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ON THE COEFFICIENTS OF DIFFERENTIATED EXPANSIONS OF DOUBLE AND TRIPLE LEGENDRE POLYNOMIALS

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Abstract. Formulae expressing the coefficients of an expansion of double Legendre polynomials which has been partially differentiated an arbitrary number of times with respect to its variables in terms of the coefficients of the original expansion are stated and proved. Extension to expansion of triple Legendre polynomials is given.

An application of how to use double Legendre polynomials for solving Poisson's equation in two variables subject to homogeneous mixed boundary conditions with the tau method is considered.

1. Introduction

Classical orthogonal polynomials are used extensively for the numerical solution of ordinary and partial differential equations in spectral and pseudospectral methods, see for example Gottlieb and Orszag [4], Haidvogel and Zang [5], Horner [6], Voigt et al. [9], Doha [1] and Doha [2].

For spectral and pseudospectral methods explicit expressions for the expansion coefficients of the derivatives in terms of the expansion coefficients of the solution are required.

A formula expressing the Chebyshev coefficients of the general order derivative of an infinitely differentiable function in terms of its Chebyshev coefficients is given by Karageorghis [7], and a corresponding formula for the Legendre coefficients is obtained by Phillips [8].

Formulae expressing the coefficients of expansions of double and triple Chebyshev polynomials which have been partially differentiated an arbitrary number of times with respect to their variables in terms of the original expansion are given in Doha [3].

In the present paper we state and prove the corresponding formulae expressing the coefficients of expansions of double and triple Legendre polynomials which have been partially differentiated any number of times with respect to their variables in terms of the coefficients of the original expansion.

In Section 2 we give some properties of double Legendre series expansions and in Section 3 we describe how they are used to solve Poisson's equation in two variables inside a square subject to the homogeneous mixed boundary conditions with the tau method as a model problem. In Section 4 we state and prove the main results of the paper which are three expressions for the coefficients of general order partial derivatives of an expansion in double Legendre polynomials in terms of the coefficients of the original expansion. Extension to expansion in triple Legendre polynomials is also considered in Section 5.

2. Some properties of double Legendre series expansions

The one variable Legendre polynomials $P_n(x)$ $(n = 0, 1, 2, \ldots)$ are a sequence of polynomials, each respectively of degree n, and may be generated by using Rodrigue's formula

$$P_n(x) = \left(-\frac{1}{2}\right)^n (1/n!) D_x^n (1 - x^2)^n$$

and are satisfying the orthogonality relation

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0, & m \neq n, \\ 2/(2n+1), & m = n. \end{cases}$$

Suppose now we are given a function $u(x)$ which is infinitely differentiable in the closed interval [-1,1], then we can write

$$u(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

and for the q-th derivative of $u(x)$

$$u^{(q)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} P_n(x).$$

Phillips [8] proved that

$$(1)\quad a_n^{(q)} = \frac{(2n+1)}{2^{q-2}(q-1)!}\sum_{i=1}^{\infty}\frac{(i+q-2)!(2n+2i+2q-3)!(n+i)!}{(i-1)!(2n+2i)!(n+i+q-2)!}a_{n+2i+q-2}.$$

Now we define the double Legendre polynomials as

$$(2)\qquad P_{m,n}(x,y) = P_m(x)P_n(y),$$

i.e. a product of two one variable Legendre polynomials, where $P_m(x)$, $P_n(y)$ are Legendre polynomials of degrees m and n in the variables x and y respectively. These polynomials are satisfying the biorthogonality relation

$$\int_{-1}^{1}\int_{-1}^{1} P_{i,j}(x,y)P_{k,\ell}(x,y)dxdy =$$

$$(3)\qquad = \begin{cases} 4 & \text{if} \quad i=j=k=\ell=0, \\ \frac{4}{(2i+1)(2j+1)} & \text{if} \quad i=k\neq 0,\ \ j=\ell\neq 0, \\ \frac{4}{2i+1} & \text{if} \quad i=k\neq 0,\ \ j=\ell=0, \\ \frac{4}{2j+1} & \text{if} \quad i=k=0,\ \ j=\ell\neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u(x,y)$ be a continuous function defined on the square $S = (-1 \leq \leq x, y \leq 1)$, and let it have continuous and bounded partial derivatives of any order with respect to its variables x and y. Then it is possible to express

$$(4)\qquad u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}P_m(x)P_n(y),$$

$$(5)\qquad D_x^p D_y^q u(x,y) = u^{(p,q)}(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)}P_m(x)P_n(y),$$

where $a_{mn}^{(p,q)}$ denote the Legendre expansion coefficients of $D_x^p D_y^q u(x,y)$ and $a_{mn}^{(0,0)} = a_{mn}$. Using the expressions

$$(6)\qquad (2m+1)P_m(x) = D_x(P_{m+1}(x) - P_{m-1}(x)),$$

$$(2n+1)P_n(y) = D_y(P_{n+1}(y) - P_{n-1}(y)) \tag{7}$$

with the assumptions

$$D_x \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p-1,q)} P_m(x)P_n(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)} P_m(x)P_n(y),$$

$$D_y \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q-1)} P_m(x)P_n(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^{(p,q)} P_m(x)P_n(y)$$

it is not difficult to derive the expressions

$$\frac{1}{2m-1} a_{m-1,n}^{(p,q)} - \frac{1}{2m+3} a_{m+1,n}^{(p,q)} = a_{mn}^{(p-1,q)} \qquad m,p \geq 1, \tag{8}$$

$$\frac{1}{2n-1} a_{m,n-1}^{(p,q)} - \frac{1}{2n+3} a_{m,n+1}^{(p,q)} = a_{mn}^{(p,q-1)} \qquad n,q \geq 1. \tag{9}$$

For computing purposes equations (8) and (9) are not easy to use, since the coefficients on the left hand sides are functions of m and n, respectively. To simplify the computing, we define a related set of coefficients $b_{mn}^{(p,q)}$ by writing

$$a_{mn}^{(p,q)} = \left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right) b_{mn}^{(p,q)} \qquad m,n \geq 0; \;\; p=q=0,1,2,\ldots \tag{10}$$

Equations (8) and (9) take the simpler forms

$$b_{m-1,n}^{(p,q)} - b_{m+1,n}^{(p,q)} = (2m+1) b_{mn}^{(p-1,q)} \qquad m,p \geq 1, \tag{11}$$

$$b_{m,n-1}^{(p,q)} - b_{m,n+1}^{(p,q)} = (2n+1) b_{mn}^{(p,q-1)} \qquad n,q \geq 1. \tag{12}$$

Repeated application of (11) keeping n and q fixed yields

$$b_{mn}^{(p,q)} = \sum_{i=1}^{\infty} (2m+4i-1) b_{m+2i-1,n}^{(p-1,q)} \qquad p \geq 1, \tag{13}$$

and the same with (12) keeping m and p fixed yields

$$b_{mn}^{(p,q)} = \sum_{j=1}^{\infty} (2n+4j-1) b_{m,n+2j-1}^{(p,q-1)} \qquad q \geq 1. \tag{14}$$

3. The tau method for Poisson's equation inside a square

Consider Poisson's equation inside the square $S = (-1 \leq x, y \leq 1)$

$$D_x^2 u(x,y) + D_y^2 u(x,y) = f(x,y), \qquad -1 \leq x, y \leq 1, \tag{15}$$

subject to the homogeneous mixed boundary conditions

$$\left.\begin{array}{ll} u + \alpha_1 D_x u = 0 & x = -1 \\ u + \alpha_2 D_x u = 0 & x = 1 \end{array}\right] \qquad -1 \leq y \leq 1, \tag{16}$$

$$\left.\begin{array}{ll} u + \beta_1 D_y u = 0 & y = -1 \\ u + \beta_2 D_y u = 0 & y = 1 \end{array}\right] \qquad -1 \leq x \leq 1 \tag{17}$$

and assume that both $u(x,y)$ and $f(x,y)$ are approximated by truncated double Legendre series

$$u(x,y) = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{mn} P_m(x) P_n(y), \tag{18}$$

$$f(x,y) = \sum_{n=0}^{N} \sum_{m=0}^{M} f_{mn} P_m(x) P_n(y), \tag{19}$$

then the Legendre tau equations for Poisson's equation (15) are given by

$$a_{mn}^{(2,0)} + a_{mn}^{(0,2)} = f_{mn}, \qquad 0 \leq m \leq M-2; \;\; 0 \leq n \leq N-2, \tag{20}$$

while the boundary conditions (16) and (17) yield

$$\left.\begin{array}{l} \sum_{m=0}^{M} (-1)^m \left[a_{mn} + \alpha_1 a_{mn}^{(1,0)}\right] = 0 \\ \sum_{m=0}^{M} \left[a_{mn} + \alpha_2 a_{mn}^{(1,0)}\right] = 0 \end{array}\right] \qquad n = 0, 1, 2, \ldots, N, \tag{21}$$

$$\left.\begin{aligned}&\sum_{n=0}^{N}(-1)^n\left[a_{mn}+\beta_1 a_{mn}^{(0,1)}\right]=0\\&\sum_{n=0}^{N}\left[a_{mn}+\beta_2 a_{mn}^{(0,1)}\right]=0\end{aligned}\right] \qquad m=0,1,2,\ldots,M. \tag{22}$$

The $2M+2N+4$ boundary conditions given by (21) and (22) are not all linearly independent, there exist four linear relations among them. Thus equations (20), (21) and (22) give $(M+1)(N+1)$ equations for the $(M+1)(N+1)$ unknowns a_{mn} $(0 \le m \le M,\ 0 \le n \le N)$.

The coefficients $a_{mn}^{(1,0)}$, $a_{mn}^{(2,0)}$, $a_{mn}^{(0,1)}$ and $a_{mn}^{(0,2)}$ of the first and second partial derivatives of the approximation $u(x,y)$ are related to the coefficients a_{mn} of $u(x,y)$ by invoking (13) with $p=1$ and $p=2$, and (14) with $q=1$ and $q=2$ respectively. In the next section we show how the coefficients of any derivatives may be expressed in terms of the original expansion coefficients. This allows us to replace $a_{mn}^{(1,0)}$, $a_{mn}^{(0,1)}$, $a_{mn}^{(2,0)}$ and $a_{mn}^{(0,2)}$ in (21), (22) and (20) by explicit expressions in terms of the a_{mn}. In this way we can set up a linear system for a_{mn} $(0 \le m \le M,\ 0 \le n \le N)$ which may be solved using standard techniques.

4. Relations between the coefficients

Theorem 1. *The coefficients $b_{mn}^{(p,q)}$ are related to the coefficients $b_{mn}^{(0,q)}$, $b_{mn}^{(p,0)}$ and the coefficients b_{mn} by*

$$b_{mn}^{(p,q)}=\frac{4}{2^p(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!(2m+2i+2p-3)!(m+i)!}{(i-1)!(2m+2i)!(m+i+p-2)!}\times \\ \times(2m+4i+2p-3)b_{m+2i+p-2,n}^{(0,q)}, \qquad p\ge 1, \tag{23}$$

$$b_{mn}^{(p,q)}=\frac{4}{2^q(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!(2n+2j+2q-3)!(n+j)!}{(j-1)!(2n+2j)!(n+j+q-2)!}\times \\ \times(2n+4j+2q-3)b_{m,n+2j+q-2}^{(p,0)}, \qquad q\ge 1, \tag{24}$$

(25)

$$b_{mn}^{(p,q)} = \frac{16}{2^{p+q}(p-1)!(q-1)!}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(2m+2i+2p-3)!(m+i)!}{(i-1)!(2m+2i)!(m+i+p-2)!}\times$$
$$\times\frac{(j+q-2)!(2n+2j+2q-3)!(n+j)!}{(j-1)!(2n+2j)!(n+j+q-2)!}(2m+4i+2p-3)\times$$
$$\times(2n+4j+2q-3)b_{m+2i+p-2,n+2j+q-2}, \qquad p,q\geq 1$$

for all $m,n\geq 0$.

In order to prove Theorem 1, the following two lemmas are required:

$$\sum_{i=1}^{M}(2m+4i-1)\frac{(M-i+p-1)!(2m+2i+2M+2p-3)!(m+i+M)!}{(M-i)!(2m+2i+2M)!(m+i+M+p-2)!}=$$

$$(26)\qquad =\frac{1}{2p}\frac{(M+p-1)!(2m+2M+2p-1)!(m+M)!}{(M-1)!(2m+2M)!(m+M+p-1)!}, \qquad m,p\geq 1,$$

$$\sum_{j=1}^{N}(2n+4j-1)\frac{(N-j+q-1)!(2n+2j+2N+2q-3)!(n+j+N)!}{(N-j)!(2n+2j+2N)!(n+j+N+q-2)!}=$$

$$(27)\qquad =\frac{1}{2q}\frac{(N+q-1)!(2n+2N+2q-1)!(n+N)!}{(N-1)!(2n+2N)!(n+N+q-1)!}, \qquad n,q\geq 1.$$

Proof of lemma (26). For $M=1$ the left hand side of (26) equals the right hand side of (26) which is

$$\frac{(p-1)!(2m+2p+1)!(m+1)!}{2(2m+2)!(m+p)!}.$$

If we apply induction on M, assuming that (26) holds, we have to show that

$$\sum_{i=1}^{M+1}(2m+4i-1)\frac{(M-i+p)!(2m+2i+2M+2p-1)!(m+i+M+1)!}{(M-i+1)!(2m+2i+2M+2)!(m+i+M+p-1)!}=$$

$$(28)\qquad =\frac{1}{2p}\frac{(M+p)!(2m+2M+2p+1)!(m+M+1)!}{M!(2m+2M+2)!(m+M+p)!}.$$

From (26) by taking $m+2$ instead of m and $j=i+1$ we get

$$\sum_{j=2}^{M+1}(2m+4j-1)\frac{(M-j+p)!(2m+2j+2M+2p-1)!(m+j+M+1)!}{(M-j+1)!(2m+2j+2M+2)!((m+j+M+p-1)!}=$$

$$=\frac{1}{2p}\frac{(M+p-1)!(2m+2M+2p+3)!(m+M+2)!}{(M-1)!(2m+2M+4)!(m+M+p+1)!}. \tag{29}$$

The left hand side of (28) becomes

$$(2m+3)\frac{(M+p-1)!(2m+2M+2p+1)!(m+M+2)!}{M!(2m+2M+4)!(m+M+p)!}+$$

$$+\sum_{i=2}^{M+1}(2m+4i-1)\frac{(M-i+p)!\,(2m+2i+2M+2p-1)!(m+i+M+1)!}{(M-i+1)!\,(2m+2i+2M+2)!(m+i+M+p-1)!}=$$

$$=(2m+3)\frac{(M+p-1)!(2m+2M+2p+1)!(m+M+2)!}{M!(2m+2M+4)!(m+M+p)!}+$$

$$+\frac{1}{2p}\frac{(M+p-1)!(2m+2M+2p+3)!(m+M+2)!}{(M-1)!(2m+2M+4)!(m+M+p+1)!}=$$

$$=\frac{1}{2p}\frac{(M+p)!(2m+2M+2p+1)!(m+M+1)!}{M!(2m+2M+2)!(m+M+p)!},$$

which completes the induction and proves the lemma (26). Lemma (27) can be proved similarly.

Proof of Theorem 1. Firstly we prove formula (23). For $p=1$ application of (13) with $p=1$ yields the required formula. Proceeding by induction, assuming that the relation is valid for p (keeping n and q fixed), we want to show that

$$b_{mn}^{(p+1,q)}=\frac{1}{2^{p-1}p!}\times \tag{30}$$

$$\times\sum_{i=1}^{\infty}\frac{(i+p-1)!(2m+2i+2p-1)!(m+i)!}{(i-1)!(2m+2i)!(m+i+p-1)!}(2m+4i+2p-1)b_{m+2i+p-1,n}^{(0,q)}.$$

From (13), replacing p by $p+1$ and assuming the validity of (23) for p,

$$b_{mn}^{(p+1,q)}=\frac{1}{2^{p-2}(p-1)!}\sum_{i=1}^{\infty}(2m+4i-1)\times \tag{31}$$

$$\times\left\{\sum_{k=1}^{\infty}\frac{(k+p-2)!(2m+4i+2k+2p-5)!}{(k-1)!(2m+4i+2k-2)!}\frac{(m+2i+k-1)!}{(m+2i+k+p-3)!}\cdot\right.$$

$$\left.\cdot(2m+4i+4k+2p-5)b_{m+2i+2k+p-4,n}\right\}.$$

Let $i+k-1=M$, then (31) takes the form

$$b_{mn}^{(p+1,q)}=\frac{1}{2^{p-2}(p-1)!}\sum_{M=1}^{\infty}\left[\sum_{\substack{i,k=1\\ i+k=M+1}}^{M}(2m+4i-1)\times\right.$$

$$\left.\frac{(k+p-2)!(2m+4i+2k+2p-5)!}{(k-1)!(2m+4i+2k-2)!}\frac{(m+2i+k-1)!}{(m+2i+k+p-3)!}(2m+4M+2p-1)\right]\times$$

$$\times b_{m+2M+p-2,n},$$

which may also be written as

$$b_{mn}^{(p+1,q)}=\frac{1}{2^{p-2}(p-1)!}\sum_{M=1}^{\infty}\left[\sum_{i=1}^{M}(2m+4i-1)\times\right.$$

$$\left.\times\frac{(M-i+p-1)!(2m+2i+2M+2p-3)!}{(M-i)!(2m+2i+2M)!}\frac{(m+i+M)!}{(m+i+M+p-2)!}\right]\times$$

$$\times(2m+4M+2p-1)b_{m+2M+p-2,n}.$$

Application of lemma (26) to the second series yields equation (30) and the proof of formula (23) is complete.

It can also bo shown that formula (24) is true by following the same procedure with (14), keeping m and p fixed. Formula (25) is obtained immediately by substituting (23) into (24). This completes the proof of the theorem.

Now substitution of (23), (24) and (25) into (10) give the relations between the coefficients $a_{mn}^{(p,q)}$, $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ and a_{mn}:

$$a_{mn}^{(p,q)}=\frac{4(2m+1)}{2^p(p-1)!}\times \tag{32}$$

$$\times\sum_{i=1}^{\infty}\frac{(i+p-2)!(2m+2i+2p-3)!(m+i)!}{(i-1)!(2m+2i)!(m+i+p-2)!}a_{m+2i+p-2,n}^{(0,q)},\qquad p\geq 1,$$

(33)
$$a_{mn}^{(p,q)} = \frac{4(2n+1)}{2^q(q-1)!}\times$$
$$\times\sum_{j=1}^{\infty}\frac{(j+q-2)!(2n+2j+2q-3)!(n+j)!}{(j-1)!(2n+2j)!(n+j+q-2)!}a_{m,n+2j+q-2}^{(p,0)}, \qquad q\geq 1,$$

(34)
$$a_{mn}^{(p,q)} = \frac{16(2m+1)(2n+1)}{2^{p+q}(p-1)!(q-1)!}\times$$
$$\times\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(2m+2i+2p-3)!(m+i)!}{(i-1)!(2m+2i)!(m+i+p-2)!}\times$$
$$\times\frac{(j+q-2)!(2n+2j+2q-3)!(n+j)!}{(j-1)!(2n+2j)!(n+j+q-2)!}a_{m+2i+p-2,n+2j+q-2}, \qquad p,q\geq 1$$

for all $m,n\geq 0$.

5. Extension to triple Legendre series expansion

Let $u(x,y,z)$ be a continuous function defined on the cube $C=(-1\leq$ $\leq x,y,z\leq 1)$, and let it have continuous and bounded partial derivatives of any order with respect to its three variables x, y, z. Then it is possible to write

(35)
$$u(x,y,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}a_{\ell mn}P_\ell(x)P_m(y)P_n(z),$$

(36)
$$u^{(p,q,r)}(x,y,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}a_{\ell mn}^{(p,q,r)}P_\ell(x)P_m(y)P_n(z).$$

Further, let

(37)
$$a_{\ell mn}^{(p,q,r)}=\left(\ell+\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)b_{\ell mn}^{(p,q,r)},$$
$$\ell,m,n\geq 0;\quad p,q,r=0,1,2,\ldots,$$

then it is not difficult to show that

$$
\begin{aligned}
b^{(p,q,r)}_{\ell-1,m,n} - b^{(p,q,r)}_{\ell+1,m,n} &= (2\ell+1)b^{(p-1,q,r)}_{\ell mn} \qquad p \geq 1,\\
b^{(p,q,r)}_{\ell,m-1,n} - b^{(p,q,r)}_{\ell,m+1,n} &= (2m+1)b^{(p,q-1,r)}_{\ell mn} \qquad q \geq 1,\\
b^{(p,q,r)}_{\ell,m,n-1} - b^{(p,q,r)}_{\ell,m,n+1} &= (2n+1)b^{(p,q,r-1)}_{\ell mn} \qquad r \geq 1,
\end{aligned} \tag{38}
$$

which, in turn, yield

$$b^{(p,q,r)}_{\ell mn} = \sum_{i=1}^{\infty}(2\ell+4i-1)b^{(p-1,q,r)}_{\ell+2i-1,m,n} \qquad p \geq 1, \tag{39}$$

$$b^{(p,q,r)}_{\ell mn} = \sum_{j=1}^{\infty}(2m+4j-1)b^{(p,q-1,r)}_{\ell,m+2j-1,n} \qquad q \geq 1, \tag{40}$$

$$b^{(p,q,r)}_{\ell mn} = \sum_{k=1}^{\infty}(2n+4k-1)b^{(p,q,r-1)}_{\ell,m,n+2k-1} \qquad r \geq 1. \tag{41}$$

Now we state the following theorem which is to be considered as an extension of Theorem 1 of Section 4.

Theorem 2. *The coefficients $b^{(p,q,r)}_{\ell mn}$ are related to the coefficients with superscripts $(0,q,r)$, $(p,0,r)$, $(p,q,0)$, $(0,0,r)$, $(0,q,0)$, $(p,0,0)$ and the coefficients $b_{\ell mn}$ by*

$$
\begin{aligned}
b^{(p,q,r)}_{\ell mn} &= \frac{4}{2^p(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!(2\ell+2i+2p-3)!(\ell+i)!}{(i-1)!(2\ell+2i)!(\ell+i+p-2)!}\times\\
&\times(2\ell+4i+2p-3)b^{(0,q,r)}_{\ell+2i+p-2,m,n}, \qquad p \geq 1,
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
b^{(p,q,r)}_{\ell mn} &= \frac{4}{2^q(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!(2m+2j+2q-3)!(m+j)!}{(j-1)!(2m+2j)!(m+j+q-2)!}\times\\
&\times(2m+4j+2q-3)b^{(p,0,r)}_{\ell,m+2j+q-2,n}, \qquad q \geq 1,
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
b^{(p,q,r)}_{\ell mn} &= \frac{4}{2^r(r-1)!}\sum_{k=1}^{\infty}\frac{(k+r-2)!(2n+2k+2r-3)!(n+k)!}{(k-1)!(2n+2k)!(n+k+r-2)!}\times\\
&\times(2n+4k+2r-3)b^{(p,q,0)}_{\ell,m,n+2k+r-2}, \qquad r \geq 1,
\end{aligned} \tag{44}
$$

$$b^{(p,q,r)}_{\ell mn} = \frac{16}{2^{p+q}(p-1)!(q-1)!}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(2\ell+2i+2p-3)!(\ell+i)!}{(i-1)!(2\ell+2i)!(\ell+i+p-2)!}\times$$

$$(45)\quad \times \frac{(j+q-2)!(2m+2j+2q-3)!(m+j)!}{(j-1)!(2m+2j)!(m+j+q-2)!}(2\ell+4i+2p-3)\times$$

$$\times\,(2m+4j+2q-3)b^{(0,0,r)}_{\ell+2i+p-2,m+2j+q-2,n}, \qquad p,q\geq 1,$$

$$b^{(p,q,r)}_{\ell mn} = \frac{16}{2^{p+r}(p-1)!(r-1)!}\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(2\ell+2i+2p-3)!(\ell+i)!}{(i-1)!(2\ell+2i)!(\ell+i+p-2)!}\times$$

$$(46)\quad \frac{(k+r-2)!(2n+2k+2r-3)!(n+k)!}{(k-1)!(2n+2k)!(n+k+r-2)!}(2\ell+4i+2p-3)\times$$

$$\times\,(2n+4k+2r-3)b^{(0,q,0)}_{\ell+2i+p-2,m,n+2k+r-2}, \qquad p,r\geq 1,$$

$$b^{(p,q,r)}_{\ell,mn} = \frac{16}{2^{q+r}(q-1)!(r-1)!}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{(j+q-2)!(2m+2j+2q-3)!(m+j)!}{(j-1)!(2m+2j)!(m+j+q-2)!}\times$$

$$(47)\quad \times \frac{(k+r-2)!(2n+2k+2r-3)!(n+k)!}{(k-1)!(2n+2k)!(n+k+r-2)!}(2m+4j+2q-3)\times$$

$$\times\,(2n+4k+2r-3)b_{\ell,m+2j+q-2,n+2k+r-2}, \qquad q,r\geq 1,$$

$$b^{(p,q,r)}_{\ell mn} =$$

$$= \frac{64}{2^{p+q+r}(p-1)!(q-1)!(r-1)!}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{(i+p-2)!(2\ell+2i+2p-3)!}{(i-1)!(2\ell+2i)!}\times$$

$$\frac{(\ell+i)!(j+q-2)!(2m+2j+2q-3)!(m+j)!(k+r-2)!(2n+2k+2r-3)!}{(\ell+i+p-2)!(j-1)!(2m+2j)!(m+j+q-2)!(k-1)!(2n+2k)!}$$

$$(48)\qquad \times\frac{(n+k)!}{(n+k+r-2)!}(2\ell+4i+2p-3)(2m+4j+2q-3)\times$$

$$\times(2n+4k+2r-3)b_{\ell+2i+p-2,m+2j+q-2,n+2k+r-2}, \qquad p,q,r\geq 1.$$

Outlines of the proof. Formula (42) can be proved by induction on p, (43) by induction on q and (44) by induction on r respectively. The substitutions of (42) into (43) and (44), and (43) into (44) give formulae (45), (46) and (47). Formula (48) is obtained by substituting (44) into (45).

The explicit formulae which relate the coefficients $a^{(p,q,r)}_{\ell mn}$ with those with superscripts $(0,q,r)$, $(p,0,r)$, $(p,q,0)$, $(0,0,r)$, $(0,q,0)$, $(p,0,0)$ and the original coefficients $a_{\ell mn}$ can simply be obtained by using the relation (37) with formulae (42)-(48).

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