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A THEOREM ON THE h-RANGE OF B_h-SEQUENCES

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1. Introduction

A sequence $A_k = \{a_1, a_2, \dots, a_k\}$ of k integers $a_1 < a_2 < \dots < a_k$ is called a B_2-sequence if the sums

$$a_i + a_j, \qquad 1 \le i \le j \le k$$

are all different (cf. [1] p.85, Def.3). A B_h-sequence A_k may similarly be defined as a sequence of k integers $a_1 < a_2 < \dots < a_k$ such that the sums

$$a_{i_1} + a_{i_2} + \dots + a_{i_h}, \qquad 1 \le i_1 \le i_2 \le \dots \le i_h \le k$$

are all different.

For a given h the set of all these sums will be called the h-fold sum and will be denoted by hA, i.e.

$$hA := \left\{ \sum_{i=1}^{k} x_i a_i \;\middle|\; x_i \in N_0, \quad \sum_{i=1}^{k} x_i = h \right\}.$$

The class of all finite and infinite B_h-sequences will be denoted by B_h. An interval $[a, b]$ will be defined as

$$[a, b] := \{m \in Z \mid a \le m \le b\}.$$

Let A_k be a B_h-sequence. Consider the largest interval

$$I_h(A_k) = [l_k, m_k] \subseteq hA_k,$$

the length of $I_h(A_k)$, which is given by $m_k - l_k$, will be referred to as the range of A_k with respect to h and will be denoted by $S_h(A_k)$ (cf. [2]). Furthermore we define

$$S_h(k) := \max_{A_k \in B_h} S_h(A_k).$$

For an arbitrary $h \geq 2$ it is obvious that

$$A_2 = \{0, 1\} \quad \text{is a } B_h\text{–sequence}, \quad \text{hence } S_h(2) \geq h.$$

It is also obvious that

$$A_3 = \{0, 1, h + 1\} \quad \text{is a } B_h\text{–sequence}, \quad \text{hence } S_h(3) \geq 2h.$$

Moreover, we can easily extend A_2 (or A_3) to a B_h-sequence A_k of k elements for any $k \geq 2$. This shows that for any fixed $k \geq 2$ and an arbitrary positive integer l there exist $h \geq 2$ and a B_h-sequence A_k such that

$$S_h(A_k) \geq l.$$

On the other hand we will show in our theorem that given a fixed integer $h \geq 2$ and an arbitrary positive integer l we can find an integer $k \geq 2$ and a B_h-sequence A_k such that

$$S_h(A_k) \geq l.$$

2. Theorem

We prove the following

Theorem. *For arbitrary positive integers* $h \geq 2, \quad k > 2$

$$S_h(k + 2) \geq S_h(k) + 1.$$

Proof. First we notice that if A_k is a B_h-sequence then

$$A_k - a_1 := \{a_i - a_1 \mid a_i \in A_k\} \subseteq N_0$$

is a B_h-sequence which contains 0 and $S_h(A_k - a_1) = S_h(A_k)$. Now let $A_k = \{a_1, a_2, \ldots, a_k\}$, $0 = a_1 < a_2 < \ldots < a_k$, $k > 2$ be a B_h-sequence such that $S_h(A_k) = S_h(k)$.

We define

(1) $$a_{k+1} := (h+2)a_k \quad \text{and} \quad a_{k+2} := m_k + 1 - (h-1)a_{k+1},$$

from which it follows that

(2) $$a_{k+2} + (h-1)a_{k+1} = m_k + 1$$

and that

(3) $$a_{k+2} = m_k + 1 - (h^2 + h - 2)a_k.$$

But

(4) $$m_k + 1 \notin hA_k$$

and

(5) $$0 \leq m_k \leq ha_k, \qquad \text{since} \quad m_k \in hA_k.$$

As a consequence of (3) and (5) we get

(6) $$-(h^2 + h - 2)a_k + 1 \leq a_{k+2} \leq -(h^2 - 2)a_k + 1.$$

Now let $A_{k+2} := A_k \cup \{a_{k+1}, a_{k+2}\}$. Then $[l_k, m_k + 1] \subseteq hA_{k+2}$ by (2), and it suffices to show that A_{k+2} is a B_h-sequence. We observe that any element in hA_{k+2} can be written in the form

$$S_x = x_1 a_{k+1} + x_2 a_{k+2} + S_{x_3}, \quad \text{where} \quad x_i \in N_0, \quad x_1 + x_2 + x_3 = h$$

and

$$S_{x_3} = a_{i_1} + a_{i_2} + \ldots + a_{i_{x_3}} \in x_3 A_k, \qquad 1 \leq i_1 \leq i_2 \leq \ldots \leq i_{x_3} \leq k.$$

Hence

(7) $$0 \leq S_{x_3} \leq x_3 a_k.$$

Let $S_y = y_1 a_{k+1} + y_2 a_{k+2} + S_{y_3}$ be another element in hA_{k+2}, where

$$S_{y_3} = a_{j_1} + a_{j_2} + \ldots + a_{j_{y_3}} \in y_3 A_k, \qquad 1 \leq j_1 \leq j_2 \leq \ldots \leq j_{y_3} \leq k.$$

Since A_k is a B_h-sequence and $0 \in A_k$, it follows that if $S_{x_3} = S_{y_3}$, then $x_3 = y_3$ and $a_{i_l} = a_{j_l}$ for $1 \leq l \leq x_3$, i.e.

$$(a_{i_1}, \ldots, a_{i_{x_3}}) = (a_{j_1}, \ldots, a_{j_{y_3}}).$$

Now we shall prove that if $S_x = S_y$, then $x_1 = y_1$, $x_2 = y_2$ and hence $S_{x_3} = S_{y_3}$, i.e. A_{k+2} is a B_h-sequence. Consider the following cases:

(I) $x_2 = y_2$

Since $S_x = S_y$, it follows that $x_1 a_{k+1} + S_{x_3} = y_1 a_{k+1} + S_{y_3}$. Hence $(x_1 - y_1)a_{k+1} = S_{y_3} - S_{x_3}$. If $x_1 \neq y_1$, say $x_1 > y_1$, i.e. $(x_1 - y_1) \geq 1$, then we get

$$(x_1 - y_1)a_{k+1} \geq a_{k+1} = (h+2)a_k$$

by definition (1), and

$$S_{y_3} - S_{x_3} \leq y_3 a_k \leq h a_k$$

by (7), which is a contradiction.

(II) $x_2 \neq y_2$, *say* $x_2 > y_2$, *i.e.* $(x_2 - y_2) \geq 1$

In this case we must have $x_2 \geq 1$ and hence $x_1 + x_3 \leq h - 1$. We consider two subcases:

(IIa) $x_1 = h - 1$

In this subcase $x_3 = 0$, $x_2 = 1$ and $y_2 = 0$. Thus $(h-1)a_{k+1} + a_{k+2} = = y_1 a_{k+1} + S_{y_3}$ since $S_x = S_y$, and hence $m_k + 1 = y_1 a_{k+1} + S_{y_3}$ by (2). This is impossible since

1. if $y_1 = 0$, it would follow that $m_k + 1 = S_{y_3} \in hA_k$ which contradicts (4);

2. if $y_2 \geq 1$, then we would have $m_k + 1 < y_1 a_{k+1} + S_{y_3}$ since $m_k \leq h a_k$ from (5) and $a_{k+1} = (h+2)a_k$ by definition (1).

(IIb) $x_1 \leq h - 2$

$S_x = S_y$ means that $x_1 a_{k+1} + x_2 a_{k+2} + S_{x_3} = y_1 a_{k+1} + y_2 a_{k+2} + S_{y_3}$, hence

$$(x_2 - y_2)a_{k+2} = (y_1 a_{k+1} + S_{y_3}) - (x_1 a_{k+1} + S_{x_3}). \tag{8}$$

It follows from (6) that

$$\text{(9)} \qquad \text{L.H.S. of (8)} \le a_{k+2} \le -(h^2-2)a_k + 1, \quad \text{since } \; x_2 - y_2 \ge 1$$

and a_{k+2} is negative.

Again this is impossible since:

1. if $x_1 = h-2$, then we would have $x_3 \le 1$ and since $y_1 a_{k+1} + S_{y_3} \ge 0$ we would get

$$\begin{aligned}\text{R.H.S. of (8)} &\ge -(x_1 a_{k+1} + S_{x_3}) = -((h-2)(h+2)a_k + S_{x_3}) \ge \\ &\ge -((h^2-4)a_k + a_k) \quad \text{by (7) since } \; x_3 \le 1,\end{aligned}$$

i.e. R.H.S. of (8)$\ge -(h^2-3)a_k$ which contradicts (9) since $a_k > 1$;

2. if $x_1 \le h-3$, then we would get

$$\begin{aligned}\text{R.H.S. of (8)} &\ge -(x_1 a_{k+1} + S_{x_3}) \ge -((h-3)(h+2)a_k + S_{x_3}) \ge \\ &\ge -((h^2-h-6)a_k + (h-1)a_k)\end{aligned}$$

by (7) since $x_3 \le h-1$, i.e. R.H.S. of (8)$\ge -(h^2-7)a_k$ which contradicts (9). This completes the proof of the theorem.

References

[1] **Halberstam H. and Roth K.F.**, *Sequences I.*, Oxford Univ. Press, Oxford, 1966.

[2] **Hofmeister G.**, Eine Verallgemeinerung des Reichweitenproblems, *Abhdlg. d.Braunschweig. Wiss. Gs.*, **XXXIII** (1982), 161-163.

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