Annales Univ. Sci. Budapest., Sect. Comp. 15 (1995) 49-63

APPROXIMATE SOLUTION OF THE INITIAL VALUE PROBLEM y'''=f(x,y) USING DEFICIENT SPLINE POLYNOMIAL

Th. Fawzy and M. Ahmed (Ismailia, Egypt)

1. Introduction

The purpose of this paper is to investigate the existence, uniqueness, consistency relations and convergence for approximating the solution of the third order initial value problem

$$(1.1)\qquad \begin{aligned} &y''' = f(x,y), \quad y(0) = y_o, \quad y'(0) = y'_o \quad \text{and} \\ &y''(0) = y''_o, \end{aligned}$$

using a spline of degree m and continuity class C^{m-3} and a step of length $H = 3h$. We deal with the interval $[0, b]$ without any loss of generality.

We will discuss a new method for the numerical solution of the Cauchy problem (1.1), where $f \in C^{m-2}([0,b] \times R)$ in some domain T, $T = \{(x,y),\ 0 \leq x \leq b\}$ and which satisfies a Lipschitz condition

$$|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|,$$

where L is a Lipschitz constant, I_k is the interval $[kH,(k+1)H]$ with $k = 0(1)N-1$ and $h = b/3N$.

We define the spline function $s(x)$, $x \in I_k$, by

$$(1.2)\qquad s(x) = \sum_{i=0}^{m-3} \frac{s_{3k}^{(i)}}{i!}(x-3kh)^i + \sum_{i=m-2}^{m} \frac{C_{i,k}}{i!}(x-3kh)^i$$

and

$$(1.3)\qquad s_{3k}^{(i)} = s^{(i)}(3kh), \qquad i = 0(1)m-3,$$

where the coefficients $C_{i,k}$ are determined so as to satisfy the equation

$$(1.4) \qquad s^{(3)}(jh) = f(jh, s(jh)), \qquad j = 3k + \varepsilon, \qquad \varepsilon = 1(1)3.$$

Put $m = 6$ in equation (1.2), then we get

$$(1.5) \qquad s(x) = \sum_{i=0}^{3} \frac{s_{3k}^{(i)}}{i!}(x - 3kh)^i + \sum_{i=4}^{6} \frac{C_{i,k}}{i!}(x - 3kh)^i.$$

Setting $x = jh$, $j = 3k + \varepsilon, \varepsilon = 1(1)3$, we get

$$(1.6) \qquad s(jh) = \sum_{i=0}^{3} \frac{s_{3k}^{(i)}}{i!}(jh - 3kh)^i + \sum_{i=4}^{6} \frac{C_{i,k}}{i!}(jh - 3kh)^i.$$

2. Existence and uniqueness

In this section we prove that under certain conditions there exists a unique spline polynomial $s(x)$ approximating the solution $y(x)$ of (1.2)-(1.3). For this purpose, we state and prove the first theorem of this section.

Theorem 2.1. *If $h = \min\{h_1, h_2, h_3\}$ then the spline function $s(x)$, given in (1.5), exists and is unique, where*

$$h_1 < \sqrt[3]{4/3L}, \qquad h_2 < 1/\sqrt[3]{L}, \qquad h_3 < \sqrt[3]{4/L}.$$

Proof. To prove the uniqueness it is enough to show that the coefficients are uniquely determined. In the interval I_k $s(x)$ is defined by (1.5). Thus, differentiating equation (1.5) and setting $j = 3k + \varepsilon$, $\varepsilon = 1(1)3$, we obtain

$$(2.1) \qquad s_{3k+\eta}^{(3)} = s_{3k}^{(3)} + \sum_{i=4}^{6} \frac{(\eta h)^{i-3}}{(i-3)!} C_{i,k}, \qquad \eta = 1(1)3.$$

By using the elimination method, the coefficients $C_{i,k}$, $i = 4, 5, 6$ in (2.1) are

$$C_{6,k} = \frac{1}{h^3}[s^{(3)}_{3k+3} - s^{(3)}_{3k} + 3s^{(3)}_{3k+1} - 3s^{(3)}_{3k+2}], \tag{2.2}$$

$$C_{5,k} = \frac{1}{h^2}[4s^{(3)}_{3k+2} + 2s^{(3)}_{3k} - 5s^{(3)}_{3k+1} - s^{(3)}_{3k+3}] \tag{2.3}$$

and

$$C_{4,k} = \frac{1}{h}\left[3s^{(3)}_{3k+1} - \frac{11}{6}s^{(3)}_{3k} - \frac{3}{2}s^{(3)}_{3k+2} + \frac{1}{3}s^{(3)}_{3k+3}\right]. \tag{2.4}$$

From equation (2.2) the coefficient $C_{6,k}$ will have the form

$$C_{6,k} = \frac{1}{h^3}[f((3k+3)h, s((3k+3)h)) - s^{(3)}_{3k} - 3f((3k+2)h, s((3k+2)h)) +$$

$$+3f((3k+1)h, s((3k+1)h))] = g_1(C_{6,k}). \tag{2.5}$$

Equation (1.6) takes another form

$$s(jh) = A(jh) + \sum_{i=4}^{6} \frac{C_{i,k}}{i!}(jh - 3kh)^i, \tag{2.6}$$

where

$$A(jh) = \sum_{i=0}^{3} \frac{s^{(i)}_{3k}}{i!}(jh - 3kh)^i, \quad j = 3k + \varepsilon, \quad \varepsilon = 1(1)3.$$

Define G_k: $R \to R$ by $C_{j,k} \to g_\nu(C_{j,k})$, $C_{j,k} \in R$, $j = 4, 5, 6$ and $\nu = 1, 2, 3$.

We show that under conditions of the theorem operator G_k is a contraction, thus having a unique fixed point, from equation (2.6) the constant $C_{6,k}$ will be

$$\begin{aligned} C_{6,k} = \frac{1}{h^3}\Bigg[& f\left((3k+3)h, A((3k+3)h) + \sum_{i=4}^{6} \frac{C_{i,k}}{i!}(3h)^i\right) - \\ & - 3f\left((3k+2)h, A((3k+2)h) + \sum_{i=4}^{6} \frac{C_{i,k}}{i!}(2h)^i\right) + \\ & + 3f\left((3k+1)h, A((3k+1)h) + \sum_{i=4}^{6} \frac{C_{i,k}}{i!}h^i\right) - s^{(3)}_{3k}\Bigg] = \\ = g_1(C_{6,k}). \end{aligned} \tag{2.7}$$

Let $C_{j,k}, C^*_{j,k} \in R$, $j = 4, 5, 6$ and their distance

$$\rho(C_{j,k}, C^*_{j,k}) = |C_{j,k} - C^*_{j,k}|, \qquad j = 4, 5, 6.$$

According to the Lipschitz condition and equation (2.7) it follows that

$$\rho(G_1(C_{6,k}), G_1(C^*_{6,k})) = |g_1(C_{6,k}) - g_1(C^*_{6,k})| \leq \frac{3}{4}h^3 L \rho(C_{6,k}, C^*_{6,k}).$$

If $\frac{3}{4}h^3 L < 1$, then

$$h = h_1 < \sqrt[3]{\frac{4}{3L}}. \tag{2.8}$$

Similarly, h_2 and h_3 for the constants $C_{5,k}$ and $C_{4,k}$ respectively

$$h = h_2 < \sqrt{\frac{1}{3L}}, \quad h = h_3 < \sqrt[3]{\frac{4}{L}}.$$

Let $h = \min\{h_1, h_2, h_3\}$, then follows that $G_k, k = 1(1)3$ is a contraction operator and hence equation (1.5) has a unique solution. This completes the proof.

3. Consistency relations

In the present section we deal with the question of consistency of the procedure so as to assure its convergence. Hence, we would have to establish necessary and sufficient conditions for the method to be convergent. Let us state this lemma.

Lemma 3.1. *The spline function $s(x)$ given in (1.5) is consistent.*

Proof. In equation (1.6) put $j = 3k + \varepsilon$, $\varepsilon = 1(1)3$, then we get

$$s_{3k+3} = \sum_{i=0}^{3} \frac{s^{(i)}_{3k}}{i!}(3h)^i + M_1, \tag{3.1}$$

$$s_{3k+2} = \sum_{i=0}^{3} \frac{s^{(i)}_{3k}}{i!}(2h)^i + M_2 \tag{3.2}$$

and

$$(3.3)\qquad s_{3k+1} = \sum_{i=0}^{3} \frac{s_{3k}^{(i)}}{i!}(h)^i + M_3,$$

where

$$(3.4)\qquad M_j = \sum_{i=4}^{6} \frac{C_{i,k}}{i!}((4-j)h)^i, \quad j = 1,2,3.$$

From equation (3.2) and (3.3)

$$(3.5)\qquad s''_{3k} = \frac{1}{h^2}(s_{3k+2} - 2s_{3k+1} + s_{3k} - s_{3k}^{(3)}h^3 + 2M_3 - M_2)$$

and

$$(3.6)\qquad s'_{3k} = \frac{1}{h}\left(2s_{3k+1} - \frac{1}{2}s_{3k+2} - \frac{3}{2}s_{3k} + \frac{1}{3}h^3 s_{3k}^{(3)} - 2M_3 + \frac{1}{2}M_2\right).$$

Substituting equations (3.5) and (3.6) in (3.1) and by using (3.4) we get

$$(3.7)\qquad s_{3k+3} - 3s_{3k+2} + 3s_{3k+1} - s_{3k} = h^3 s_{3k}^{(3)} + \frac{1}{4}[6h^4 C_{4,k} - 5h^5 C_{5,k} + 3h^6 C_{6,k}].$$

Substituting equations (2.2), (2.3) and (2.4) in equation (3.7) we get

$$s_{3k+3} - 3s_{3k+2} + 3s_{3k+1} - s_{3k} = h^3\left[\frac{52}{4}f_{3k+1} - \frac{38}{4}f_{3k+2} + \frac{10}{4}f_{3k+3} - \frac{20}{4}f_{3k}\right].$$

Hence the associative polynomials $\rho(\xi)$ and $\sigma(\xi)$ are

$$\rho(\xi) = \xi^3 - 3\xi^2 + 3\xi - 1$$

and

$$\sigma(\xi) = \frac{10}{4}\xi^3 - \frac{38}{4}\xi^2 + \frac{52}{4}\xi - \frac{20}{4}.$$

Obviously,

$$\rho(1) = \rho^{(1)}(1) = \rho^{(2)}(1) = 0, \quad \rho^{(3)}(1) = 3!\sigma(1).$$

Hence, the method is constant for $m = 6$ and the condition of stability is fulfilled, since the roots of $\rho(\xi)$ lie on the unit circle and are of multiplicity 3; thus the method is convergent at $m = 6$.

4. Convergence of spline approximant

We are now in a position to measure the order of convergence of the method. To begin with, let us introduce the following

Lemma 4.1. *If* $\|s(jh) - y(jh)\| < kh^p$ *and* $s^{(3)}(jh) = f(jh, s(jh))$ *for* $j = 3k + \varepsilon$, $\varepsilon = 1(1)3$ *and* $k = 0(1)N - 1$, *then there exists a constant* k^* *such that*

$$\|s(jh) - y(jh)\| < k^* h^p \quad \text{and} \quad \|s^{(3)}(jh) - y^{(3)}(jh)\| < k^* h^p.$$

Proof. This is an immediate consequence of the Lipschitz condition.

Lemma 4.2. *Let* $y \in C^{m+1}[0, b]$ *and* $s(x)$ *be the spline function of degree* m *having its knots at the points* x_{3k}, $k = 1(1)n - 1$ *defined above such that the conditions*

$$(4.1) \qquad \|s^{(r)}(x_{3k}) - y^{(r)}(x_{3k})\| = O(h^{p_r}), \qquad r = 0(1)m - 3$$

$$(4.2) \qquad \|C_{m-i,k} - y^{(m-i)}(x_{3k})\| = O(h^{p_{m-i}}), \quad i = 1, 2$$

and

$$(4.3) \qquad \|s^{(m-i)}(x) - y^{(m-i)}(x)\| = O(h^{i+1}), \quad i = 0(1)2, x_{3k} < x < x_{3k+3}, \; k = 0(1)N - 1$$

are satisfied. Then,

$$(4.4) \qquad \|s(x) - y(x)\| = O(h^p), \quad x \in [0, b],$$

where

$$(4.5) \qquad p = \min_{r=0(1)m} (r + p_r), \quad p_{m-i} = i + 1, \quad i = 0(1)2.$$

Furthermore,

$$(4.6) \qquad \|s^{(m-i)}(x) - y^{(m-i)}(x)\| = O(h^{i+1}), \quad i = 0(1)2, \quad x \in [0, b].$$

Proof. The proof is by induction. Let $x_{3k} < x < x_{3k+3}$ and expanding by Taylor's theorem with $\omega = x - x_{3k} \leq 3h$ we obtain

$$(4.7) \qquad y(x) = \sum_{r=0}^{m-1} \frac{\omega^r}{r!} y^{(r)}(x_{3k}) + \frac{\omega^m}{m!} y^{(m)}(\xi_1), \qquad x_{3k} < \xi_1 < x$$

and

$$(4.8)\qquad s(x)=\sum_{r=0}^{m-3}\frac{\omega^r}{r!}s^{(r)}(x_{3k})+\sum_{r=1}^{2}\frac{\omega^{m-r}}{(m-r)!}C_{m-r,k}+\frac{\omega^m}{m!}s^{(m)}(\xi_2).$$

Note that $s^{(m)}(x)$ is constant for $x_{3k}<x<x_{3k+3}$. Subtracting (4.7) from (4.8) and since $\omega=x-x_{3k}\leq 3h$, we obtain

$$\begin{aligned}\|s(x)-y(x)\|\leq&\sum_{r=0}^{m-3}\frac{(3h)^r}{r!}\|s^{(r)}(x_{3k})-y^{(r)}(x_{3k})\|+\\&+\sum_{r=1}^{2}\frac{(3h)^{m-r}}{(m-r)!}\|C_{m-r,k}-y^{(m-r)}(x_{3k})\|+\\&+\frac{(3h)^m}{m!}\|s^{(m)}(\xi_2)-y^{(m)}(\xi_1)\|.\end{aligned}$$

In view of (4.1), (4.2) and (4.3), this establishes (4.4). To prove equation (4.6) it is sufficient by virtue of equation (4.4) to consider the nodal points x_{3k}, $k=1(1)N-1$. By equation (4.3) and by the usual arithmetic mean, we define the functions $s^{(m-i)}(x)$ which are piecewise continuous on $[0,b]$ as

$$s^{(m-i)}(x_{3k})=\frac{1}{2}\left[s^{(m-i)}\left(x_{3k}-\frac{1}{2}h\right)+s^{(m-i)}\left(x_{3k}+\frac{1}{2}h\right)\right],$$
$$k=1(1)N-1,i=0(1)2.$$

Since

$$y^{(m-i)}\left(x_{3k}-\frac{1}{2}h\right)=y^{(m-i)}(x_{3k})-\frac{1}{2}hy^{(m+1-i)}(\xi_1),\ x_{3k}-\frac{h}{2}<\xi_1<x_{3k},$$
$$y^{(m-i)}\left(x_{3k}+\frac{1}{2}h\right)=y^{(m-i)}(x_{3k})+\frac{1}{2}hy^{(m+1-i)}(\xi_2),\ x_{3k}<\xi_2<x_{3k}+\frac{1}{2}h$$

and consequently,

$$s^{(m-i)}(x_{3k})=y^{(m-i)}(x_{3k})+O(h^{i+1})$$

and then

$$\|s^{(m-i)}(x)-y^{(m-i)}(x)\|=O(h^{i+1}),\quad i=0(1)2,\quad x\in[0,b],$$

which completes the proof.

Lemma 4.3. *Let $s(x)$ be the spline function defined in (1.6) and let $m = 6$, then there exists a constant K such that*

$$\|s(jh) - y(jh)\| < Kh^7, \qquad j = 1(1)3.$$

Proof. For $x \in [0, 3h]$ the equation (1.5) at $m = 6$ becomes

$$s(x) = \sum_{i=0}^{3} \frac{x^i}{i!} y_o^{(i)} + \sum_{i=4}^{6} \frac{x^i}{i!} C_{i,0}, \quad x = jh, \quad j = 1(1)3. \tag{4.9}$$

From the Taylor expansion of order 7 we have

$$y(jh) = \sum_{i=0}^{6} \frac{(jh)^i}{i!} y_0^{(i)} + \frac{(jh)^7}{7!} y^{(7)}(\xi), \quad j = 1(1)3, \quad 0 < \xi < 3h. \tag{4.10}$$

From equations (4.9) and (4.10)

$$s(jh) - y(jh) = \sum_{i=4}^{6} \frac{(jh)^i}{i!} \left(C_{i,0} - y_o^{(i)} \right) - \frac{(jh)^7}{7!} y^{(7)}(\xi), \quad 0 < \xi < 3h. \tag{4.11}$$

Put $k = 0$ in equation (2.7), then we get

$$\begin{aligned} C_{6,0} = \frac{1}{h^3} \Bigg[& f\left(3h, A(3h) + \sum_{i=4}^{6} \frac{C_{i,0}}{i!}(3h)^i\right) - 3f\Big(2h, A(2) + \\ & + \sum_{i=4}^{6} \frac{C_{i,0}}{i!}(2h)^i\Big) + 3f\left(h, A(h) + \sum_{i=4}^{6} \frac{C_{i,0}}{i!} h^i\right) - \\ & - s_o^{(3)} \Bigg] = g_1(C_{6,0}). \end{aligned} \tag{4.12}$$

Similarly $C_{5,0}$ and $C_{4,0}$ are given by

$$\begin{aligned} C_{5,0} = \frac{1}{h^2} \Bigg[& 4f\left(2h, A(2h) + \sum_{i=4}^{6} \frac{C_{i,0}}{i!}(2h)^i\right) - 5f\Big((h, A(h) + \\ & + \sum_{i=4}^{6} \frac{C_{i,0}}{i!}(h)^i\Big) - f\left(3h, A(3h) + \sum_{i=4}^{6} \frac{C_{i,0}}{i!}(3h)^i\right) + \\ & + 2s_o^{(3)} \Bigg] = g_2(C_{5,0}). \end{aligned} \tag{4.13}$$

and

$$C_{4,0} = \frac{1}{h}\left[3f\left(h, A(h) + \sum_{i=4}^{6}\frac{C_{i,0}}{i!}h^i\right) - \frac{3}{2}f\left(2h, A(2h) + \sum_{i=4}^{6}\frac{C_{i,0}}{i!}h^i\right) - \right.$$

$$(4.14) \qquad \left. -\frac{1}{3}f\left(3h, A(3h) + \sum_{i=4}^{6}\frac{C_{i,0}}{i!}(3h)^i\right) - \frac{11}{6}s_o^{(3)}\right] = g_3(C_{4,0}),$$

where $A(\nu h) = \sum_{i=0}^{3}\frac{s_o^{(3)}}{i!}(\nu h)^i$, $\nu = 1(1)3$.

The proof of the lemma is reduced to showing that $C_{4,0}$, $C_{5,0}$ and $C_{6,0}$ are uniformly bounded as $h \to 0$.

The function $g_1(C_{6,0})$ is a contraction if $h < \sqrt[3]{4/3L}$. In particular for $h < \sqrt[3]{1/3L}$ we have

$$|g_1(C_{6,0}) - g_1(C^*_{6,0})| < \frac{3}{4}h^3L\rho(C_{6,0}, C^*_{6,0}) < \frac{1}{4}|C_{6,0} - C^*_{6,0}|.$$

Taking $C^*_{6,0} = 0$, we obtain

$$|g_1(C_{6,0})| - |g_1(0)| \leq |g_1(C_{6,0}) - g_1(0)| < \frac{3}{4}h^3L|C_{6,0}| < \frac{1}{4}|C_{6,0}|.$$

Put $g_1(C_{6,0}) = C_{6,0}$, then we have

$$|C_{6,0}| - |g_1(0)| \leq \frac{1}{4}|C_{6,0}|$$

and

$$|C_{6,0}| < \frac{4}{3}|g_1(0)|.$$

From equations (4.10) and (4.12)

$$g_1(0) = \frac{1}{h}[3y^{(3)}(h) - 3y^{(3)}(2h) + y^{(3)}(3h) - y_o^{(3)} + O(h^4)] \leq M.$$

For some constant M, since uniform spacing is required over the interval $[0, b]$, there is only a finite number of possible values of h between $\sqrt[3]{1/3L}$ and $\sqrt[3]{4/3L}$, so that $C_{6,0}$ is uniformly bounded for all $h < \sqrt[3]{4/3L}$.

Similarly, $C_{5,0}$ and $C_{4,0}$ are uniformly bounded for all $h < \sqrt[3]{1/L}$ and $h < \sqrt[3]{4/L}$ respectively as $h \to 0$.

Since $C_{4,0}, C_{5,0}$ and $C_{6,0}$ are uniformly bounded as $h \to 0$, from equation (4.11) we get

$$s(jh) - y(jh) = O(h^4).$$

Our purpose is to improve this inequality as follows:

Applying Lemma 4.1 with $p = 2$ and from equation (4.9)

$$y_o^{(3)} + (jh)C_{4,0} + \frac{(jh)^2}{2!}C_{5,0} + \frac{(jh)^3}{3!}C_{6,0} - y^{(3)}(jh) = O(h^2), \quad j = 1(1)3.$$

Thus, from the Taylor expansion we can obtain

$$||C_{4,0} - y_o^{(4)}|| = O(h).$$

Thus equation (4.11) becomes

$$||s(jh) - y(jh)|| = O(h^5), \qquad j = 1(1)3. \tag{4.15}$$

Applying Lemma 4.1 with $p = 3$ and by using the same steps, we get

$$||C_{4,0} - y_0^{(4)}|| = O(h^2) \quad \text{and} \quad ||C_{5,0} - y_0^{(5)}|| = O(h)$$

and thus

$$||s(jh) - y(jh)|| = O(h^6), \qquad j = 1, 2, 3.$$

Applying Lemma 4.1 with $p = 4$ and by using the same steps as before, we come to

$$||C_{4,0} - y_o^{(4)}|| = O(h^3),$$

$$||C_{5,0} - y_o^{(5)}|| = O(h^2),$$

and

$$||C_{6,0} - y_o^{(6)}|| = O(h).$$

From equation (4.11)

$$||s(jh) - y(jh)|| = O(h^7), \qquad j = 1, 2, 3$$

which completes the proof of the lemma.

Lemma 4.3 shows that the starting value $s(3h)$ has error $O(h^7)$, and the following relations hold by Lemma 4.1 with $p = 7$

$$s(jh) = y(jh) + O(h^7), \tag{4.16}$$

and

$$s^{(3)}(jh) = y^{(3)}(jh) + O(h^7), \quad j = 3k + \varepsilon, \quad \varepsilon = 1(1)3.$$

In the sequel we shall prove that $s(x)$ and its derivatives converge to $y(x)$ and its derivatives. Thus we state the following

Theorem 4.1. *Let $f \in C^4([0,b] \times R)$, then*

$$||s^{(i)}(x) - y^{(i)}(x)|| = O(h^{7-i}), \qquad i = 4,5,6.$$

Proof. By using equations (1.5) and (4.16), the coefficients $C_{4,k}$, $C_{5,k}$ and $C_{6,k}$ are

$$C_{4,k} = \frac{1}{h^4}\left[\frac{24}{81}y_{3k+3} - \frac{9}{2}y_{3k+2} + 72y_{3k+1} - \frac{575}{9}hy_{3k}^{(1)} - \frac{85}{3}h^2y_{3k}^{(2)} - \frac{3661}{54}y_{3k} - \frac{22}{3}h^3y_{3k}^{(3)}\right] + O(h^3), \tag{4.17}$$

$$C_{5,k} = \frac{1}{h^5}\left[-\frac{20}{9}y_{3k+3} + 30y_{3k+2} - 300y_{3k+1} + \frac{740}{3}hy_{3k}^{(1)} + 100y_{3k}^{(2)}h^2 + 20h^3y_{3k}^{(3)} + \frac{2450}{9}y_{3k}\right] + O(h^2), \tag{4.18}$$

and

$$C_{6,k} = \frac{1}{h^6}\left[\frac{40}{9}y_{3k+3} - 45y_{3k+2} + 360y_{3k+1} - \frac{850}{3}hy_{3k}^{(1)} - 110y_{3k}^{(2)}h^2 - 20y_{3k}^{(3)}h^3 - \frac{2875}{9}y_{3k}\right] + O(h). \tag{4.19}$$

Clearly, $y_{3k+i}, i = 1(1)3$ can take the form

$$y_{3k+i} = \sum_{r=0}^{6} \frac{(ih)^r}{r!}y_{3k}^{(r)} + O(h^7), \qquad i = 1(1)3. \tag{4.20}$$

Using equation (4.20), (4.17), (4.18) and (4.19) we get respectively

$$\begin{aligned} C_{4,k} &= y_{3k}^{(4)} + O(h^3), \\ C_{5,k} &= y_{3k}^{(5)} + O(h^2), \\ C_{6,k} &= y_{3k}^{(6)} + O(h). \end{aligned} \tag{4.21}$$

By using Taylor's expansion of order 7 to equations (4.21) we get

$$
\begin{aligned}
&C_{4,k} = \sum_{i=0}^{2} \frac{(x_{3k}-x)^i}{i!} y^{(4+i)}(x) + \frac{(x_{3k}-x)^i}{3!} y^{(7)}(\xi) + O(h^3), \\
(4.22)\quad &C_{5,k} = \sum_{i=0}^{1} \frac{(x_{3k}-x)^i}{i!} y^{(5+i)}(x) + \frac{(x_{3k}-x)^2}{2!} y^{(7)}(\xi) + O(h^2), \\
&C_{6,k} = y^{(6)}(x) + (x_{3k}-x) y^{(7)}(\xi) + O(h), \\
&\qquad\qquad \xi \in (x_{3k}, x), \quad |x - x_{3k}| < 3h.
\end{aligned}
$$

From equation (1.5) and assuming that $x \in (x_{3k}, x_{3k+3})$

$$
(4.23)\qquad \left\{
\begin{aligned}
&s^{(4)}(x) = C_{4,k} + (x - x_{3k})C_{5,k} + \frac{(x-x_{3k})}{2!} C_{6,k} \\
&s^{(5)}(x) = C_{5,k} + (x - x_{3k})C_{6,k} \\
&s^{(6)}(x) = C_{6,k}.
\end{aligned}
\right\}
$$

Using (4.23) and (4.22) we obtain

$$
\begin{aligned}
s^{(4)}(x) - y^{(4)}(x) &= O(h^3), \\
s^{(5)}(x) - y^{(5)}(x) &= O(h^2),
\end{aligned}
$$

and

$$
s^{(6)}(x) - y^{(6)}(x) = O(h)
$$

which completes the proof.

Theorem 4.2. *Let $f \in C^4([0,b] \times R)$, then there exist constants K_i such that $||s^{(i)}(x) - y^{(i)}(x)|| < K_i h^{7-i}$, $i = 0(1)3$, $x \in [0,b]$.*

Proof. Let $i = 0$, $p = 7$ and $m = 6$ in equations (4.4) and (4.6), then

$$
||s(x) - y(x)|| = O(h^7).
$$

To prove the theorem 4.2 at $i = 1, 2, 3$, let us consider the following

Lemma 4.4. *Assume further that the following conditions are satisfied*

$$
(4.24)\qquad ||s^{(r)}(x_{3k}) - y^{(r)}(x_{3k})|| = O(h^{m-r+1}), \quad r = 1(1)m-3, k = 0(1)n-1
$$

$$
(4.25)\qquad ||C_{m-i,k} - y^{(m-i)}(x_{3k})|| = O(h^{i+1}), \; i = 1, 2
$$

and

(4.26) $$||s^{(m-i)}(x) - y^{(m-i)}(x)|| = O(h^{i+1}), i = 0, 1, 2,$$
$$x_{3k} < x < x_{3k+3}, k = 0(1)N-1.$$

Then,

(4.27) $$||s^{(i)}(x) - y^{(i)}(x)|| = O(h^m),\ i = 1, 2 \text{ and } 3.$$

Proof. To prove Lemma 4.4 let $m = 6$ in equations (4.7) and (4.8). Then we get

(4.28) $$y(x) = \sum_{r=0}^{5} \frac{\omega^r}{r!} y^{(r)}(x_{3k}) + \frac{\omega^6}{6!} y^{(6)}(\xi),\ x_{3k} < \xi < x$$

and

(4.29) $$s(x) = \sum_{r=0}^{3} \frac{\omega^r}{r!} s^{(r)}(x_{3k}) + \sum_{r=1}^{2} \frac{\omega^{6-r}}{(6-r)!} C_{6-r,k} + \frac{\omega^6}{6!} s^{(6)}(\xi),$$
$$x_{3k} < \xi < x.$$

Subtract the first derivative of equation (4.29) from the first derivative of equation (4.28), then we have

$$\begin{aligned} ||s^{(1)}(x) - y^{(1)}(x)|| \le & \sum_{r=1}^{3} \frac{\omega^{r-1}}{(r-1)!} ||s^{(r)}(x_{3k}) - y^{(r)}(x_{3k})|| + \\ & + \sum_{r=1}^{2} \frac{\omega^{5-r}}{(5-r)!} ||C_{6-r,k} - y^{(6-r)}(x_{3k})|| + \\ & + \frac{\omega^5}{5!} ||s^{(5)}(\xi) - y^{(5)}(\xi)||, \end{aligned}$$

where $\omega = (x - x_{3k}) \le 3h$.

By using equations (4.27), (4.28) and (4.29) we get

$$||s^{(1)}(x) - y^{(1)}(x)|| = O(h^6).$$

Similarly the proof could be easily completed for $i = 2$ and 3.

Theorem 4.3. *Let $m = 6$ in equation (1.6), then*

$$||s^{(i)}(x) - y^{(i)}(x)|| = O(h^{7-i}), \quad i = 0(1)6.$$

Proof. From Theorems 4.1 and 4.2 the proof is clear.

References

[1] **Lénárd M. and Székelyhidi L.**, Functional-differential equations by spline functions, *Annales Univ. Sci. Bud. Sect. Comp.*, **3** (1982), 25-32.

[2] **Loscalso F.R. and Talbot T.D.**, Spline function approximations for solutions of ordinary differential equations, *SIAM J. Numer. Anal.*, **4** (3) (1967), 433-445.

[3] **Loscalso F.R.**, An introduction to the application of spline functions to initial value problems, in *"Theory and Applications of Spline Functions"*, Academic Press, New York, 1969, 37-64.

[4] **Micula Gh.**, Approximate solution of the differential equation $y'' = f(x,y)$ with spline functions, *Mat. of Computation*, **27** (124) (1973), 807-816.

[5] **Mülthei H.N.**, Splineapproximationen von beliebigem Defekt zur numerischen Lösung von gewöhnlichen Differentialgleichungen, Teil I-III., *Numer. Math.*, **32** (1979), 147-157, 343-358, **34** (1980), 143-154.

[6] **Sallam S. and El-Hawary H.M.**, A deficient spline function approximation to systems of first-order differential equations, *Appl. Math. Modelling*, **7** (5) (1983), 385-392.

[7] **Sallam S. and El-Hawary H.M.**, A deficient spline function approximation to systems of first-order differential equations II., *Appl. Math. Modelling*, **8** (2) (1984), 128-132.

[8] **Fawzy Th.**, Spline functions and the Cauchy problems, *Annal. Univ. Sci. Bud. Sect. Math.*, **24** (1981), 57-62.

[9] **Fawzy Th.**, Approximate solution of the initial value problem for ordinary differential equation, *Annales. Univ. Sci. Bud. Sect. Comp.*, **5** (1984), 83-86.

(Received November 13, 1991)

Th.Fawzy and M.Ahmed
Mathematical Department
Faculty of Science
Suez Canal University
Ismailia, Egypt