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# A SPLINE METHOD FOR APPROXIMATE SOLUTION OF THE INITIAL VALUE PROBLEM $y^{(n)}(x) = f(x, y(x), y', \dots, y^{(n-1)}(x))$

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**Abstract.** Approximate solution is constructed in the form of spline function for the Cauchy problem regarding an $n$-th order nonlinear ordinary differential equation: $y^{(n)}(x) = f(x, y(x), y'(x), ..., y^{(n-1)}(x))$; $y(x_0) = y_0$, $y'(x_0) = y_0^1$, ..., $y^{(n-1)}(x_0) = y_0^{n-1}$, where $n \geq 1$ integer, $f \in C^2$. The spline is of degree $m = n + 1$ and of class $C^n$. The method has a local truncation error $O(h^{n+2})$ in $y$ and the corresponding derivatives are approximated by the derivatives of the constructed splines $S_m$ as follows: $|y^{(j)}(x) - S_m^{(j)}(x)| = O(h^{m+1-j})$, $j = 1, 2, ..., m$. The existence and uniqueness of the approximate spline function are proved. Some numerical examples are given for illustration. This method is a modified version of the methods given by Callender [3], Loscalzo, Talbot [10,11] and Micula [12,13] and it can also be regarded as a slightly modified Taylor's expansion method of order $m$.

## 1. Introduction

Approximate solution of ordinary differential equations with spline functions was investigated by several authors. The first order Cauchy problem $y' = f(x, y)$ was discussed by Callender [3], Loscalzo and Talbot [10,11], and Mülthei [14,15]. Callender presented a single step method of order $m$ ($m \geq 2$), where the spline is of class $C^1$, but of degree $m$. The method given in the

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paper of Loscalzo and Talbot is equivalent to a linear multistep method and is convergent only if $m = 2, 3$. A general procedure is presented by Mülthei for the above problem. In this procedure several well-known spline approximation methods are included as special cases, additional knots are admitted in every subinterval. Micula [12,13] studied the approximation of the solution of the problem $y'' = f(x, y)$, i.e. a nonlinear second order differential equation, but only when the first derivative is absent. The problem $y'' = f(x, y, y')$ was solved by Sharma and Gupta [17] with a one-step method based upon the Lobatto four-point quadrature formula. Fawzy [4,5] solved the same problem using spline functions but his approximation process consists of two stages. For the solution of the initial value problem $y^{(n)}(x) = f(x, y, y')$ Fawzy and Soliman proposed a spline method in [6], but there is no connection with the method presented in this paper.

Consider the following $n$-th order initial value problem:

$$y^{(n)}(x) = f(x, y(x), \dots, y^{(n-1)}(x)), \tag{1.1}$$

$$y^{(i)}(x_0) = y_0^i, \quad i = 0, 1, \dots, n-1.$$

When the function $f$ on the right belongs to the class $C^k$ $(k = 0, 1, 2, \dots)$ then the solution of (1.1) belongs to the class of functions $C^{n+k}$. A natural claim is that similar property holds for the approximate solution of (1.1). In this paper we shall use splines of degree $m = n + 1$ by assumption that $S_m \in C^n$.

To solve numerically the problem (1.1) it has usually to be reduced to a system of first order differential equations (see [7,8,19]). Furthermore, sometimes in physical applications it is important to approximate also the higher derivatives of the solution.

It is well known that direct using of the Taylor expansion method in general is not recommended for the first order differential equations or for the system of equations (Henrici [8] p.66) because it has to be operated simultaneously not only with the function $f$ but also with its higher derivatives. There is a good review in [1] which shows the possibilities of using step-by-step methods.

The method proposed in this paper for the solution of nonlinear $n$-th order ordinary differential equation is a modified version and a generalization of the methods given in [3,10-13] and it can also be considered as a slight modification of the Taylor's expansion. Furthermore, it has the advantage over the discrete method that it gives a global approximation of the solution and also permits the study of the behaviour of derivatives of the approximate solution.

## 2. Description of the method

We construct a single step method for the $n$-th order nonlinear ordinary differential equations using a spline of degree $m$ and continuity class $C^{m-1}$. Here $m$ and the order of approximation depend on the order of the differential equation $n$ ($n \geq 1$ integer, $m = n + 1$). Consider the nonlinear differential equation

$$y^{(n)}(x) = f(x, y, y', ..., y^{(n-1)}), \tag{2.1}$$

where $f \in C^2([x_0, b] \times R^n)$ and $x_0, b \in R$, $x_0 < b$. We assign to the equation (2.1) the initial conditions:

$$y^{(i)}(x_0) = y_0^i, \qquad i = 0, ..., n-1, \tag{2.2}$$

where $y_0^i, i = 0, ..., n-1$ are preassigned values. Let $D := \{(x, y, ..., y^{(n-1)}) \mid x_0 \leq x \leq b\}$ and $f \in C^2(D)$ furthermore, let $L$ be the Lipschitz coefficient for which

$$\begin{aligned} &|f(x, y_1, y_1', ..., y_1^{(n-1)}) - f(x, y_2, y_2', ..., y_2^{(n-1)})| \leq \\ &\leq L(|y_1 - y_2| + |y_1' - y_2'| + ... + |y_1^{(n-1)} - y_2^{(n-1)}|). \end{aligned} \tag{2.3}$$

In accordance with these conditions there exists only one $y(x)$ solution of the initial value problem (2.1), (2.2) (see [8], Theorem 4.1). We construct a polynomial spline function $S(x)$ of degree $m = n + 1$ approximating $y(x)$ and its derivatives. (Instead of $S_m$ we write $S$ which denotes a spline of order $m$.) For this purpose let $h$ be the stepsize, $h = (b - x_0)/N$ ($N \in \mathbb{N}$) and we define in each subinterval $[x_{i-1}, x_i]$ $i = 1, ..., N$ the components of $S$ by

$$p_i(x) = a_i^m (x - x_{i-1})^m + a_i^{m-1}(x - x_{i-1})^{m-1} + ... + a_i^1(x - x_{i-1}) + p_{i-1}. \tag{2.4}$$

$p_{i-1}$ is known approximated value of the exact solution $y$ at the point of $x_{i-1}$ and the coefficients $a_i^j$, $i = 1, ..., N$, $j = 1, ..., m$ are needed to be determined. (We consider equidistant mesh points only for the sake of simplicity but the method is applicable with variable stepsizes too.) For $i = 1$ let

$$p_0 = y_0, \quad a_1^j = \frac{y_0^j}{j!}, \quad j = 1, ..., m-2, \tag{2.5}$$

$$a_1^{m-1} = \frac{f(x_0, y_0, y_0^1, \dots, y_0^{n-1})}{(m-1)!}. \tag{2.6}$$

From the condition $S \in C^{(m-1)}$ it follows that

$$p_i^{(j)}(x_i) = p_{i+1}^{(j)}(x_i), \quad i = 1, \dots, N-1, \quad j = 0, \dots, m-1. \tag{2.7}$$

These equalities give us recursion formulae for $a_i^j$:

(2.8)

$$a_{i+1}^1 = m a_i^m h^{m-1} + (m-1) a_i^{m-1} h^{m-2} + \dots + a_i^1,$$

$$2! a_{i+1}^2 = m(m-1) a_i^m h^{m-2} + (m-1)(m-2) a_i^{m-1} h^{m-3} + \dots + 2 a_i^2,$$

$$3! a_{i+1}^3 = m(m-1)(m-2) a_i^m h^{m-3} + (m-1)(m-2)(m-3) a_i^{m-1} h^{m-4} + \dots + 3! a_i^3,$$

$$(m-2)! a_{i+1}^{m-2} = m(m-1)\dots 3 a_i^m h^2 + (m-1)! a_i^{m-1} h + (m-2)! a_i^{m-2},$$

$$(m-1)! a_{i+1}^{m-1} = m! a_i^m h + (m-1)! a_i^{m-1},$$

where $i = 1, \dots, N-1$.

We now calculate the remaining unknown coefficients of $S$. Integrating (2.1) from $x_{i-1}$ to $x_i$ we get

$$y^{(n-1)}(x_i) - y^{(n-1)}(x_{i-1}) = \int_{x_{i-1}}^{x_i} f(x, y, y', \dots, y^{(n-1)})\, dx.$$

Let us replace $y^{(n-1)}(x_i)$ by $p_{i+1}^{(n-1)}(x_i)$, $y^{(n-1)}(x_{i-1})$ by $p_i^{(n-1)}(x_{i-1})$ and in the integrand $y^{(j)}(x)$ by $p_i^{(j)}(x)$, $j = 0, 1, \dots, n-1$. It follows that

$$(m-2)! a_{i+1}^{m-2} - (m-2)! a_i^{m-2} = \int_{x_{i-1}}^{x_i} f(x, p_i, p_i', \dots, p_i^{(n-1)})\, dx.$$

On the base of equation which corresponds to the smoothness condition $S^{(m-2)} \in C$ we get implicit nonlinear equations for $a_i^m$ :

$$m(m-1)\dots 3 a_i^m h^2 + (m-1)! a_i^{m-1} h =$$

$$(2.9) \qquad = \int_{x_{i-1}}^{x_i} f(x, p_i(x), p_i'(x), ..., p_i^{(n-1)}(x))\, dx,$$

where $i = 1, ..., N$. By this construction we can determine the spline function $S(x)$ of degree $m = n + 1$, for which the unknown coefficients can be computed by (2.5), (2.6), by equations in (2.8) and (2.9).

## 3. Existence, uniqueness and convergence of the approximate solution

There is the only one approximate spline solution.

**Theorem 1.** *If $h < \frac{3}{L+1}$ then the spline function $S$ given by the above construction exists and is unique.*

**Proof.** Define $p_i(x)$ in the subinterval $[x_{i-1}, x_i]$, $i = 1, ..., N$ as in (2.4) where the coefficients $p_{i-1}$, $a_i^j$, $j = 1, ..., m-1$ are uniquely determined by the continuity conditions. We prove that $a_i^m$ may be uniquely determined from (2.9). By rearrangement we get

$$(3.1) \qquad a_i^m = g_i(a_i^m) := \frac{1}{m(m-1)...3h^2} \times$$

$$\times \int_{x_{i-1}}^{x_i} [f(x, p_i(x), p_i'(x), ..., p_i^{(n-1)}(x)) - (m-1)! a_i^{(m-1)}]\, dx,$$

$i = 1, ..., N$. The right-hand side of this equation contains also the unknown $a_i^m$ in $p_i^{(j)}$, $j = 0, 1, ..., n-1$.

Define $G_i : R \to R$ by $a_i^m \to g_i(a_i^m)$, $a_i^m \in R$. We show that the operator $G_i$ is a contraction, thus it has a unique fixed point.

Let $a_i^m, \tilde{a}_i^m \in R$, and their distance $\rho(a_i^m, \tilde{a}_i^m) = |a_i^m - \tilde{a}_i^m|$. According to the Lipschitz condition (2.3) follows that

$$\rho(G_i(a_i^m), G_i(\tilde{a}_i^m)) = |g_i(a_i^m) - g_i(\tilde{a}_i^m)| \le \frac{Lh(3^n - h^n)}{3^n(3-h)} \rho(a_i^m, \tilde{a}_i^m).$$

So if $Lh/(3-h) < 1$, then $G_i$ is a contraction operator and (3.1) has a unique solution.

**Remark 1.** It is readily seen that Theorem 1 gives the stepsize restriction $h < 3/L$ for the first order equation.

Now we study the approximation of a solution of the problem (2.1), (2.2) by the constructed spline. In order to estimate the approximation error we assume that the derivatives of the solution $y$ are bounded and that there exist bounds for the partial derivatives up to the order two for the function $f$.

**Theorem 2.** *Let $f \in C^2(D)$ and let $S$ be the spline function approximating the solution $y$ of the problem* (2.1), (2.2). *Then for any $h < 1/3LC$, where $C := 1/3! + ... + 1/(n+2)!$ holds*

$$|y^{(j)}(x_i) - S^{(j)}(x_i)| \le K h^{m+1-j}, \quad j = 0, 1, ..., m, \tag{3.2}$$

*where the constant $K$ does not depend on $h$.*

**Proof.** We proceed by induction on $i$. For $i = 1$ the coefficients of the polynomial $p_1(x)$ are determined by (2.5) and (2.6). Moreover, from equation (2.9) we get

$$a_1^m = \frac{1}{m(m-1)...3h^2} \times \tag{3.3}$$

$$\times \int_{x_0}^{x_1} [f(x, p_1(x), p_1'(x), ..., p_1^{(n-1)}(x)) - f(x_0, y_0, y_0^1, ..., y_0^{n-1})]\, dx.$$

Considering the Taylor's expansion

$$f(x, p_1(x), p_1'(x), ..., p_1^{(n-1)}(x)) - f(x_0, y_0, y_0^1, ..., y_0^{n-1}) =$$

$$= f'(x_0, y_0, y_0^1, ..., y_0^{n-1})(x - x_0) + O((x - x_0)^2),$$

a short calculation gives that

$$a_1^m = f'(x_0, y_0, y_0^1, ..., y_0^{n-1})/m! + O(h) =$$

$$= y^{(m)}(x_0)/m! + O(h).$$

Now let $i = k$ and suppose that $a_k^j - y^{(j)}(x_{k-1})/j! = O(h^{m+1-j}), j = 1, ..., m$. From equations (2.8) we get

$$a_{k+1}^1 = y'(x_k) + O(h^m),$$

(3.4) $$2!a_{k+1}^2 = y''(x_k) + O(h^{m-1}),$$

$$(m-1)!a_{k+1}^{m-1} = y^{(m-1)}(x_k) + O(h^2).$$

On the base of the last estimates it follows from (3.1) that

$$a_{k+1}^m = \frac{1}{m(m-1)...3h^2}\times$$

$$\times \int_{x_k}^{x_{k+1}} \{[f(x, p_{k+1}(x), p'_{k+1}(x), ..., p_{k+1}^{(n-1)}(x)) - y^{(n)}(x_k)] + O(h^2)\}\, dx.$$

As in the case $i = 1$, it can be calculated for $i = k+1$ that

(3.5) $$a_{k+1}^m = f'(x_k, y_k, y'_k, ..., y_k^{(n-1)})/m! + O(h) = y^{(m)}(x_k)/m! + O(h),$$

whenever $3LCh < 1$, where $C := 1/3!+1/4!+...+1/(n+2)!$. From the construction of the spline function (2.4) it follows immediately that $a_i^j = p_i^{(j)}(x_{i-1})/j!$, for $j = 1, ..., m,\quad i = 1, ..., N$. This proves (3.2) for $j = 1, ..., m$. To prove (3.2) for $j = 0$ we have to apply (2.4), estimates proved just above and the Taylor's expansion for $y$ with the remainder term of order $(m+1)$:

$$y(x_i) = y(x_{i-1}) + y'(x_{i-1})h + \frac{y''(x_{i-1})}{2!}h^2 + ... + \frac{y^{(m)}(x_{i-1})}{m!}h^m + O(h^{m+1}),$$

$$p_i(x_i) = p_{i-1} + a_i^1 h + a_i^2 h^2 + ... + a_i^m h^m, \quad i = 1, ..., N.$$

Using (3.4) – (3.5) follows (3.2) for $j = 0$ because $S(x_i) = p_i(x_i)$.

**Remark 2.** From Theorem 2 follows that this method can be regarded as a modified Taylor expansion method of order $m$, provided that starting values $y_0^j$ are given exactly or with appropriate accuracy.

**Theorem 3.** *If $f \in C^2(D)$ and $S$ is the spline function approximating the solution of problem* (2.1), (2.2), *then for any $h < 1/3LC$ and $x \in [x_0, b]$,*

$$|y^{(j)}(x) - S^{(j)}(x)| \le K_1 h^{m+1-j}, \quad j = 0, 1, ..., m,$$

*where the constant $K_1$ is independent on $h$.*

**Proof.** Let be $x \in [x_i, x_{i-1}]$. Taking into consideration that the function $S^{(m)}$ is constant in $[x_i, x_{i-1}]$, furthermore on the base of Theorem 2 and by the help of Taylor's formula for $y^{(m)}$ it follows that

$$S^{(m)}(x) - y^{(m)}(x) =$$

$$= [S^{(m)}(x) - S^{(m)}(x_i)] + [S^{(m)}(x_i) - y^{(m)}(x_i)] + [y^{(m)}(x_i) - y^{(m)}(x)] = O(h).$$

The Taylor's expansion leads for the function $S^{(m-1)} - y^{(m-1)}$ to

$$S^{(m-1)}(x) - y^{(m-1)}(x) = [S^{(m-1)}(x_i) - y^{(m-1)}(x_i)] +$$

$$+ [S^{(m-1)}(\xi) - y^{(m-1)}(\xi)](x - x_i) = O(h^2) + O(h)(x - x_i) = O(h^2),$$

where $\xi \in (x_i, x)$.
Similarly we can also get the other estimates. Finally

$$S'(x) - y'(x) = [S'(x_i) - y'(x_i)] + [S''(\xi') - y''(\xi')](x - x_i) =$$

$$= O(h^m) + O(h^{m-1})(x - x_i) = O(h^3),$$

where $\xi' \in (x_i, x)$ and

$$S(x) - y(x) = [S(x_i) - y(x_i)] + [S'(\eta) - y'(\eta)](x - x_i) =$$

$$= O(h^{m+1}) + O(h^m)(x - x_i),$$

where $\eta \in (x_i, x)$. This completes the proof.

**Remark 3.** The problems of stability of the spline method proposed here will be investigated in a following paper.

## 4. Applications of the method

Let us consider the first order initial value problem $y'(x) = f(x, y)$, $y(0) = y_0$. Loscalzo and Talbot [10, 11] proved for this problem the following

**Proposition 1.** *If $f(x, y) \in C^2$ in $T := \{(x, y) : 0 \le x \le b\}$, then a constant $K_2$ can be determined such that for all $h < \frac{2}{L}$*

$$|S_2(x) - y(x)| < K_2 h^2, \qquad |S_2'(x) - y'(x)| < K_2 h^2,$$

$$|S_2''(x) - y''(x)| < K_2 h,$$

*if $x \in [0, b]$, provided we define the value of the step function $S_2''(x)$ at a knot $x_k$, by the usual arithmetic mean.*

Micula [12] proved for the cubic spline approximation of the Cauchy problem regarding a second order differential equation $y'' = f(x, y)$, $y(0) = y_0$, $y'(0) = y_0^1$ the next

**Proposition 2.** *If $f \in C^3([0, b] \times R)$ and $S$ is the cubic spline function approximating the solution of problems $y'' = f(x, y)$, $y(0) = y_0$, $y'(0) = y_0^1$ then there exists a constant $K$ such that, for any $h < (6/L)^{1/2}$ and $x \in [0, b]$,*

$$|S(x) - y(x)| < Kh^3, \qquad |S'(x) - y'(x)| < Kh^2,$$

$$|S''(x) - y''(x)| < Kh^2, \qquad |S'''(x) - y'''(x)| < Kh,$$

*provided $S'''(x_k)$ is given by the usual arithmetic mean.*

We can say that Theorem 3 is an extension of the Proposition 2 for the second order Cauchy problem.

**Remark 4.** The method is applicable also for the system of differential equations but in this case instead of (2.9) a system of nonlinear equations is needed to be solved.

In the following section some numerical examples are given for second, third and fourth order initial value problems.

## 5. Numerical results

**The algorithm:** From the initial values we get:

$$p_0 := y_0,$$

$$a_1^1 := y_0^1,$$

$$a_1^2 := y_0^2/2!,$$

$$a_1^{m-1} := y_0^{m-1}/(m-1)!.$$

In every iteration step has to be computed:

1. $a_i^m$ from the implicit equation (2.9) by the help of the values $p_{i-1}, a_i^1, \ldots, a_i^{m-1}$,
2. $p_i(x_i)$ from the expression (2.4) $(p_i(x_i) = y_i)$,
3. $a_{i+1}^j$, $j = 1, \ldots, m-1$ from the equations (2.8) by the help of the values of $a_i^1, \ldots, a_i^m$, $i = 1, \ldots, N$.

Numerical results (presented below) have been obtained on an IBM AT/386 compatible computer with programs in PASCAL and with extended variables. For the illustration the following differential equations were solved.

**Example 1.** (Babuška, Práger, Vitásek [2], Example 3.4)

$$y'' = -y, \quad 0 \le x \le 1, \quad y(0) = 0, \quad y'(0) = 1.$$

*Maximum absolute and relative errors of approximate solution and its derivatives.*

(For the sake of brevity we apply the following notations: $1.54 - 5$ denotes $1.54 \times 10^{-5}$.)

| | $h = 0.1$ | | $h = 0.01$ | | $h = 0.001$ | | $h = 0.0001$ | |
|---|---|---|---|---|---|---|---|---|
| | abs | rel | abs | rel | abs | rel | abs | rel |
| $y$ | 4.05-7 | 5.56-7 | 4.05-11 | 5.55-11 | 4.04-15 | 5.55-15 | 3.16-17 | 3.87-17 |
| $y'$ | 1.75-7 | 3.25-7 | 1.75-11 | 3.24-11 | 1.75-15 | 3.23-15 | 4.92-17 | 9.11-17 |
| $y''$ | 7.02-4 | 8.35-4 | 7.01-6 | 8.33-6 | 7.01-8 | 8.33-8 | 7.01-10 | 8.33-10 |
| $y'''$ | 4.16-2 | 7.70-2 | 4.20-3 | 7.78-3 | 4.21-4 | 7.79-4 | 4.20-5 | 7.79-5 |

The absolute errors of the numerical results are given in [2] for the methods of Störmer and Adams-Bashforth as follows: for $h = 0.01$ : $3.27 \times 10^{-6}$ and $3.12 \times 10^{-7}$; for $h = 0.001$ : $3.535 \times 10^{-4}$ and $3.6 \times 10^{-8}$; and at last for $h = 1/20000$ : $0.158$ and $8.99 \times 10^{-7}$. The derivatives were not approximated.

**Example 2.** (Fawzy, Soliman [6], Example 2.)

$$y''' = -y - x, \quad 0 \le x \le 1, \quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 1.$$

The exact solution is: $y(x) = e^{-x} - x$.

*Maximum absolute and relative errors of approximate solution and its derivatives.*

| | $h = 0.1$ | | $h = 0.01$ | | $h = 0.001$ | | $h = 0.0001$ | |
|---|---|---|---|---|---|---|---|---|
| | abs | rel | abs | rel | abs | rel | abs | rel |
| $y$ | 3.82-7 | 5.25-6 | 3.82-11 | 5.76-9 | 3.81-15 | 1.14-11 | 8.03-17 | 4.86-13 |
| $y'$ | 1.33-6 | 7.22-7 | 1.38-10 | 7.23-11 | 1.39-14 | 7.23-15 | 2.16-17 | 1.57-17 |
| $y''$ | 2.19-7 | 5.95-7 | 2.19-11 | 5.94-11 | 2.19-15 | 5.95-15 | 1.43-17 | 2.96-17 |
| $y'''$ | 1.59-3 | 2.88-3 | 1.66-5 | 3.08-5 | 1.67-7 | 3.10-7 | 1.67-9 | 3.10-9 |
| $y^{(IV)}$ | 6.26-2 | 9.18-2 | 6.63-3 | 9.49-3 | 6.66-4 | 9.53-4 | 6.67-5 | 9.53-5 |

The absolute errors in [6] are as follows: at $x = 0.4$ for $y : 1.0 \times 10^{-9}$, $y' : 4.5 \times 10^{-8}$, $y'' : 2.84 \times 10^{-7}$ and $y''' : 1.0 \times 10^{-9}$, but the stepsize was not explained.

**Example 3.** (Rutishauser [16])

$$y^{(IV)} = y, \quad 0 \le x \le 10, \quad y(0) = y'(0) = y''(0) = y'''(0) = 1.$$

*Maximum absolute and relative errors of approximate solution and its derivatives.*

| $h = 0.1$ | $x = 0.1$ | | $x = 1.0$ | | $x = 5.0$ | | $x = 10.0$ | |
|---|---|---|---|---|---|---|---|---|
| | abs | rel | abs | rel | abs | rel | abs | rel |
| $y$ | 1.44-9 | 1.30-9 | 3.68-7 | 1.36-7 | 8.85-5 | 5.96-7 | 2.42-2 | 1.10-6 |
| $y'$ | 5.77-8 | 5.22-8 | 8.57-7 | 3.15-7 | 1.03-4 | 6.96-7 | 2.65-2 | 1.20-6 |
| $y''$ | 1.45-6 | 1.31-6 | 9.71-7 | 3.57-7 | 1.38-4 | 9.32-7 | 3.17-2 | 1.44-6 |
| $y'''$ | 1.27-9 | 1.15-9 | 9.18-8 | 3.38-8 | 7.18-5 | 4.84-7 | 2.18-2 | 9.87-7 |
| $y^{(IV)}$ | 1.75-3 | 1.59-3 | 1.43-3 | 5.26-4 | 1.23-1 | 8.26-4 | 1.83+1 | 8.31-4 |
| $y^{(V)}$ | 7.10-2 | 6.42-2 | 1.17-1 | 4.30-2 | 7.28+0 | 4.91-2 | 1.08+3 | 4.92-2 |

| $h = 0.01$ | $x = 0.1$ | | $x = 1.0$ | | $x = 5.0$ | | $x = 10.0$ | |
|---|---|---|---|---|---|---|---|---|
| | abs | rel | abs | rel | abs | rel | abs | rel |
| $y$ | 2.85-13 | 2.58-13 | 3.70-11 | 1.63-11 | 9.01-9 | 6.07-11 | 2.48-6 | 1.13-10 |
| $y'$ | 5.77-12 | 5.22-12 | 8.60-11 | 3.16-11 | 1.05-8 | 7.09-11 | 2.71-6 | 1.23-10 |
| $y''$ | 5.84-12 | 5.29-12 | 9.81-11 | 3.61-11 | 1.41-8 | 9.48-11 | 3.24-6 | 1.47-10 |
| $y'''$ | 5.09-15 | 4.61-15 | 1.12-11 | 4.11-12 | 7.46-9 | 5.03-11 | 2.25-6 | 1.02-10 |
| $y^{(IV)}$ | 8.76-7 | 7.93-7 | 1.43-5 | 5.27-6 | 1.23-3 | 8.28-6 | 1.84-1 | 8.33-6 |
| $y^{(V)}$ | 3.85-3 | 3.48-3 | 1.20-2 | 4.38-3 | 7.39-1 | 4.98-3 | 1.10+2 | 4.99-3 |

The exact result at $x = 10$ : $y = 22026.4657948067$. Our results: for $h = 0.1$ : 22026.4900, for $h = 0.01$ : 22026.4657972859. The modified Euler method in [14] gives for $h = 0.1$ : 19865.18 and for $h = 0.01$ : 21794.34.

The problem was solved also with the stepsize $h = 0.001$. The result is in this case at $x = 10$ the following: $y(10) = 22026.4657948070$, the absolute error: 2.44 − 10, the relative error: 1.11 − 14.

**Example 4.** (Jain, Goel [9], Problem 1.)

$$y'' = -ky', \quad 0 \le x \le 1, \quad y(0) = 0, \quad k = 1, 10, 30, 50, 100.$$

The exact solution is: $y(x) = (1 - e^{-kx})/k$.

*Maximum absolute and relative errors of approximate solution and its derivatives.*

| $h = 10^{-2}$ | $k = 1$ | | $k = 10$ | | $k = 30$ | | $k = 50$ | | $k = 100$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | abs | rel | abs | rel | abs | rel | abs | rel | abs | rel |
| $y$ | 1.4-10 | 1.4-8 | 1.9-6 | 1.9-6 | 3.8-2 | - | - | - | - | - |
| $y'$ | 1.4-10 | 2.7-10 | 1.9-5 | 4.3-1 | 1.1+0 | - | - | - | - | - |
| $y''$ | 1.7-5 | 4.0-5 | 2.3-1 | 5.1+2 | 4.6+3 | - | - | - | - | - |
| $y'''$ | 6.6-3 | 1.1-2 | 4.6+1 | 1.0+4 | 8.7+5 | - | - | - | - | - |
| $h = 10^{-3}$ | | | | | | | | | | |
| $y$ | 1.4-14 | 1.4-11 | 2.0-10 | 1.4-8 | 4.1-6 | 1.2-4 | 1.5-2 | 7.5-1 | - | - |
| $y'$ | 1.4-14 | 2.8-14 | 2.0-9 | 4.3-5 | 1.2-4 | - | 7.5-1 | - | - | - |
| $y''$ | 1.7-7 | 4.0-7 | 2.3-3 | 5.2+0 | 5.0+1 | - | 1.8+5 | - | - | - |
| $y'''$ | 6.7-4 | 1.1-3 | 4.7+0 | 1.0+3 | 9.9+4 | - | 3.6+8 | - | - | - |
| $h = 10^{-4}$ | | | | | | | | | | |
| $y$ | 2.6-17 | 1.4-14 | 2.0-14 | 1.4-11 | 4.1-10 | 1.2-8 | 1.5-6 | 7.5-5 | - | - |
| $y'$ | 2.3-17 | 6.3-17 | 2.0-13 | 4.3-9 | 1.2-8 | - | 7.5-5 | - | - | - |
| $y''$ | 1.7-9 | 4.0-9 | 2.3-5 | 5.2-2 | 5.0-1 | - | 1.8+3 | - | - | - |
| $y'''$ | 6.7-5 | 1.1-4 | 4.7-1 | 1.0+2 | 9.9+3 | - | 3.6+7 | - | - | - |
| $h = 10^{-5}$ | | | | | | | | | | |
| $y$ | 4.7-16 | 3.9-15 | 2.5-18 | 1.4-14 | 4.1-14 | 1.2-12 | 1.5-10 | 7.5-9 | 2.1-2 | 2.1+0 |
| $y'$ | 4.7-16 | 1.3-15 | 2.1-17 | 4.4-13 | 1.2-12 | 1.3+1 | 7.5-9 | - | 2.1+0 | - |
| $y''$ | 1.7-11 | 4.0-11 | 2.3-7 | 5.2-4 | 5.0-3 | - | 1.8+1 | - | 2.5+9 | - |
| $y'''$ | 6.7-6 | 1.1-5 | 4.7-2 | 1.0+1 | 9.9+2 | - | 3.6+6 | - | 5.0+14 | - |

It seems that the method approximates badly the higher derivatives of the solution for great $k$ because of the singularity of problem. The next example shows how to correct this handicap.

**Example 5.**

The problem, like in the Example 4., was solved by a slight modification of our method. The modification is as follows: the coefficient of the highest order term in (2.4) was computed by arithmetic mean, i.e. $(a_{i-1}^m + a_i^m)/2$ instead of $a_i^m$ (now $m = 3$).

*Maximum absolute and relative errors of approximate solution and its derivatives.*

| $h = 10^{-2}$ | $k = 1$ | | $k = 10$ | | $k = 30$ | | $k = 50$ | | $k = 100$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | abs | rel | abs | rel | abs | rel | abs | rel | abs | rel |
| $y$ | 1.3-5 | 2.1-5 | 5.0-4 | 5.0-3 | 1.5-3 | 4.5-2 | 2.5-3 | 1.2-1 | 6.6-3 | 6.6-1 |
| $y'$ | 1.8-5 | 5.1-5 | 1.9-3 | 5.5-2 | 2.3-2 | - | 6.6-2 | - | 2.9-1 | - |
| $y''$ | 8.1-5 | 8.4-5 | 6.6-2 | 4.9-2 | 1.3+0 | - | 5.1+0 | - | 3.3+1 | - |
| $y'''$ | 6.7-3 | 6.9-3 | 6.5+0 | 8.8-2 | 1.6+2 | - | 6.5+2 | - | 3.9+3 | - |
| $h = 10^{-3}$ | | | | | | | | | | |
| $y$ | 1.3-7 | 2.1-7 | 5.1-6 | 5.0-5 | 1.5-5 | 4.5-4 | 2.5-5 | 1.3-3 | 5.0-5 | 5.0-3 |
| $y'$ | 1.8-7 | 5.0-7 | 1.8-5 | 5.0-4 | 1.7-4 | 1.4-2 | 4.6-4 | 6.7-2 | 1.9-3 | 7.1-1 |
| $y''$ | 8.3-7 | 8.3-7 | 8.1-4 | 4.5-4 | 2.1-2 | 1.3-2 | 9.3-2 | 6.5-2 | 6.6-1 | 7.0-1 |
| $y'''$ | 6.7-4 | 6.7-4 | 6.7-1 | 6.9-3 | 1.8+1 | 2.2-2 | 8.3+1 | 3.9-2 | 6.5+2 | 6.1-1 |
| $h = 10^{-4}$ | | | | | | | | | | |
| $y$ | 1.3-9 | 2.1-9 | 5.0-8 | 5.0-7 | 1.5-7 | 4.5-6 | 2.5-7 | 1.3-5 | 5.0-7 | 5.0-5 |
| $y'$ | 1.8-9 | 5.0-9 | 1.8-7 | 5.0-6 | 1.7-6 | 1.4-4 | 4.6-6 | 6.3-4 | 1.8-5 | 5.1-3 |
| $y''$ | 8.3-9 | 8.3-9 | 8.3-6 | 4.4-6 | 2.2-4 | 1.3-4 | 1.0-3 | 6.1-4 | 8.1-3 | 5.0-3 |
| $y'''$ | 6.7-5 | 6.7-5 | 6.7-2 | 6.7-4 | 1.8+0 | 8.8-3 | 8.3+0 | 3.4-3 | 6.7+1 | 6.9-3 |
| $h = 10^{-5}$ | | | | | | | | | | |
| $y$ | 1.3-11 | 2.1-11 | 5.0-10 | 5.0-9 | 1.5-9 | 4.5-8 | 2.5-9 | 1.3-7 | 5.0-9 | 5.0-7 |
| $y'$ | 1.8-11 | 5.0-11 | 1.8-9 | 5.0-8 | 1.7-8 | 1.4-6 | 4.6-8 | 6.3-6 | 1.8-7 | 5.0-5 |
| $y''$ | 8.3-11 | 8.3-11 | 8.3-8 | 4.4-8 | 2.3-6 | 1.3-6 | 1.0-5 | 6.1-6 | 8.3-5 | 5.0-5 |
| $y'''$ | 6.7-6 | 6.7-6 | 6.7-3 | 6.7-5 | 1.8-1 | 2.0-4 | 8.3-1 | 3.3-4 | 6.7+0 | 6.7-4 |

Jain and Goel [9] have got the following results by their fourth order method:

| $h = \frac{1}{32}$ | $k = 1$ | | $k = 10$ | | $k = 30$ | | $k = 50$ | | $k = 100$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | abs | rel | abs | rel | abs | rel | abs | rel | abs | rel |
| $y$ | - | - | 4.9-7 | - | 1.4-5 | - | 6.2-5 | - | - | - |
| $y'$ | - | - | 4.9-6 | - | 4.2-4 | - | 3.1-3 | - | - | - |

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