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TWO-DIMENSIONAL BLOCK-PULSE FUNCTIONS SERIES SOLUTION OF A SYSTEM OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

Rafat Riad (Cairo, Egypt)

1. Introduction

Since Corrington [1] constructed Walsh tables for solving higher-order differential equations and Chen and Hsiao [2] developed the Walsh operational matrix for solving state equations, the Walsh operational method has been successfully applied to various problems. Shih and Han [3] have shown the two-dimensional Walsh series solution of a system of first-order partial differential equations (PDE). The key step in this solution was the inversion of a $kmn \times kmn$ matrix. Some difficulties might occur in obtaining the inverse of a square matrix of size $kmn \times kmn$, specially, if k, m and n are large values. Block-pulse functions (b.p.f.) and Walsh functions are closely related. As basis function in an approximation the two sets of functions lead to the same results. Chen, Tsay and Wu [4] and Goplasami and Deekshatulu [5] introduced b.p.f. for solutions of distributed systems and identification problems.

This paper simplifies, enormously, the method of Shih and Han [3] by using b.p.f. instead of Walsh functions as the basis to solve a system of first-order PDE. The difference between the two methods is not in the final results, but rather in their computation. Depending on the special properties of the operational matrix for b.p.f., which is simpler than the Walsh operational matrix, an algorithm is established to reduce the key step from the inverse of $kmn \times kmn$ matrix to the inverse of $k \times k$ matrix.

As a start, the b.p.f. are introduced and their properties briefly summarized [4], [6].

2. Introduction to b.p.f.

A set of b.p.f. on unit interval [0,1) is defined as follows: For each integer i, $0 \le i < m$, $m \in \mathbb{P} = \{1, 2, \ldots\}$, the function $\varphi_i(t)$ is given by

$$\varphi_i(t) = \begin{cases} 1 & \text{for } \dfrac{i}{m} \le t < \dfrac{i+1}{m}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

This set of functions can be concisely described by an m-vector $\Phi_{(m)}(t)$ with $\varphi_i(t)$ as its i-th component. It is well known that a function f which is integrable in (0,1) can be approximated as

$$f(t) \simeq \sum_{i=0}^{m-1} a_i \varphi_i(t), \tag{2}$$

where the coefficients a_i are given by

$$a_i = m \int_{\frac{i}{m}}^{\frac{i+1}{m}} f(t)dt, \qquad 0 \le i < m \tag{3}$$

$$= \text{ avarage value of } f(t) \text{ over the interval } \frac{i}{m} \le t < \frac{i+1}{m}.$$

The b.p.f. satisfies the following properties

$$\varphi_i(t)\varphi_j(t) = \delta_{ij}\varphi_i(t) \tag{4}$$

and

$$\int_0^1 \varphi_i(t)\varphi_j(t) = \frac{1}{m}\delta_{ij}, \tag{5}$$

where δ_{ij} denotes the Kronecker δ symbol.

It is known [4], [6] that

$$\int_0^t \Phi_{(m)}(\lambda)d\lambda \simeq B_{(m\times m)}\Phi_{(m)}(t), \tag{6}$$

where the operational matrix for integration $B_{(m\times m)}$ is given by

(7)
$$B_{(m\times m)} = \frac{1}{m}\begin{bmatrix} \frac{1}{2} & 1 & 1 & & 1 \\ 0 & \frac{1}{2} & 1 & & 1 \\ 0 & 0 & \frac{1}{2} & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \frac{1}{2} \end{bmatrix}.$$

3. Two-dimensional b.p.f. series approximation

A function of two independent variables $y(x,t)$ which is integrable in $x \in$ $\in [0,1)$ and $t \in [0,1)$ can be approximately represented by a b.p.f. series of size n with respect to t as follows

(8)
$$y(x,t) \simeq \sum_{i=0}^{n-1} y_i(x)\varphi_i(t).$$

Using the orthogonal property of b.p.f. in (5), the coefficient functions $y_i(x)$ of (8) become

(9)
$$y_i(x) = n \int_{\frac{i}{n}}^{\frac{i+1}{n}} y(x,t)dt \qquad (i = 0, 1, \ldots, n-1).$$

Similarly, a b.p.f. series approximation of $y_i(x)$ gives

(10)
$$y_i(x) \simeq \sum_{j=0}^{m-1} \varphi_j(x) y_{ji},$$

where y_{ji} are coefficients obtained by

(11)
$$y_{ji} = m \int_{\frac{i}{m}}^{\frac{i+1}{m}} y_i(x)dx = mn \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{i}{n}}^{\frac{i+1}{n}} y(x,t)dtdx.$$

Combination of equations (8) and (10) yields

$$y(x,t) \simeq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \varphi_j(x) y_{ji} \varphi_i(t). \tag{12}$$

Defining the b.p.f. vectors

$$\Phi_{(n)}(t) = [\varphi_0(t),\ \varphi_1(t),\ \ldots,\ \varphi_{n-1}(t)]^T, \tag{13}$$

$$\Phi_{(m)}(x) = [\varphi_0(x),\ \varphi_1(x),\ \ldots,\ \varphi_{m-1}(x)]^T, \tag{14}$$

where "T" means transpose, and coefficient matrix Y of $m \times n$ dimension

$$Y = \begin{bmatrix} y_{00} & y_{01} & y_{0,n-1} \\ y_{10} & y_{11} & y_{1,n-1} \\ \vdots & \vdots & \vdots \\ y_{m-1,0} & y_{m-1,1} & y_{m-1,n-1} \end{bmatrix}, \tag{15}$$

equation (12) is written in matrix form as

$$y(x,t) \simeq \Phi^T_{(m)}(x) Y \Phi_{(n)}(t). \tag{16}$$

This is the two-dimensional b.p.f. series approximation of $y(x,t)$.

From (6) the integration of b.p.f. vectors (13) and (14) gives, respectively,

$$\int_0^t \Phi_{(n)}(t')dt' \simeq B_{(n\times n)} \Phi_{(n)}(t), \tag{17}$$

$$\int_0^x \Phi_{(m)}(x')dx' \simeq B_{(m\times m)} \Phi_{(m)}(x). \tag{18}$$

4. Solution of system of first-order PDE

Consider k simultaneous first-order partial equations as follows

$$\frac{\partial y_i(x,t)}{\partial t} + \sum_{j=1}^{k} c_{ij} \frac{\partial y_j(x,t)}{\partial x} = \sum_{j=1}^{k} a_{ij} y_j(x,t) + \sum_{j=1}^{\ell} b_{ij} u_j(x,t), \tag{19}$$

$$i = 1, 2, \ldots, k,$$

and the boundary conditions are

$$y_i(x,0) = g_i(x) \tag{20}$$

$$(i = 1, 2, \ldots, k)$$

$$y_i(0,t) = f_i(t) \tag{21}$$

where c_{ij}, a_{ij} and b_{ij} are constants and $u_j(x,t)$ $(j = 1, 2, \ldots, \ell)$ are input functions. For solving this problem by the b.p.f. approach we use the technique introduced by Shih and Han [3] to solve this problem by the Walsh functions. Integration of (19) with respect to t and x gives

$$\int_0^t \int_0^x \frac{\partial y_i}{\partial t'} dx'dt' + \sum_{j=1}^{k} c_{ij} \int_0^t \int_0^x \frac{\partial y_j}{\partial x'} dx'dt' = \sum_{j=1}^{k} a_{ij} \int_0^t \int_0^x y_j \, dx'dt' +$$

$$+ \sum_{j=1}^{\ell} b_{ij} \int_0^t \int_0^x u_j \, dx'dt' \qquad (i = 1, 2, \ldots, k). \tag{22}$$

From (16) the two-dimensional b.p.f. series approximation of $y_i(x,t)$ is given by

$$y_i(x,t) \simeq \Phi^T_{(m)}(x) Y^{(i)} \Phi_{(n)}(t), \tag{23}$$

where

$$Y^{(i)} = \begin{bmatrix} Y_1^{(i)} & Y_2^{(i)} & & Y_n^{(i)} \end{bmatrix} \qquad (i = 1, 2, \ldots, k) \tag{24}$$

are $m \times n$ matrices with unknown components $y_{pq}^{(i)}$. Notice that $Y_j^{(i)}$ is the j-th column of $Y^{(i)}$ $(j = 1, 2, \ldots, n)$ and of m-dimension. Likewise,

$$u_j(x,t) \simeq \Phi_{(m)}^T(x) U^{(j)} \Phi_{(n)}(t) \qquad (j = 1, 2, \ldots, \ell), \tag{25}$$

where $U^{(j)}$ $(j = 1, 2, \ldots, \ell)$ are $m \times n$ matrices with known components $u_{pq}^{(j)}$

$$u_{pq}^{(j)} = mn \int_{\frac{p}{m}}^{\frac{p+1}{m}} \int_{\frac{q}{n}}^{\frac{q+1}{n}} u_j(x,t)\,dt\,dx. \tag{26}$$

A b.p.f. series approximation of $g_i(x)$ of (20) is

$$g_i(x) \simeq \Phi_{(m)}^T(x) \left[g_0^{(i)},\ g_1^{(i)}, \ldots, g_{m-1}^{(i)} \right]^T, \tag{27}$$

where

$$g_p^{(i)} = m \int_{\frac{p}{m}}^{\frac{p+1}{m}} g_i(x)\,dx. \tag{28}$$

From the definition of the b.p.f. (27) can be written as

$$g_i(x) \simeq \Phi_{(m)}^T(x) G^{(i)} \Phi_{(n)}(t) \qquad (i = 1, 2, \ldots, k), \tag{29}$$

where $G^{(i)}$ is an $m \times n$ matrix each column of which is

$$\left[g_0^{(i)},\ g_1^{(i)},\ \ldots,\ g_{m-1}^{(i)} \right]^T$$

Similarly, a b.p.f. series approximation of $f_i(t)$ of (21) can be written as

$$f_i(t) \simeq \Phi_{(m)}^T(x) F^{(i)} \Phi_{(n)}(t) \qquad (i = 1, 2, \ldots, k), \tag{30}$$

where $F^{(i)}$ is an $m \times n$ matrix each row of which is $\left[f_0^{(i)},\ f_1^{(i)}, \ldots, f_n^{(i)} \right]$ and

$$f_q^{(i)} = n \int_{\frac{q}{n}}^{\frac{q+1}{n}} f_i(t)\,dt. \tag{31}$$

Using the b.p.f. series approximations (23), (25), (29) and (30) and equations (17), (18), (20) and (21), the four terms of (22) can be evaluated, respectively, as follows

$$\text{term}(1) = \int_0^x (y_i(x',t) - y_i(x',0))dx' \simeq$$

(32)
$$\simeq \int_0^x \left(\Phi^T_{(m)}(x')Y^{(i)}\Phi_{(n)}(t) - \Phi^T_{(m)}(x')G^{(i)}\Phi_{(n)}(t)\right) dx' \simeq$$

$$\simeq \Phi^T_{(m)}(x)\left(B^T_{(m\times m)}Y^{(i)} - B^T_{(m\times m)}G^{(i)}\right)\Phi_{(n)}(t),$$

$$\text{term}(2) = \sum_{j=1}^{k} c_{ij} \int_0^t (y_j(x,t') - y_j(0,t'))\,dt' \simeq$$

(33)
$$\simeq \sum_{j=1}^{k} c_{ij} \int_0^t \left(\Phi^T_m(x)Y^{(j)}\Phi_{(n)}(t') - \Phi^T_{(m)}(x)F^{(j)}\Phi_{(n)}(t')\right) dt' \simeq$$

$$\simeq \Phi^T_{(m)}(x)\left[\sum_{j=1}^{k} c_{ij}\left(Y^{(j)}B_{(n\times n)} - F^{(j)}B_{(n\times n)}\right)\right]\Phi_{(n)}(t),$$

(34)
$$\text{term}(3) \simeq \Phi^T_{(m)}(x)\left[\sum_{j=1}^{k} a_{ij}B^T_{(m\times m)}Y^{(j)}B_{(n\times n)}\right]\Phi_{(n)}(t),$$

(35)
$$\text{term}(4) \simeq \Phi^T_{(m)}(x)\left[\sum_{j=1}^{\ell} b_{ij}B^T_{(m\times m)}U^{(j)}B_{(n\times n)}\right]\Phi_{(n)}(t).$$

Substitution of (32)-(35) into (22) gives an equation of the following form

(36)
$$\Phi^T_{(m)}(x)[\ldots]\Phi_{(n)}(t) = 0.$$

Since (36) is valid for any x and t in the domain of consideration, the quantity in the brackets should be equal to zero. That is

(37)
$$B^T_{(m\times m)}Y^{(i)} + \sum_{j=1}^{k} c_{ij}Y^{(j)}B_{(n\times n)} - \sum_{j=1}^{k} a_{ij}B^T_{(m\times m)}Y^{(j)}B_{(n\times n)} =$$

$$= B^T_{(m\times m)}G^{(i)} + \sum_{j=1}^{k} c_{ij}F^{(j)}B_{(n\times n)} + \sum_{j=1}^{\ell} b_{ij}B^T_{(m\times m)}U^{(j)}B_{(n\times n)}$$

$$(i = 1, 2, \dots, k).$$

The equation (37) is a set of k algebraic equations for the unknown matrices $Y^{(i)}$, $i = 1, 2, \dots, k$. An explicit form for the solution of $Y^{(i)}$ $(i = 1, 2, \dots, k)$ can be obtained using the Kronecker product formulae as follows. First, multiply equation (37) by $B^{-1}_{(n\times n)}$ from the right

$$B^T_{(m\times m)} Y^{(i)} B^{-1}_{(n\times n)} + \sum_{j=1}^{k} c_{ij} Y^{(i)} - \sum_{j=1}^{k} a_{ij} B^T_{(m\times m)} Y^{(j)} = Q^{(i)}, \tag{38}$$

where

$$Q^{(i)} = B^T_{(m\times m)} G^{(i)} B^{-1}_{(n\times n)} + \sum_{j=1}^{k} c_{ij} F^{(j)} + \sum_{j=1}^{\ell} b_{ij} B^T_{(m\times m)} U^{(j)}, \tag{39}$$

$$i = 1, 2, \dots, k$$

are known $m \times n$ matrices. The first column of $Q^{(i)}$ may be defined as $Q^{(i)}_1$; the second column as $Q^{(i)}_2$, etc., as we defined for $Y^{(i)}$ in (24). Let the components of $Y^{(i)}$ be rearranged into an mn vector z_i

$$z_i = \begin{bmatrix} Y^{(i)}_1 \\ Y^{(i)}_2 \\ \vdots \\ Y^{(i)}_n \end{bmatrix}. \tag{40}$$

Similarly, the mn vector w_i is

$$w_i = \begin{bmatrix} Q^{(i)}_1 \\ Q^{(i)}_2 \\ \vdots \\ Q^{(i)}_n \end{bmatrix}. \tag{41}$$

Then (38) becomes

$$\left[B^T_{(m\times m)} \otimes (B^{-1}_{(n\times n)})^T\right] z_i + \sum_{j=1}^{k} c_{ij} z_j - \sum_{j=1}^{k} a_{ij} \left[B^T_{(m\times m)} \otimes I_{(n\times n)}\right] z_j = w_i,$$

$$(i = 1, 2, \dots, k) \tag{42}$$

where $I_{(n\times n)}$ is an $n \times n$ identity matrix. Here $A \otimes D$ is the Kronecker product defined as

$$(43) \qquad A \otimes D = \begin{bmatrix} d_{11}A & d_{12}A & & d_{1n}A \\ d_{21}A & d_{22}A & & d_{2n}A \\ \vdots & \vdots & & \vdots \\ d_{n1}A & d_{n2}A & & d_{nn}A \end{bmatrix},$$

where $D = (d_{ij})$ is $n \times n$ matrix. Taking the transpose of (42) we have

$$z_i^T \left[B_{(m\times m)} \otimes B_{n\times n}^{-1}\right] + \sum_{j=1}^{k} c_{ij} z_j^T - \sum_{j=1}^{k} a_{ij} z_j^T \left[B_{(m\times m)} \otimes I_{(n\times n)}\right] = w_i^T$$

$$(44) \qquad (i = 1, 2, \ldots, k).$$

Notice that z_i^T and w_i^T are $1 \times mn$ row vectors. Letting

$$(45) \qquad Z = \begin{bmatrix} z_1^T \\ z_2^T \\ \vdots \\ z_k^T \end{bmatrix}, \qquad W = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_k^T \end{bmatrix}$$

Z and W are $k \times mn$ matrices. The k equations of (44) are combined together

$$(46) \quad Z\left[B_{(m\times m)} \otimes B_{(n\times n)}^{-1}\right] + C_{(k\times k)}Z - A_{(k\times k)}Z\left[B_{(m\times m)} \otimes I_{(n\times n)}\right] = W$$

where

$$C_{k\times k} = [c_{ij}], \qquad A_{(k\times k)} = [a_{ij}]$$

are coefficient matrices of PDE (19).

Let the i-th column of Z be $(Z)_i$, and the i-th column of W be $(W)_i$ $(i = 1, 2, \ldots, mn)$ and rearrange Z and W into kmn vectors in the same way as in (40). Then, using the Kronecker product formula, (46) may be finally written as

$$(47) \qquad \begin{bmatrix} (Z)_1 \\ (Z)_2 \\ \vdots \\ (Z)_{mn} \end{bmatrix} = E^{-1} \begin{bmatrix} (W)_1 \\ (W)_2 \\ \vdots \\ (W)_{mn} \end{bmatrix},$$

where

(48) $$E =$$

$$= I_{(k\times k)} \otimes B^T_{(m\times m)} \otimes (B^{-1}_{(n\times n)})^T + C_{(k\times k)} \otimes I_{(mn\times mn)} - A_{(k\times k)} \otimes B^T_{(m\times m)} \otimes I_{(n\times n)}$$

is $kmn \times kmn$ matrix. Equation (47) gives solution $Y^{(i)}$, $i = 1, 2, \ldots, k$. Once $Y^{(i)}$ are determined the approximation of $y_i(x,t)$ can be obtained.

If we use (47) directly to determine the solution, some difficulties might occur in obtaining the inverse of a square matrix $kmn \times kmn$, specially, if k, m and n are large values. In fact the b.p.f. is more fundamental than the Walsh functions and the operational matrix B is simpler than the operational matrix derived from Walsh functions. We derive a recursive algorithm to reduce the key step in (47) from the inverse of the $kmn \times kmn$ matrix to the inverse of $k \times k$ matrix and then the solution $Y^{(i)}$ $(i = 1, 2, \ldots, k)$ is easily obtained. The algorithm is as follows.

First, we note that the matrix $B_{(n\times n)}$ is triangular matrix

(49) $$B_{(n\times n)} = \frac{1}{2n}\begin{bmatrix} 1 & 2 & 2 & & 2 \\ 0 & 1 & 2 & & 2 \\ 0 & 0 & 1 & & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}.$$

An elementary calculation shows that $B^{-1}_{(n\times n)}$ has the following form

(50) $$B^{-1}_{(n\times n)} = 2n\begin{bmatrix} r_1 & r_2 & r_3 & & r_n \\ 0 & r_1 & r_2 & & r_{n-1} \\ 0 & 0 & r_1 & & r_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & r_1 \end{bmatrix},$$

where r_i $(i = 1, 2, \ldots, n)$ are obtained by the recursive formulae

(51) $$r_1 = 1,$$ $$r_i = -2\sum_{j=1}^{i-1} r_j \qquad (i = 2, 3, \ldots, n).$$

Substituting (50) into (48) and using the definition of the Kronecker product of matrices in (43), the matrix E can be written as

$$E = \begin{bmatrix} P_1 & 0 & 0 & & 0 \\ P_2 & P_1 & 0 & & 0 \\ P_3 & P_2 & P_1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ P_n & P_{n-1} & P_{n-2} & \dots & P_1 \end{bmatrix} \tag{52}$$

where

$$P_1 = 2nI_{(k\times k)} \otimes B^T_{(m\times m)} + C_{(k\times k)} \otimes I_{(m\times m)} - A_{(k\times k)} \otimes B^T_{(m\times m)} \tag{53}$$

and

$$P_i = 2nr_i I_{(k\times k)} \otimes B^T_{(m\times m)} \qquad (i = 2, 3, \dots, n) \tag{54}$$

are $km \times km$ matrices and 0 is $km \times km$ zero matrix. The matrix E has a special form and its inverse is easily obtained by

$$E^{-1} = \begin{bmatrix} R_1 & 0 & 0 & & 0 \\ R_2 & R_1 & 0 & & 0 \\ R_3 & R_2 & R_1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ R_n & R_{n-1} & R_{n-2} & & R_1 \end{bmatrix},$$

where R_i $(i = 1, 2, \dots, n)$ are $km \times km$ matrices determined by the following recursive formulae

$$R_1 = P_1^{-1}, \tag{56}$$

$$R_i = -\sum_{j=1}^{i-1} R_1 P_{i-j+1} R_j \qquad (i = 2, 3, \dots, n). \tag{57}$$

Substituting (54) into (57), we have

$$R_i = -2nR_1[I_{(k\times k)} \otimes B^T_{(m\times m)}] \left(\sum_{j=1}^{i-1} r_{i-j+1} R_j \right) \tag{58}$$

$$(i = 2, 3, \dots, n).$$

Now, the inverse of matrix P_1 can be obtained in the same way. Indeed, from (53) and (7)

$$(59) \qquad P_1 = \begin{bmatrix} H_1 & 0 & 0 & & 0 \\ H & H_1 & 0 & & 0 \\ H & H & H_1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ H & H & H & & H_1 \end{bmatrix},$$

where

$$(60) \qquad H_1 = \frac{n}{m} I_{(k\times k)} + C_{(k\times k)} - \frac{1}{2m} A_{(k\times k)}$$

and

$$(61) \qquad H = \frac{2n}{m} I_{(k\times k)} - \frac{1}{m} A_{(k\times k)}$$

are $k \times k$ matrices and 0 is $k \times k$ zero matrix. Consequently,

$$(62) \qquad P_1^{-1} = \begin{bmatrix} S_1 & 0 & 0 & & 0 \\ S_2 & S_1 & 0 & & 0 \\ S_3 & S_2 & S_1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ S_m & S_{m-1} & S_{m-2} & & S_1 \end{bmatrix},$$

where S_i $(i = 1, 2, \ldots, m)$ are $k \times k$ matrices determined by the following recursive formulae

$$(63) \qquad S_1 = H_1^{-1},$$

$$(64) \qquad S_i = -S_1 H \left(\sum_{j=1}^{i-1} S_j \right) \qquad (i = 2, 3, \ldots, m).$$

This completes the derivation of inverse of matrix E. The solution $Y^{(i)}$ $(i = = 1, 2, \ldots, k)$ is easily found by substituting E^{-1} into (47), namely

$$(65) \qquad \begin{bmatrix} (Z)_{(j-1)m+1} \\ (Z)_{(j-1)m+2} \\ \vdots \\ (Z)_{jm} \end{bmatrix} = \sum_{i=1}^{j} R_{j-i+1} \begin{bmatrix} (W)_{(i-1)m+1} \\ (W)_{(i-1)m+2} \\ \vdots \\ (W)_{im} \end{bmatrix}$$

$$(j = 1, 2, \dots, n).$$

The final result will be identical to the result obtained by Shih and Han [3]. Obviously, the number of critical operations involved in the present method, using b.p.f., is much smaller than in the Walsh series method. Therefore, we have saved computing time, storage and, consequently, have reduced the round-off errors significantly.

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Rafat Riad
Mathematical Department
Faculty of Education
Ain Shams University
Cairo, Egypt