

A CONSTRUCTIVE METHOD FOR UNIFORM APPROXIMATION BY MEANS OF LAGRANGE-INTERPOLATION IN THE SPACE OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

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To the memory of Professor József Mogyoródi

G.Grünwald [1] has proved that for every function $f(x) \in C^1[-1, +1] = \{f; \exists f'(x) \in C[-1, +1]\}$ the Hermite-Fejér [2] interpolation polynomials of degree $(2n - 1)$

$$(1) \quad H_{2n-1}[f, f', x] = \sum_{k=1}^n f(x_k^{(n)})h_k(x) + \sum_{k=1}^n f'(x_k^{(n)})\hat{h}_k(x)$$

converge uniformly on the closed interval $[-1, +1]$ for any strongly normal system of nodal points

$$\{x_k^{(n)}\}_{k=1}^n; \quad (n = 1, 2, \dots) \quad \text{if } n \rightarrow \infty.$$

The aim of this note is to give such a sequence of polynomials $\{Q_n[f, f']\}$ getting by integration of the interpolation polynomials $L_{n-1}[f', x]$ of $f'(x)$ which has the same property as the above mentioned Fejér polynomials have, for a rather extended class of nodal point systems.

Theorem. *Let be $f(x) \in C^1[a, b] = \{x; -\infty < a < x < b < +\infty\}$ and let us consider a system of nodal points*

$$(2) \quad a < x_1^{(n)} < x_2^{(n)} < \dots < x_k^{(n)} < \dots < x_n^{(n)} < b$$

$$(n = 1, 2, \dots)$$

generated by the roots of $\omega_n(x)$, where $\{\omega_n(x)\}_{n=0}^\infty$ is the system of the orthogonal polynomials with respect to a given weight function $\rho(x) \in L[a, b]$ satisfying the additional condition that $[\rho(x)]^{-1} \in L[a, b]$.

If we denote by

$$(3) \quad P_{n-1}(x) = L_{n-1}[f', x] = \sum_{k=1}^n f'(x_k^{(n)}) \ell_k(x)$$

the Lagrange interpolation polynomials of $f'(x)$, then the polynomials

$$(4) \quad Q_n(x) = \int_a^x P_{n-1}(t) dt + f(a)$$

of degree n converge uniformly to $f(x)$ on the interval $[a, b]$ as $n \rightarrow +\infty$.

Proof. For any fixed point $x \in [a, b]$ we get from (4) that

$$(5) \quad f(x) - Q_n(x) = \int_a^x (f'(t) - P_{n-1}(t)) dt = \int_a^x (f'(t) - L_{n-1}[f', t]) dt$$

and therefore

$$(6) \quad |f(x) - Q_n(x)| \leq \int_a^x |f'(t) - L_{n-1}[f', t]| dt \leq \int_a^b |f'(t) - L_{n-1}[f', t]| dt.$$

Let us denote by $H_{n-1}(t)$ the best approximating polynomial of degree $\leq (n-1)$ for the function $f'(t) \in C[a, b]$ on the interval $[a, b]$, then we get from the inequality

$$(7) \quad |f'(t) - L_{n-1}[f', t]| \leq |f'(t) - H_{n-1}(t)| + |H_{n-1}(t) - L_{n-1}[f', t]| = |f'(t) - H_{n-1}(t)| + |L_{n-1}[H_{n-1}, t] - L_{n-1}[f', t]|$$

that the inequality (6) can be reduced to the following one

$$(8) \quad |f(x) - Q_n(x)| \leq \int_a^b |f'(t) - H_{n-1}(t)| dt + \int_a^b |L_{n-1}[H_{n-1}, t] - L_{n-1}[f', t]| dt \leq E_{n-1}(f')(b-a) + \int_a^b |L_{n-1}[H_{n-1}, t] - L_{n-1}[f', t]| dt$$

where $E_{n-1}(f') = \max_{[a,b]} |f'(t) - H_{n-1}(t)| \rightarrow 0$ ($n \rightarrow \infty$).

And so the only task is to estimate from above the second term in the right of the inequality (8).

It can be estimated by the Schwarz inequality as follows

$$\begin{aligned}
 & \int_a^b |L_{n-1}[H_{n-1}, t] - L_{n-1}[f', t]| dt = \\
 & = \int_a^b (|L_{n-1}[H_{n-1}, t] - L_{n-1}[f', t]| \sqrt{\rho(t)}) \frac{1}{\sqrt{\rho(t)}} dt \leq \\
 (9) \quad & \leq \left[\int_a^b (L_{n-1}[H_{n-1}, t] - L_{n-1}[f', t])^2 \rho(t) dt \right]^{\frac{1}{2}} \left[\int_a^b \frac{1}{\rho(t)} dt \right]^{\frac{1}{2}} = \\
 & = \sqrt{C} \left[\int_a^b (L_{n-1}[H_{n-1}, t] - L_{n-1}[f', t])^2 \rho(t) dt \right]^{\frac{1}{2}}
 \end{aligned}$$

where we have used that $\int_a^b \frac{1}{\rho(t)} dt = C$ is a finite positive number. If we write the last right side of (9) in the form

$$\begin{aligned}
 & \sqrt{C} \left[\int_a^b \left(\sum_{k=1}^n [H_{n-1}(x_k^{(n)}) - f'(x_k^{(n)})] \ell_k(t) \right)^2 \rho(t) dt \right]^{\frac{1}{2}} = \\
 (10) \quad & = \sqrt{C} \left[\sum_{k=1}^n (H_{n-1}(x_k^{(n)}) - f'(x_k^{(n)}))^2 \int_a^b \ell_k^2(t) \rho(t) dt \right]^{\frac{1}{2}}
 \end{aligned}$$

where we have used that the fundamental polynomials $\{\ell_k(t)\}_{k=1}^n$ are orthogonal [4] with respect to the weight function $\rho(t)$ on the interval $[a, b]$, i.e.

$$\int_a^b \ell_k(t) \ell_i(t) \rho(t) dt = 0 \quad \text{if } k \neq i,$$

and so we get from (8), (9) and (10) that

$$\begin{aligned}
 & |f(x) - Q_n(x)| \leq \\
 & \leq E_{n-1}(f')(b-a) + \sqrt{C} \left[E_{n-1}^2(f') \sum_{k=1}^n \int_a^b \ell_k^2(t) \rho(t) dt \right]^{\frac{1}{2}} = \\
 & = E_{n-1}(f')(b-a) + \sqrt{C} E_{n-1}(f') \left[\int_a^b \left(\sum_{k=1}^n \ell_k^2(t) \right) \rho(t) dt \right]^{\frac{1}{2}} = \\
 (11) \quad & = E_{n-1}(f') \left[(b-a) + \sqrt{C} \left\{ \int_a^b \left(\sum_{k=1}^n \ell_k(t) \right)^2 \rho(t) dt \right\}^{\frac{1}{2}} \right] = \\
 & = E_{n-1}(f') \left[(b-a) + \sqrt{C} \left(\int_a^b \rho(t) dt \right)^{\frac{1}{2}} \right]
 \end{aligned}$$

regarding the fact that $\left(\sum_{k=1}^n \ell_k(t) \right)^2 \equiv 1$, and so for every $x \in [a, b]$

$$(12) \quad |f(x) - Q(x)| \leq E_{n-1}(f') \left[(b-a) + \left(\int_a^b \frac{1}{\rho(t)} \int_a^b \rho(t) dt \right)^{\frac{1}{2}} \right]$$

and so our theorem is proved, and at the same time the inequality (12) gives us an order of magnitude with respect to

$$(13) \quad \|f(x) - Q_n(x)\|_{C[a,b]} = \max_{[a,b]} |f(x) - Q_n(x)| = O(E_{n-1}(f')).$$

Corollaries. (I) If we write the definition (4) in the form

$$A_n[f] = \int_a^x \left(\sum_{k=1}^n f'(x_k^{(n)}) \ell_k(t) \right) dt + f(a)$$

then we can see that $Q_n(x) = A_n[f]$ is a polynomial operator in the space $C^1[a, b]$ of degree n and if we consider this operator as

$$A_n[f] \in C^1[a, b] \rightarrow C[a, b]$$

mapping $C^1[a, b]$ into $C[a, b]$, then

$$\|f - A_n[f]\|_{C[a,b]} \xrightarrow{n \rightarrow \infty} 0 \quad \text{as } n \rightarrow \infty.$$

(II) If $\rho_{(\alpha, \beta)}(x) = (1-x)^\alpha(1+x)^\beta$ and $[a, b] = [-1, +1]$, i.e. $\omega_n(x) = J_n^{(\alpha, \beta)}(x)$ is the n -th Jacobian polynomial, then we can apply our theorem if $-1 < \alpha, \beta < +1$ because in this case both of the integrals

$$\int_{-1}^{+1} \rho(x) dx = \int_{-1}^{+1} (1-x)^\alpha(1+x)^\beta dx \quad \text{and}$$

$$\int_{-1}^{+1} \frac{1}{\rho(x)} dx = \int_{-1}^{+1} (1-x)^{-\alpha}(1+x)^{-\beta} dx$$

are finite.

(III) If we consider the initial value problem

$$(14) \quad y' = f(x), \quad y(a) = A$$

where $f(x) \in C[a, b]$, then for the solution $y = \varphi(x)$ of (14) we can apply our method and the approximating polynomials

$$(15) \quad \varphi_n(x) = A + \int_a^x \left(\sum_{k=1}^n f(x_k^{(n)}) \ell_k(t) \right) dt$$

of n -th order converge uniformly to $\varphi(x)$ on the interval $[a, b]$, and

$$(16) \quad \max_{[a,b]} |\varphi_n(x) - \varphi(x)| = O(E_{n-1}(f)).$$

Specially, if $f(x) \in \text{Lip}^{\alpha > 0}[a, b]$ then in the right of (16) we can write $O\left(\frac{1}{n^\alpha}\right)$,

and if $f(x)$ is differentiable then we may write $O\left(\frac{1}{n}\right)$ [3].

References

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