that Q is contractive) consider the difference

$$|Q(x) - Q(y)|(t) \le \tag{2.15}$$

$$\le \int_a^t \int_a^{\mu_1} \int_a^{\mu_2} \cdots \int_a^{\mu_{m-1}} \Big| f\left(\mu_m, \{x^{(j)}(\mu_m)\}_{j=0}^{m-1},\ IKX\right) -$$

$$- f\left(\mu_m, \{y^{(j)}(\mu_m)\}_{j=0}^{m-1},\ IKY\right) \Big| d\mu_m d\mu_{m-1} \ldots d\mu_1.$$

According to the Lipschitz condition (2.7) and (2.8) the above inequality becomes

$$|Q(x) - Q(y)|(t) \le$$

$$\leq \ell_2 \int_a^t \int_a^{\mu_1} \int_a^{\mu_2} \cdots \int_a^{\mu_{m-1}} \left(\sum_{j=0}^{m-1} |x^{(j)}(\mu_m) - y^{(j)}(\mu_m)| + \right. \tag{2.16}$$

$$\left. +\ell_1 I \left(\sum_{j=0}^{m-1} |x^{(j)}(\mu_m) - y^{(j)}(\mu_m)| \right) \right) d\mu_m d\mu_{m-1} \ldots d\mu_1.$$

Multiplying each term of the r.h.s. of (2.16) by the the product $e^{-\nu\mathcal{L}(t-a)} \times$ $\times e^{\nu\mathcal{L}(t-a)}$ we obtain

$$|Q(x) - Q(y)|(t) \leq \tag{2.17}$$

$$\leq \mathcal{L} \int_a^t \int_a^{\mu_1} \int_a^{\mu_2} \cdots \int_a^{\mu_{m-1}} \left(\sum_{j=0}^{m-1} |x^{(j)}(s) - y^{(j)}(s)| e^{-\nu\mathcal{L}(s-a)} e^{\nu\mathcal{L}(s-a)} + \right.$$

$$\left. +\mathcal{L} I \left(\sum_{j=0}^{m-1} |x^{(j)}(s) - y^{(j)}(s)| e^{-\nu\mathcal{L}(s-a)} e^{\nu\mathcal{L}(s-a)} \right) \right) ds d\mu_{m-1} \ldots d\mu_1.$$

Now taking the maximum at the r.h.s. of (2.17)

$$|Q(x) - Q(y)|(t) \leq \tag{2.18}$$

$$\leq \mathcal{L} \int_a^t \int_a^{\mu_1} \int_a^{\mu_2} \cdots \int_a^{\mu_{m-1}} \left\{ \left[\max_s \sum_{j=0}^{m-1} |x^{(j)}(s) - y^{(j)}(s)| e^{-\nu\mathcal{L}(s-a)} \right] e^{\nu\mathcal{L}(s-a)} + \right.$$

$$\left. +\mathcal{L} I \left(\left[\max_s \sum_{j=0}^{m-1} |x^{(j)}(s) - y^{(j)}(s)| e^{-\nu\mathcal{L}(s-a)} \right] e^{\nu\mathcal{L}(s-a)} \right) \right\} ds d\mu_{m-1} \ldots d\mu_1,$$

which, according to the definition of norm (2.11), leads to

$$|Q(x) - Q(y)|(t) \leq \tag{2.19}$$

$$\leq \mathcal{L} \|x - y\| \left(\int_a^t \int_a^{\mu_1} \int_a^{\mu_2} \cdots \int_a^{\mu_{m-1}} (e^{\nu\mathcal{L}(s-a)} + \mathcal{L} I e^{\nu\mathcal{L}(s-a)}) ds d\mu_{m-1} \ldots d\mu_1 \right),$$

noting that the term $\mathcal{L} I e^{\nu\mathcal{L}(\tau-a)}$ is performed as follows

$$\mathcal{L} I e^{\nu\mathcal{L}(\tau-a)} = \mathcal{L} \int_a^s e^{\nu\mathcal{L}(\tau-a)} d\tau = \frac{1}{\nu}(e^{\nu\mathcal{L}(s-a)} - 1). \tag{2.20}$$

Substituting this in (2.19) and performing the integration we come to

$$|Q(x)-Q(y)|(t) \leq \tag{2.21}$$

$$\leq \mathcal{L}\|x-y\| \left(\left(\frac{1}{(\nu\mathcal{L})^m} + \frac{1}{\nu^{m+1}\mathcal{L}^m} \right) \left(e^{\nu\mathcal{L}(t-a)} - 1 \right) - \right.$$

$$\left. - \sum_{i=1}^{m-1} \frac{(t-a)^i}{i!(\nu\mathcal{L})^{m-1}} - \sum_{i=1}^{m-1} \frac{(t-a)^i}{i!\nu^{m+1-i}\mathcal{L}^{m-1}} - \frac{(t-a)^m}{m!\nu} \right).$$

Multiplying both sides of (2.21) by $e^{-\nu\mathcal{L}(t-a)}$ and since (for $\mathcal{L} \geq 1$ and $\nu \geq 2$)

$$\frac{\mathcal{L}}{(\nu\mathcal{L})^m} \leq \frac{1}{\nu^m}, \qquad \frac{\mathcal{L}}{\nu^{m+1}\mathcal{L}^m} \leq \frac{1}{\nu^{m+1}}$$

(2.21) becomes

$$e^{-\nu\mathcal{L}(t-a)}|Q(x)-Q(y)|(t) \leq \tag{2.22}$$

$$\leq \|x-y\| \left(\left(\frac{1}{\nu^m} + \frac{1}{\nu^{m+1}} \right) \left(1 - e^{-\nu\mathcal{L}(t-a)} \right) - \right.$$

$$\left. - \sum_{i=1}^{m-1} \frac{(t-a)^i e^{-\nu\mathcal{L}(t-a)}}{i!\nu^{m-i}\mathcal{L}^{m-1-i}} - \sum_{i=1}^{m-1} \frac{(t-a)^i e^{-\nu\mathcal{L}(t-a)}}{i!\nu^{m+1-i}\mathcal{L}^{m-1-i}} - \mathcal{L}\frac{(t-a)^m}{m!\nu} e^{-\nu\mathcal{L}(t-a)} \right)$$

for $m \geq 2$, it can be majored as

$$e^{-\nu\mathcal{L}(t-a)}|Q(x)-Q(y)|(t) \leq \tag{2.23}$$

$$\leq \|x-y\| \left(\left(\frac{1}{\nu^m} + \frac{1}{\nu^{m+1}} \right) \left(1 - \min_t e^{-\nu\mathcal{L}(t-a)} \right) - \right.$$

$$- \min_t \sum_{i=1}^{m-1} \frac{(t-a)^i e^{-\nu\mathcal{L}(t-a)}}{i!\nu^{m-1}\mathcal{L}^{m-1-i}} - \min_t \sum_{i=1}^{m-1} \frac{(t-a)^i e^{-\nu\mathcal{L}(t-a)}}{i!\nu^{m+1-i}\mathcal{L}^{m-i-1}} -$$

$$\left. - \mathcal{L} \min_t \frac{(t-a)^m}{m!\nu} e^{-\nu\mathcal{L}(t-a)} \right).$$

It implies

$$e^{-\nu\mathcal{L}(t-a)}|Q(x)-Q(y)|(t) \leq \frac{1+\nu}{\nu^{m+1}} \left(1 - e^{-\nu\mathcal{L}(b-a)} \right) \|x-y\|. \tag{2.24}$$

Now the r.h.s. of (2.24) is constant, i.e. independent of t, thus (2.24) is valid for every $t \in [a,b]$, whence for the maximum of its l.h.s. we have

$$\max_t e^{-\nu\mathcal{L}(t-a)}|Q(x)-Q(y)|(t) \le \frac{1+\nu}{\nu^{m+1}}\left(1-e^{-\nu\mathcal{L}(b-a)}\right)\|x-y\|. \tag{2.25}$$

By the definition of norm (2.11) the above inequality is equivalent to

$$\|Q(x)-Q(y)\| \le \frac{1+\nu}{\nu^{m+1}}\left(1-e^{-\nu\mathcal{L}(b-a)}\right)\|x-y\|. \tag{2.26}$$

It is clear that $0 < 1-e^{-\nu\mathcal{L}(b-a)} < 1$ for any finite $\nu \ge 2$ and $\mathcal{L} \ge 1$, the term $0 < \frac{1+\nu}{\nu^{m+1}} = \frac{3}{2^{m+1}} < 1$ for $\nu = 2$, it is enough to make the coefficient $0 < \frac{4}{2^{m+1}}\left(1-e^{-2\mathcal{L}(b-a)}\right) < 1$ for $Q(x)t$ to be a contraction operator; thus the Banach fixed point theorem is applicable to guarantee the existence of a unique solution for (2.1) in the domain (2.9).

Remark. In the theorem we use the term

(*)

$$M := M3\sum_{j=1}^{m}\frac{(b-a)^{j-1}}{j!} \quad \text{with} \quad M3 = \max(|c_j|, M1, M2), \quad j = 1,\ldots,m-1,$$

where $M1$ and $M2$ are the upper bounds of K and f respectively.

We notice that, as it was clear from the theorems,

(**)

$$\text{for } j=1 \;\; M := M3 = \max(M1, M2), \quad \text{for } j=2 \;\; M := M3\left(1+\frac{b-a}{2}\right),$$

which can be used to specify

$$\delta := \min\left(b-a,\ \frac{T}{M}\right)$$

(for further details on M and T see the theorems). However, if the order of NVIDE m is fairly large, one may use

$$(***) \quad M := M3e^{b-a}, \quad \text{with} \quad M3 = \max(|c_j, M1, M2), \quad j = 1,\ldots,m-1$$

instead of (*).

3. Example

Let us consider the NVIDE

$$x^{(5)}(t) = 1 - e^{2t} - x^{(4)}(t) + 2\int_0^t e^{2\tau}(x^{(3)}(\tau))^2 d\tau, \tag{3.1}$$

$t \in [0, b]$, $x(0) = 1$, $x'(0) = 1$, $x''(0) = 3$, $x'''(0) = -1$, $x^{(4)}(0) = 1$, where $x^*(t) = e^{-t} + t^2$.

Obviously (3.1) satisfies the conditions of Theorem 2 for $m = 5$ in $B1$ and $B2$

$$B1: \quad 0 \le t \le b, \quad 0 \le \tau \le b, \quad |u_j - c_j| \le T_j < \infty, \tag{3.2}$$

$$B2: \quad 0 \le t \le b, \quad |u_j - c_j| \le T_j < \infty, \quad |u_5| \le T_5 < \infty, \tag{3.3}$$

$$j = 0, 1, 2, 3, 4; \qquad c_j = x^{(j)}(0).$$

We have the following Lipschitz condition

$$\begin{aligned} &|K(t, \tau, x(\tau), \dots, x^{(4)}(\tau)) - K(t, \tau, y(\tau), \dots, y^{(4)}(\tau))| = \\ &= |2e^{2t}(x^{(3)}(\tau) + y^{(3)}(\tau))(x^{(3)}(\tau) - y^{(3)}(\tau))| \le \\ &\le 4T_* e^{2b}|x^{(3)}(\tau) - y^{(3)}(\tau)| \le 4T_* e^{2b}\sum_{j=0}^{4}|x^{(j)}(\tau, \lambda) - y^{(j)}(\tau, \lambda)| \end{aligned} \tag{3.4}$$

for every $(t, \tau, \dots, x^{(4)}(\tau))$ and $(t, \tau, y(\tau), \dots, y^{(4)}(\tau))$, where $T_* = \max(T_j) + 3$, $j = 0, 1, 2, 3, 4$.

$$|f(t, x(t), \dots, x^{(4)}(t), IKX) - f(t, y(t), \dots, y(4)(t), IKY)| =$$

$$= |1 - e^{2t} - x^{(4)}(t) + IKX - 1 + e^{2t} + y^{(4)}(t) - IKY| \le$$

$$\le |x^{(4)}(t) - y^{(4)}(t)| + I|KX - KY| \le \tag{3.5}$$

$$\le \sum_{j=0}^{4}|x^{(j)}(t) - y^{(j)}(t)| + 4T_* e^{2b} I \sum_{j=0}^{4}|x^{(j)}(t) - y^{(j)}(t)|$$

for every $(t, x(t), \ldots, x^{(4)}(t), IKX)$ and $(t, y(t), \ldots, y^{(4)}(t), IKY)$ in $B2$. Therefore D is equipped with the norm

$$\|x\| := \max_t e^{-\nu \mathcal{L} t} \sum_{j=0}^{4} |x^{(j)}(t)|, \tag{3.6}$$

where $\mathcal{L} := \max(\ell_1, \ell_2, 1) = \max(1, 4T_* e^{2b}, 1) = 4T_* e^{2b}$, $a = 0$ and $\nu \geq 2$. Moreover, we have the following estimates for the bounds of K and f:

$$|K(t, \tau, x(\tau), \ldots, x^{(4)}(\tau))| \leq 2e^{2b} T_*^2 \quad \text{in} \quad B1, \tag{3.7}$$

$$|f(t, x(t), \ldots, x^{(4)}(t), IKY)| \leq 2e^{2b} T_*^2 b + T_* \quad \text{in} \quad B2. \tag{3.8}$$

Thus $M3 = 2e^{2b} T_*^2 b + T_*$.

Integrating both sides of (3.1) from 0 to t five times we reach the equivalent integral equation

$$x(t) = 1 + t + 3\frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \tag{3.9}$$

$$+ \int_0^t \int_0^{\mu_1} \int_0^{\mu_2} \int_0^{\mu_3} \int_0^{\mu_4} \left(1 - e^{2s} - x^{(4)}(s) + 2 \int_0^s e^{2\tau} (x^{(3)}(\tau))^2 d\tau \right) ds d\mu_4 d\mu_3 d\mu_2 d\mu_1.$$

Hence $Q(X)t :=$ the r.h.s. of (3.9).

Using (3.7) the inequality (2.14) gives

$$|IKX| \leq I|K(t, \tau, x(\tau), \ldots, x^{(4)}(\tau))| \leq 2e^{2b} T_*^2 t \leq M3\delta \leq M\delta \leq T. \tag{3.10}$$

Similarly, from inequality (2.13) using (3.8) and (3.9) we have

$$|Q(X)t - 1| \leq t + 3\frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} +$$

$$+ (2e^{2b} T_*^2 b + T_*) \int_0^t \int_0^{\mu_1} \int_0^{\mu_2} \int_0^{\mu_3} \int_0^{\mu_4} ds d\mu_4 d\mu_3 d\mu_2 d\mu_1 \leq \tag{3.11}$$

$$\leq (2e^{2b} T_*^2 b + T_*) \sum_{j=1}^{5} \frac{t^j}{j!} \leq M\delta \leq T.$$

Therefore $Q: E \to E$. The difference $|Q(X) - Q(Y)|(t)$ after applying (3.5) becomes

$$(3.12) \qquad |Q(X) - Q(Y)|(t) \leq 4T_* e^{2b} \cdot$$

$$\int_0^t \int_0^{\mu_1} \int_0^{\mu_2} \int_0^{\mu_3} \int_0^{\mu_4} \left(\sum_{j=0}^{4} |x^{(j)}(s) - y^{(j)}(s)| + I \sum_{j=0}^{4} |x^{(j)}(s) - y^{(j)}(s)| \right) ds d\mu_4 \ldots d\mu_1.$$

(3.12) is just the inequality (2.16) for $m = 5$, thus by the same method (3.12) will end at

$$\|Q(X) - Q(Y)\| \leq \frac{1+\nu}{\nu^6} \left(1 - e^{\nu \mathcal{L} b}\right) \|x - y\|$$

for $\nu \geq 2$ and $\mathcal{L} = 4T_* e^{2b}$. Hence the problem (3.1) has a unique solution $x(t) \in C^{(5)}[0, b]$ in E mentioned above.

Conclusions. 1. These theorems are valid for the parametrized and nonparametrized linear and nonlinear VIDE, where the kernel K may depend nonlinearly on its arguments, but the integral operator I should be linear.

2. By posing the weight function $e^{-r(t)}$ not only the existence of the solution is guaranteed, but its uniqueness as well.

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