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RECENT RESULTS OF RANKING METHODS BASED ON FUZZY PREFERENCE RELATIONS

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Abstract. This paper presents some important outranking methods in fuzzy preference modelling. The main properties of outranking methods are introduced. Then methods based on entering and leaving flow, net flow, maxmin and method defined by t-norms (or t-conorms) are characterized.

1. Introduction

Suppose you want to compare a number of alternatives taking into consideration different points of view, for example several criteria or the opinion of several voters. A common practice in such situation is to assign to each ordered pair (a, b) of alternatives a number expressing the credibility of the statement "a is at least as good as b", i.e. the sum of weights of the criteria prefering a to b or the percentage of voters declaring that a is favoured. When the different points of view taken into account are conflictual, it may not be easy to compare the alternatives on the basis of these numbers.

The ranking methods we discuss in this paper are wellknown in the literature for several years – the Coopeland ranking method (see Goodman [7]) is more than fifty years old – but the extensive researches of these methods started only in the last decade. We present the characterizations of the net flow method, t-norm and t-conorm methods. These results have high theoretical importance and establish the future applications of fuzzy preference modelling.

Let A be a finite set of objects called "alternatives" with n elements ($n >= 2$). We denote the alternatives by $c_1, c_2, \dots, c_n$. A fuzzy preference relation R is defined on A associating each ordered pair (a, b) of alternatives with a number from $[0, 1]$ expressing the truth value of the statement "a is no worse than b". A ranking method is a function assigning a crisp partial or complete ranking $<= (R)$ on A to any fuzzy relation R on A. (A binary relation R is partial ranking if it is reflexive and transitive. R is complete ranking if it is partial ranking and trichotom.) For any $a, b \in A$ $a <= (R)b$ expresses that b is as good as or better than a.

Throughout the paper we denote by $= (R)$ and $< (R)$ the symmetric and asymmetric parts of $<= (R)$, i.e. for all $a, b \in A$ [$a = (R)b$ iff ($a <= (R)b$ and $b <= (R)a$)] and [$a < (R)b$ and not $b <= (R)a$)].

Several ranking methods are based on scoring functions $S(a, R)$, which assign a real number to each alternative and these methods rank them according to their score, i.e.

$$a <= (R)b \quad \text{if and only if} \quad S(a, R) <= S(b, R). \tag{1}$$

For example the maxmin method is based on

$$S_M(a, R) = \min\{R(a, c) : \quad c \in A\},$$

and the scoring function of the classical net flow method (see Brans and Vincke [4], Fishburn [5], Goodman [7]) is given by

$$S_{NF}(a, R) = \sum (R(a, c) - R(c, a)), \quad c \in A \setminus \{a\}.$$

The method based on entering and leaving flows is defined as follows. Let

$$L(a, R = \sum R(a, c), \quad c \in A \setminus \{a\} \text{ be the leaving flow and}$$

$$E(a, R) = \sum R(c, a), \quad c \in A \setminus \{a\} \text{ be the entering flow.}$$

In this case $a <=_{L/E} (R)$ holds if and only if

$$L(a, R) >= L(b, R) \text{ and } E(a, R) <= E(b, R).$$

Let the values $R(a, c)$ be ordered for any $a \in A$ and denoted by $R_1(a) >= R_2(a) >= \ldots >= R_{n-1}(a)$. In this case let $a \in A$ be lexicographically greater than the alternative b ($a >_{lm} b$), if one of the following conditions is fulfilled

$$R_1(a) > R_1(b); \tag{i}$$

(*ii*) there exist $j <= n - 1$ such that

$$R_i(a) = R_i(b), \quad i = 1, 2, \ldots, j - 1 \text{ and } R_j(a) > R_j(b).$$

The ranking method $<=_{lm}$ is called the lexicographical maximum method. The methods based on weak t-norms are determined by the score function
(2)
$$S_T(c_i, R) = T(R(c_i, c_1),\ R(c_i, c_2), \dots, R(c_i, c_{i-1}), R(c_i, c_{i+1}), \dots, R(c_i, c_n)),$$

where $T : [0,1]^{n-1} \to [0,1]$ is symmetrical function, nondecreasing in each of the arguments and

$$T(x_1, x_2, \dots, x_{n-2}, 0) = 0,$$
$$T(1, 1, \dots, 1, x) = x \quad \text{for any} \quad x_1, x_2, \dots, x_{n-2}, x \in [0,1].$$

In this case T is said to be a weak t-norm. Similarly, the methods based on weak t-conorms are determined by the score function
(3)
$$S_S(c_i, R) = S(R(c_i, c_1), R(c_i, c_2), \dots, R(c_i, c_{i-1}), R(c_i, c_{i+1}), \dots, R(c_i, c_n)),$$

where $S : [0,1]^{n-1} \to [0,1]$ is weak t-conorm, i.e. it is symmetrical, nondecreasing in each of the arguments and

$$T(x_1, x_2, \dots, x_{n-2}, 1) = 1,$$
$$T(0, 0, \dots, 0, x) = x \ \text{ for any } \ x_1, x_2, \dots, x_{n-2},\ x \in [0,1] \ \text{ hold.}$$

The t-norms and t-conorms (associative weak t-norms and t-conorms) have many applications (see Weber [9]), but in our context weak t-norms and t-conorms are more suitable. Those functions were introduced by Fodor (see [6]).

2. The net flow method and the method based on entering and leaving flows

At first let us see the basic properties of ranking methods.

A ranking method is said to be monotonic if it does not respond "in the wrong direction" to a modification on R. Formally, $<= (R)$ is monotonic if for all fuzzy relations on A and all $a, b \in A$

$$a >= (R)b \quad \Rightarrow \quad a >= (R')b,$$

where R' is identical to R except that $[R(a,c) < R'(a,c)$ or $R(c,a) > R'(c,a)$ for some $c \in A \setminus a]$ or $[R(b,d) > R'(b,d)$ for some $c \in A \setminus a]$.

A ranking method $<= (R)$ is strongly monotonic if for all fuzzy relations on A and all $a, b \in A$

$$a >= (R)b \quad \Rightarrow \quad a > (R')b,$$

where R' is as before.

We say that a ranking method $<= (R)$ is non-discrimatory if for all fuzzy relations R on A and all $a, b \in A$

$$[R(a,b) = R(b,a) \text{ and } R(a,c) = R(b,c),\ R(c,a) = R(c,b) \text{ for all } c \in A\backslash\{a,b\}]$$

implies $a = (R)b$.

A ranking method $<= (R)$ is neutral if for all σ permutations on A, for all fuzzy relations R on A and all $a, b \in A$

$$a <= (R)b \quad \Leftrightarrow \quad \sigma(a) <= (R^\sigma)\sigma(b),$$

where R^σ is defined by $R^\sigma(\sigma(a), \sigma(b)) = R(a,b)$ for all $a, b \in A$.

In order to introduce the following axioms let us recall some well known definitions used in graph theory.

A digraph $G = (X, U)$ consists of a set of nodes X and a set of arcs $U \subseteq X^2$. We say that x is the initial extremity and y is the final extremity of the arc $u = (x, y) \in U$.

A circuit (resp. a cycle) of length r in a digraph is an ordered collection of arcs $(u_1, u_2, \ldots, u_r)$ such that for $i = 1, 2, \ldots, r$ the initial extremity of u_i is the final extremity of u_{i-1} and the final extremity of u_i is the initial extremity of u_{i+1} (resp. $u_i \neq u_{i-1}$, one of the extremities of u_i is an extremity of u_{i-1} and the other an extremity of u_{i+1}), where u_O is interpreted as u_r and u_{r+1} as u_1. A circuit (resp. a cycle) is elementary if and only if each node is extremity of maximum two arcs of the circuit (resp. the cycle). An arc u_i in a cycle is forward if its common extremity with u_{i-1} is its initial extremity and backward otherwise. A cycle is alternated if every forward arc in the cycle is followed by a backward one and vice versa..

Let us consider the digraphs

$$G_1 = (A, U_1), \quad U_1 = \{(a,b) : \ a, b \in A, \ a \neq b\} \quad \text{and}$$
$$G_2 = (A^+ A^-, U_2), \quad U_2 = \{(a^+, b^-) : \ a^+ \in A^+, \ b^- \in A^-, \ a \neq b\},$$

where A^+ and A^- are disjunct duplications of A.

It is obvious that there is a one-to-one correspondence between fuzzy relations on A and valuations between 0 and 1 of G_1 and G_2 in the following way. In the case $u = (a, b)$ (resp. $u = (a^+, B^-)$) let $v_{1R}(u)$ (resp. $v_{2U}(u)$) is equal to $R(a, b)$.

A transformation on an elementary circuit (resp. a cycle) consists in adding the same positive or negative quantity to the valuations of the arcs in the circuit (resp. adding it to the forward arcs and subtracting it from the backward arcs in the cycle). A transformation is admissible if the transformed valuations are still between 0 and 1. When an admissible transformation is applied to the graph associated with a fuzzy relation R, the fuzzy relation R' is obtained and we say that R' has been obtained from R through an admissible transformation.

A ranking $<= (R)$ method is independent of circuits (resp. cycles) if R' can be obtained from R through an admissible transformation on an elementary circuit of length 2 or 3 (resp. elementary cycle of length 4 or 6), it implies $<= (R) =<= (R')$ for all fuzzy relations R and R'.

Bouyssou in [1] characterized the net flow method by

Theorem 1. *The net flow method is the only ranking method that is neutral, strongly monotonic and independent of cicuits.*

The method based on entering and leaving flows has been characterized by the following (see Bouyssou and Perny [2])

Theorem 2. *If a partial ranking method $<= (R)$ is non-discriminatory, monotonic and independent of alternated cycles, then for all fuzzy relations R on A and $a, b \in A$, $a <=_{L/E} b$ implies $a <= (R)b$. Furthermore, if $<= (R)$ is strongly monotonic, then for all valued relations R on A and all $a, b \in A$ $a <_{L/E} b$ implies $a < (R)b$.*

3. The maxmin method

Let us consider a fuzzy relation R. We say that R' is obtained from R through a translation if for any $a, b \in A$ and real number k

$$R'(a, c) = R(a, c) + k, \quad c \in A \setminus \{a\};$$
$$R'(b, c) = R(b, c) + k, \quad c \in A \setminus \{a\};$$

and $R' = R$ otherwise.

A translation is said admissible if the values R' are still between 0 and 1.

A ranking method $<= (R)$ is independent of translations if R' can be obtained from R through an admissible translation implying $<= (R) =< +(R')$ for all fuzzy relations R and R'.

We say that a ranking method is reversible if for all fuzzy relations R, $a, b \in A$, in the case $a >= (R)b$ for every $c \in A$ there exists a fuzzy relation R' such that $R' = R$ except $R'(a,c) <= R(a,c)$ and $b >= (R')a$.

A ranking method is strictly reversible if for all fuzzy relations R, $a, b \in A$ in the case $a >= (R)b$ and $R(b,d) > 0$ for every $d \in A \setminus \{b\}$, for every $c \in A$ there exists a fuzzy relation R' such that $R' = R$ except $R'(a,c) <= R(a,c)$ and $b >= (R')a$.

Pirlot (see [8]) proved the following theorems.

Theorem 3. *If a ranking method is monotonic, independent of translations and reversible, the (i)-(iii) are fufilled*

$$(i) \qquad =_M (R) \subseteq = (R),$$
$$(ii) \qquad <_M (R) \supseteq < (R),$$
$$(iii) \qquad <=_M (R) \subseteq <= (R).$$

Theorem 4. *If a ranking method is monotonic, independent of translations and strictly reversible, then (i)-(iii) are fulfilled*

$$(i) \qquad =_M (R) \supseteq = (R),$$
$$(ii) \qquad <_M (R) \subseteq < (R),$$
$$(iii) \qquad <=_M (R) \supseteq <= (R).$$

4. Methods based on t-norms or t-conorms

A ranking method is said to be complete if $<= (R)$ is a complete ranking on A for any fuzzy relation R. Obviously, all methods based on scoring functions are complete.

The leaving flow $L_{ci,R}$ of a $c_i \in A$ is defined by

$$L_{ci,R} = (R(c_i, c_1), R(c_i, c_2), \ldots, R(c_i, c_{i-1}), R(c_i, c_{i+1}), \ldots, R(c_i, c_n)).$$

$L_{a,R}$ and $L_{b,R}$ are said to be equal ($L_{a,R} = L_{b,R}$) if there exists a permutation σ on $L_{a,R}$ such that $R(a, c_k) = \sigma(R(b, c_k))$.

Similarly, the entering flow $E_{ci,R}$ of a $c_i \in A$ is defined by

$$E_{ai,R} = (R(c_1, c_i), R(c_2, c_i), \ldots, R(c_{i-1}, c_i), R(c_{i+1}, c_i), \ldots, R(c_n, c_i)).$$

$E_{a,R}$ and $E_{b,R}$ are said to be equal ($E_{a,R} = E_{b,R}$) if there exists a permutation σ on $E_{a,R}$ such that $R(c_k, a) = \sigma(R(c_k, b))$.

A ranking method $<= (R)$ depends on leaving flow if for all fuzzy relations R on A and $a, b \in A$ $L_{a,R} = L_{b,R}$ implies $a = (R)b$. For example the maximum, the lexicographical maximum, the t-norm and the t-conorm methods depend on the leaving flow.

A ranking method $<=$ depends on flow if for all fuzzy relations R on A and $a, b \in A$ $[L_{a,R} = L_{b,R}$ and $E_{a,R} = E_{b,R}]$ implies $a = (R)b$. Obviously, if a method depends on leaving flow, then it depends on flow.

The following theorems express a basic property of ranking methods that are monotonic, complete and depend on leaving flow. The proofs of these theorems are offered in [3].

Theorem 5. *If a ranking method $<= (R)$ is monotonic, complete and depends on leaving flow, then there exists a scoring function $S(a, R)$ which generates $<= (R)$ by (1).*

The direct consequence of the former theorem is that the lexicographical maximum method is based on a scoring function.

The statement of the former theorem is true in a more general form.

Theorem 6. *If a ranking method $<= (R)$ is monotonic, complete and depends on flow, then there exists a scoring function $S(a, R)$ which generates $<=$ by (1).*

The proof of Theorem 5 constructs a score function which is not necessarily a weak t-norm (or weak t-conorm) of the leaving flow, even if the ranking method is based on a weak t-norm T' (or t-conorm S'). It is obvious that if $T' = \Phi(T)$, where the function $\Phi[0,1] \to [0,1]$ is strictly monotonic, then $<=_T = <=_{T'}$.

The next procedure assigns a weak t-norm T and a weak t-conorm S to a given ranking method $<= (R)$.

Let the point $(s_1, s_2, \dots, s_{n-1}) \in [0,1]^{n-1}$ be fixed. For any $t \in [0,1]$ a fuzzy relation R_t is defined by

$$\begin{aligned}
&R_t(c_1, c_i) = s_{i-1}, && i = 2, 3, \dots, n;\\
&R_t(c_2, c_i) = 1, && i = 1, 3, \dots, n-1;\\
&R_t(c_2, c_n) = t, && \text{and}\\
&R_t(c_k, c_i) = 0, && k = 3, 4, \dots, n;\ \ i = 1, 2, \dots, n;\ \ k \neq n.
\end{aligned}$$

Now we define the function $T:\ [0,1]^{n-1} \to [0,1]$ by

$$T(s_1, s_2, \dots, s_{n-1}) = \inf\{t:\ c_1 <= (R_t)c_2\}. \tag{4}$$

T is well defined because the method $<=$ depends on leaving flow. Let

$$S_T(c_i, R) = T(R(c_i, c_1), R(c_i, c_2), \dots, R(c_i, c_{i-1}), R(c_i, c_{i+1}), \dots, R(c_i, c_n))$$

and $<=_T$ is defined by (1).

A ranking method is said to be strictly 1-monotonic if for all $a, b \in A$ with leaving flows $L_a = \{1, 1, \dots, 1, r\}$ and $L_b = \{1, 1, \dots, 1, s\}$, $r < s$ implies $a < (R)b$.

It is easy to check that the method $<=_T$ is strictly 1-monotonic if and only if the function defined by (5) fulfills $T(1, 1, \dots, 1, t) = t$ for all $t \in [0,1]$. In this case T is a symmetrical weak t-norm.

An S-method can be assigned to a given ranking method in a similar way. Let the point $(s_1, s_2, \dots, s_{n-1}) \in [0,1]^{n-1}$ be fixed. For any $t \in [0,1]$ a fuzzy relation R_t is defined by

$$\begin{aligned}
&R_t(c_1, c_i) = s_{i-1}, && i = 2, 3, \dots, n,\\
&R_t(c_2, c_i) = 0, && i = 1, 3, \dots, n-1,\\
&R_t(c_2, c_n) = t && \text{and}\\
&R_t(c_k, c_i) = 0, && k = 3, 4, \dots, n;\ \ i = 1, 2, \dots, n;\ \ k \neq i.
\end{aligned}$$

Now we define the function $S:\ [0,1]^{n-1} \to [0,1]$:

$$S(s_1, s_2, \dots, s_{n-1}) = \sup\{t:\ \ c_1 >= (R_t)c_2\}. \tag{5}$$

Let

$$S_S(c_i, R) = S(R(c_i, c_1), R(c_i, c_2), \dots, R(c_i, c_{i-1}), R(c_i, c_{i+1}), \dots, R(c_i, c_n)),$$

and $<=_S$ is defined by (1).

A ranking method is said to be strictly 0-monotonic if for all $a, b \in A$ with leaving flows $L_a = \{0, 0, \ldots, 0, r\}$ and $L_b = \{0, 0, \ldots, 0, s\}$, $r < s$ implies $a < (R)b$.

It is easy to check that the method $<=_S$ is strictly 0-monotonic if and only if the function defined by (5) fulfils $S(0, 0, \ldots, 0, t) = t$ for all $t \in [0, 1]$. In this case S is a symmetrical weak t-conorm.

The following theorem characterizes the t- and s-methods. The proof of the theorem is in [3].

Theorem 7. *A ranking method $<=$ is a T-method (or an S-method) if and only if it is monotonic, complete, depends on leaving flow, strictly 1-monotonic (strictly 0-monotonic), moreover if for any fuzzy relation R on A and $a, b \in A$ $a < (R)b$ holds then there exists an alternative c with leaving flow $L_c = \{1, 1, \ldots, 1, t\}$ ($L_c = \{0, 0, \ldots, 0, t\}$ such that*

$$a \; <= (R) \; c \; < (R) \; b \quad (a \; < (R) \; c \; <= (R) \; b).$$

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