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THE VALUE DISTRIBUTION OF AN ADDITIVE FUNCTION

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Dedicated to Professor Karl-Heinz Indlekofer on his fiftieth birthday

1. Introduction

In this paper we consider a family of completely additive functions $\beta = \beta_q$ and, through the application of existing theorems in probabilistic number theory, determine the global distribution of $\beta(n)$, $\beta(p+1)$ and $\beta_2(n^2+1)$. In addition we determine the mean value of $\beta(n^2+a)$.

For each prime q the functions $\beta = \beta_q$ are defined by

$$\text{(1.1)} \quad \begin{aligned} &\text{i)}\ \beta(p) = \begin{cases} k & \text{when } q^k \| (p+1), \\ 0 & \text{when } (q, p+1) = 1; \end{cases} \\ &\text{ii)}\ \beta(n) = \sum_{p^\alpha \| n} \alpha\beta(p). \end{aligned}$$

As is the case for the well known functions ω and Ω, the values of β are nonnegative. But in contrast there are many primes p for which $\beta(p) = 0$ while there also exist primes with the property that $\beta(p) = (\log p)/(\log q)$. It is this last property which requires one to use sieve methods and estimates for the number of primes in arithmetic progressions with moduli of the form q^k.

In what follows the letters p and q refer to primes, the letter ε denotes a small positive number. If we let $N_x(n;\ldots)$ denote the number of positive integers not exceeding x which have the property ... then $\nu_x(n;\ldots) := [x]^{-1} \cdot N_x(n;\ldots)$. Similarly if $N_x(p;\ldots)$ denotes the number of positive primes not exceeding x with the property ... then $\nu_x(p;\ldots) := [\pi(x)]^{-1} \cdot N_x(p;\ldots)$. The standard Gaussian law $\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{z} e^{-t^2/2} dt$ is denoted by $\Phi(z)$. Furthermore for $k \geq 2$, $\log_k x = \log(\log_{k-1} x)$ indicates the iterated logarithm. In general the letter c will denote an arbitrary constant, not the same in each instance. With this notation we will prove the following three distribution results for the function $\beta = \beta_q$.

Let c_1 and c_2 be two constants with values

$$c_1 = \frac{q}{(q-1)^2} \quad \text{and} \quad c_2 = \frac{q(q+1)}{(q-1)^3} \tag{1.2}$$

then

Theorem 1. *For any real number z*

$$\nu_x(n;\ \beta(n) - c_1 \log_2 x \leq z(c_2 \log_2 x)^{1/2}) \to \Phi(z) \quad (x \to \infty).$$

Theorem 2. *For any real number z*

$$\nu_x(p;\ \beta(p+1) - c_1 \log_2 x \leq z(c_2 \log_2 x)^{1/2}) \to \Phi(z) \quad (x \to \infty).$$

Theorem 3. *For $\beta = \beta_2$ and any real number z*

$$\nu_x(n;\ \beta(n^2+1) \log_2 x \leq z(\log_2 x)^{1/2}) \to \Phi(z) \quad (x \to \infty).$$

The mean value which we obtain can be stated as

Theorem 4. *For a positive integer a and $\beta = \beta_q$*

$$[x]^{-1} \sum_{n \leq x} \beta(n^2 + a) = C \log_2 x (1 + o(1))$$

where $C = \begin{cases} 1 & \textit{when } a = 1 \textit{ and } q = 2, \\ \dfrac{q}{(q-a)^2} & \textit{otherwise.} \end{cases}$

Before I end this introduction I must express my deep gratitude to Professor I. Kátai who suggested this work and supported me with much needed advice and encouragement.

2. Some preliminary lemmas

We begin by determining the quantities

$$(2.1)\qquad A(x) := \sum_{p\le x} \frac{\beta(p)}{p} \quad \text{and} \quad \beta^2(x) := \sum_{p\le x} \frac{\beta^2(p)}{p}.$$

In order to do this we need an estimate for the number of primes in arithmetical progressions with moduli q^k which holds uniformly for $q^k < x^{1/3}$. From work by Iwaniec [5] the following estimate can be obtained.

Let $\mathcal{P}$ be a finite set of primes and $\mathcal{D} = \left\{ D = \prod_{p\in\mathcal{P}} p^\alpha,\ \alpha \ge 0 \right\}$ be a set of integers. For $(1, D) = 1$ and a fixed constant c the estimate

$$(2.2)\qquad \pi(x, D, 1) = \frac{x}{\phi(D)\log\ x}\{1 + O(\exp(-c\sqrt{\log\ x}))\}$$

holds uniformly for $D \in \mathcal{D}$ and $D < x^{1/3}$.

We can now prove

Lemma 1. *For fixed q and k the estimate*

$$\sum_{\substack{p\le x\\ p\equiv -1 (\mathrm{mod}\ q^k)}} \frac{1}{p} = \frac{\log_2 x}{\phi(q^k)} + O\left(\frac{k\ \log\ q}{q^k}\right)$$

holds uniformly for $q^k < x^{1/3}$.

Proof. In order to use (2.2), we distunguish between small and large primes and write

$$(2.3)\qquad \sum_{\substack{p\le x\\ p\equiv -1 (\mathrm{mod}\ q^k)}} \frac{1}{p} = \sum_{p\le q^{3k}} \frac{1}{p} + \sum_{q^{3k}<p\le x} \frac{1}{p} = S_1 + S_2,$$

where the same congruence condition holds in all sums. Since $p+1 = nq^k$ and $\frac{1}{p} \le \frac{2}{p+1}$ it follows that

$$(2.4)\qquad S_1 \le \frac{2}{q^k} \sum_{n\le q^{2k}} \frac{1}{n} \le 8\frac{k\log\ q}{q^k}.$$

To evaluate S_2 we use partial summation and (2.2). Then

$$S_2 = \int_{q^{3k}}^{x} \frac{1}{u^2}\pi(u, q^k, -1)du + \frac{\pi(x, q^k, -1)}{x} = \tag{2.5}$$

$$= \int_{q^{3k}}^{x} \frac{u}{u^2 \log\ u\phi(q^k)}du + E = \frac{1}{\phi(q^k)}(\log_2 x - \log_2 q^{3k}) + E.$$

For $q^k < x^{1/3}$

$$E = O\left\{\int_{q^{3k}}^{x} \frac{u}{u^2 \log\ u\phi(q^k)} \exp(-c\sqrt{\log\ u})du\right\} + \frac{\pi(x, q^k, -1)}{x} =$$

$$= O\left\{\frac{1}{\phi(q^k)\sqrt{\log\ q^{3k}}}\right\}. \tag{2.6}$$

Combining (2.3), (2.4), (2.5) and (2.6) the lemma follows.

It is now easy to obtain the first three moments of $\beta(n)$.

Lemma 2. *For q fixed and $c_j = \sum_{k=1}^{\infty} \frac{k^j}{q^k}$*

$$\sum_{p \le x} \frac{\beta^j(p)}{p} = c_j \log_2 x + O(1).$$

Proof. Let N be a large integer with $q^N < x^{1/3}$. Then

$$\sum_{p \le x} \frac{\beta^j(p)}{p} = \sum_{k=1}^{N-1} k^j \sum_{\substack{p \le x \\ q^k \| p+1}} \frac{1}{p} + \sum_{\substack{p \le x \\ p \equiv -1 (mod\ q^N)}} \frac{\beta^j(p)}{p} = S_1 + S_2.$$

To evaluate S_1 we observe that $q^k \| p + 1$ means that $p \equiv -1(mod\ q^k)$ and $p \equiv -1(mod\ q^{k+1})$. Since $\beta(p) < \log x$ application of Lemma 1 results in

$$S_1 = \sum_{k=1}^{N-1} k^j \log_2 x \left(\frac{1}{\phi(q^k)} - \frac{1}{\phi(q^{k+1})}\right) + O\left(\sum_{k=1}^{N-1} k^j \frac{k \log\ q}{q^k}\right)$$

and $S_2 \ll \log^j x \left(\frac{\log_2 x}{\phi(q^N)} + O\left(\frac{N \log q}{q^N} \right) \right)$.

Choosing N such that $q^N = [(\log x)^{j+1}]$ it follows that

$$S_1 = c_j \log_2 x + O(1) \quad \text{and} \quad S_2 = o(1).$$

In particular Lemma 2 estabilishes that

$$A(x) = \frac{q}{(q-1)^2} \log_2 x + O(1) \quad \text{and} \quad B^2(x) = \frac{q(q+1)}{(q-1)^3} \log_2 x + O(1).$$

Lemma 3. *If χ is a quadratic character modulo a and $(a, q) = 1$ then*

$$\sum_{\substack{p \le x \\ p \equiv -1 (\text{mod } q^k)}} \frac{\chi(p)}{p} = O\left(\frac{k \log q}{q^k} \right)$$

where the implied O constant depends upon a.

Proof. By the Chinese Remainder Theorem there exist integers $r_1, r_2, \dots, r_{\phi(a)/2}, n_1, \dots, n_{\phi(a)/2}$ such that

$$\chi(p) = \begin{cases} +1 & \text{if} \quad p \equiv r_i \ (\text{mod } aq^k), \\ -1 & \text{if} \quad p \equiv n_i \ (\text{mod } aq^k). \end{cases}$$

Therefore

$$\sum_{\substack{p \le x \\ p \equiv -1 (\text{mod } q^k)}} \frac{\chi(p)}{p} = \sum_{i=1}^{\phi(a)/2} \left(\sum_{\substack{p \le x \\ p \equiv r_i (\text{mod } aq^k)}} \frac{1}{p} - \sum_{\substack{p \le x \\ p \equiv n_i (\text{mod } aq^k)}} \frac{1}{p} \right).$$

Evaluating the inner sums on the right hand side as in Lemma 2 using (2.2) we obtain the lemma.

For each polynomial $f(n)$ the multiplicative function $\rho = \rho_f$ is defined on the positive integers by

$$\rho_f(d) = \sum_{\substack{f(n) \equiv 0 (\text{mod } d) \\ 1 \le n \le d}} 1. \tag{2.7}$$

It is well known that

$$\sum_{p \le x} \frac{\rho(p)}{p} = t \log_2 x + O_f(1)$$

where t is the number of irreducible components of $f(n)$. A comparable estimate holds for the irreducible polynomial $f(n) = n^2 + a$ when the sum is restricted to primes $p \equiv -1 (\text{mod } q^k)$.

Lemma 4. *Let $f(n) = n^2 + a$ where a is a positive integer. For $a > 1$ and all q or $a = 1$ and $q > 2$*

$$\sum_{\substack{p \leq x \\ p \equiv -1 (\text{mod } q^k)}} \frac{\rho(p)}{p} = \frac{\log_2 x}{\phi(q^k)} + O\left(\frac{k \log q}{q^k}\right).$$

Proof. Since $\rho(p) = 1 + \left(\frac{-a}{p}\right)$ when $f(n) = n^2 + a$ it follows from Lemma 1 that it is sufficient to show

$$\sum_{\substack{p \leq x \\ p \equiv -1 (\text{mod } q^k)}} \frac{\left(\frac{-a}{p}\right)}{p} = O\left(\frac{k \log q}{q^k}\right). \tag{2.8}$$

Without loss of generality it may be assumed that a is squarefree.

For $a = 1$ the Jacobi symbol $\left(\frac{-a}{p}\right) = \left(\frac{-1}{p}\right)$ is a quadratic character modulo 4. Since $q \neq 2$, (2.8) follows from Lemma 3.

For $a > 1$ and odd we apply the law of quadratic reciprocity and obtain $\left(\frac{-a}{p}\right) = (-1)^{(a-1)/2}\left(\frac{p}{a}\right)$ which is a quadratic character modulo a. When $(a, q) = 1$ the result follows from Lemma 3. When $a = qb$ and $p \equiv -1 (\text{mod } q^k)$

$$\left(\frac{p}{a}\right) = \left(\frac{p}{q}\right)\left(\frac{p}{b}\right) = \left(\frac{-1}{q}\right)\left(\frac{p}{b}\right) = (-1)^{(q-1)/2}\left(\frac{p}{b}\right),$$

which is a quadratic character modulo b and since $(b, q) = 1$ Lemma 3 gives the result in this case also.

When a is even $\left(\frac{-a}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{-b}{p}\right)$ which is a quadratic character modulo $8b$. If $(q, 8b) = 1$ one can apply Lemma 3 and (2.8) follows. The case $(q, 8b) > 1$ is left to the reader.

The following lemma is analogous to Lemma 2.

Lemma 5. *For $f(n) = n^2 + a$ the sum*

$$\sum_{p \leq x} \frac{\beta^j(p)\rho(p)}{p} = c_j \log_2 x + O(1)$$

$$\text{where} \quad c_j = \begin{cases} 1 & \text{when } a = 1 \text{ and } q = 2, \\ \sum_{k=1}^{\infty} \frac{k^j}{q^k} & \text{otherwise.} \end{cases}$$

Proof. When $a = 1$ and $q = 2$, $\rho(p) = 1$ iff $p \equiv 1 \pmod 4$. In this case $\beta(p) = 1$. Therefore by Lemma 1

$$\sum_{p \le x} \frac{\beta^j(p)\rho(p)}{p} = \sum_{\substack{p \le x \\ p \equiv 1 \pmod 4}} \frac{2}{p} = \log_2 x + O(1).$$

In all other cases the proof follows from Lemma 4 and is left to the reader.

3. Proofs of distribution results

The theorems will be established by verifying that existing results in probabilistic number theory apply to the function $\beta(n)$, which will not be too difficult due to Lemma 2 and 5.

Proof of Theorem 1. In [2] we find a result due to Kubilius and Shapiro which reads:

Let $f(n)$ be a strongly additive function (i.e. $f(p^\alpha) = f(p)$ for all $\alpha \ge 1$) and define

$$(3.1) \qquad A(x) = \sum_{p \le x} \frac{f(p)}{p} \quad \text{and} \quad B(x) = \left(\sum_{p \le x} \frac{f^2(p)}{p} \right)^{1/2} \ge 0.$$

If for all $\varepsilon > 0$

$$(3.2) \qquad \frac{1}{B^2(x)} \sum_{\substack{p \le x \\ |f(p)| > \varepsilon B(x)}} \frac{f^2(p)}{p} \to 0 \quad (x \to \infty)$$

then for every real number z

$$\nu_x\Big(f(n) - A(x) \le z\, B(x)\Big) \to \Phi(z) \quad (x \to \infty).$$

This result can easily be extended to additive functions when $B(x)$ is unbounded, which is the case for the function $\beta(n)$. Since $\beta(p) \ge 0$, the fact that condition (3.2) holds for $\beta(n)$ follows by applying Lemma 2 which results in

$$\frac{1}{B^2(x)} \sum_{\substack{p \le x \\ |\beta(p)| > \varepsilon B(x)}} \frac{\beta^2(p)}{p} < \frac{1}{\varepsilon B^3(x)} \sum_{p \le x} \frac{\beta^3(p)}{p} \ll \frac{\log_2 x}{(\log_2 x)^{3/2}}.$$

Because $\Phi(z)$ is continuous and $B(x)$ is unbounded $A(x)$ and $B(x)$ can be replaced by $\frac{q}{(q-1)} \log_2 x$ and $\left(\frac{q(q+1)}{(q-1)^3} \log_2 x\right)^{1/2}$ respectively and Theorem 1 follows.

Proof of Theorem 2. The Kubilius class H is defined to be the set of strongly additive functions with the property that for all $y > 0$

$$\frac{B(x^y)}{B(x)} \to 1 \quad (x \to \infty). \tag{3.3}$$

The distribution result for $\beta(n)$ on the shifted primes now follows from a result by Barban, Vinogradov and Linnik [1] which can be stated as follows.

Let $f(n)$ be in the Kubilius class H and $A(x)$ and $B(x)$ defined as in (3.1). If

$$\frac{1}{B^2(x)} \sum_{\substack{p \le x \\ f(p) \le u B(x)}} \frac{f^2(p)}{p} \to \begin{cases} 1 & \text{for } u \ge 0 \\ 0 & \text{for } u < 0 \end{cases} \tag{3.4}$$

then for each real number z

$$\nu_x\Big(f(p+1) - A(x) \le z\, B(x)\Big) \to \Phi(z) \quad (x \to \infty).$$

For $\beta(n)$ the condition (3.3) obviously holds. Again, since $B(x)$ is unbounded, the fact that $\beta(n)$ is not strongly additive does not matter. Finally condition (3.4) is also satisfied because $\beta(p) \ge 0$ for all p. This means that for $u < 0$ the limit in (3.4) is obviously 0 while for $u \ge 0$ condition (3.4) is equivalent to (3.2). Again, since $\Phi(z)$ is continuous and $B(x)$ is unbounded, Theorem 2 can be inferred.

Proof of Theorem 3. A result of Halberstam [3] can be stated as follows:

Let $f(n)$ be a strongly additive function with $f(0) = 0$. Let $h(n)$ be a polynomial in $\mathbb{Z}[x]$ and $\rho = \rho_h$ be defined by (2.7). Define

$$A_1(x) = \sum_{p \le x} \rho(p) \frac{f(p)}{p},$$

$$B_1(x) = \left(\sum_{p \le x} \rho(p) \frac{f^2(p)}{p}\right)^{1/2} \ge 0,$$

$$\mu_x = \max_{p \le x} \mid f(p) \mid B_1(x)^{-1}.$$

If

$$\text{(3.5)} \qquad \text{(assuming that } B_1(x) > 0) \qquad \mu_x \to 0 \text{ as } x \to \infty$$

then for all real numbers z

$$\nu_x\Big(f(h(n)) - A_1(x) \le zB_1(x)\Big) \to \Phi(z) \text{ as } x \to \infty.$$

For unbounded $B_1(x)$ this result can be extended to all additive functions. When $h(n) = n^2 + 1$ and $f(n) = \beta(n) = \beta_2(n)$ it can be seen from Lemma 5 that

$$A_1(x) = B_1^2(x) = \log_2 x + O(1).$$

Since $\beta_2(p)$ can be as large as $(\log p)/(\log 2)$ it seems that condition (3.5) is not satisfied. This can be easily overcome by defining

$$\text{(3.6)} \qquad \beta^*(p) = \begin{cases} \beta(p) & \text{when } p \equiv 1 \ (mod\ 4), \\ 0 & \text{otherwise.} \end{cases}$$

Hence the theorem of Halberstam can be applied to $\beta^*(n)$ but since $n^2 + 1 \equiv$ $\equiv 0$ $(mod\ p)$ only has solutions if $p \equiv 1$ $(mod\ 4)$ the values of $\beta(n^2 + 1)$ and $\beta^*(n^2 + 1)$ are the same for all n and Theorem 3 follows immediately.

4. Proof of Theorem 4

While in Theorem 3 we applied the restriction $a = 1$ and $q = 2$, no such restriction is necessary when we consider the mean value of $\beta(n^2 + a)$. For any positive integer a the sum

$$S = \sum_{n \le x} \beta(n^2 + a) =$$

$$= \sum_{p \le x} \rho(p)\beta(p)\left[\frac{x}{p}\right] + \sum_{\substack{p^\alpha \le x^2 + a \\ \alpha \ge 2}} \rho(p^\alpha)\beta(p^\alpha)\left[\frac{x}{p^\alpha}\right] + \sum_{x < p \le x \log x} \rho(p)\beta(p) +$$

$$+ \sum_{x \log x < p \le x^2 + a} \rho(p)\beta(p) = S_1 + S_2 + S_3 + S_4.$$

Taking $j = 1$ in Lemma 5 we obtain the main term of S namely

$$S_1 = x \sum_{p \le x} \frac{\rho(p)\beta(p)}{p} + O\Big(\sum_{p \le x} \rho(p)\beta(p)\Big) = C\ x\ \log_2 x + O(x) \tag{4.1}$$

$$\text{where} \quad C = \begin{cases} 1 & \text{when } a = 1 \text{ and } q = 2, \\ \dfrac{q}{(q-1)^2} & \text{otherwise.} \end{cases}$$

Since $\rho(p^\alpha) \le 4$ and $\beta(p) \le \log\, p$ it follows that

$$S_2 \ll \sum_{p^2 \le x^2 + a} \log\ p \sum_{\alpha=2}^{\log\ x} \frac{\alpha}{p^\alpha} \ll x \sum_{p \le x} \frac{\log p}{p(p-1)} = O(x). \tag{4.2}$$

When evaluating S_3 we observe that $\beta(p) < 2 \log x$ and choose an integer K such that $q^K = [x^\delta]$ with $0 < \delta < 1$. Then

$$S_3 \ll \sum_{k=1}^{K} k \cdot \sum_{\substack{x < p \le x \log x \\ p \equiv -1 \pmod{q^k}}} 1 + \log\ x \cdot \sum_{\substack{x < p \le x \log x \\ p \equiv -1 \pmod{q^{K+1}}}} 1 \ll \tag{4.3}$$

$$\ll \sum_{k=1}^{K} k \cdot \pi(x \log x, q^k, -1) + \log\ x \cdot \pi(x \log x, q^{K+1}, -1) = O(x).$$

To evaluate S_4 we choose $m = (\log_3 x)^2$ and $M = 4 \log_2 x$ and set $S_4 = \sum_1 + \sum_2 + \sum_3$. In $\sum_1$ we consider only those primes for which $\beta(p) \le m$, in $\sum_2$ the primes for which $m < \beta(p) \le M$ are considered and in $\sum_3$, the remaining primes in the interval $[x \log x,\ x^2 + a]$.

Since there are at most x large primes p with $\rho(p) \ne 0$,

$$\sum\nolimits_1 \le 2mx = 2x(\log_3 x)^2. \tag{4.4}$$

In $\sum_3$ we replace $\beta(p)$ by $\log x$ so that

$$\sum\nolimits_3 \le 2 \log\ x \sum_{\substack{x \log x < p \le x^2 + a \\ p \equiv -1 \pmod{q^M}}} 1.$$

For $n^2 + a = \nu P$, $P \equiv -1(\text{mod } q^M)$ and $P > x\log x$ it follows that $\nu < x/\log x$, $n^2 + a \equiv -\nu(\text{mod } \nu q^M)$ and $P > q^M$. Therefore

$$\sum\nolimits_3 \ll \log x \sum_{\nu < x/\log x} \rho(\nu q^M)\left[\frac{x}{\nu q^M}\right] \ll \tag{4.5}$$

$$\ll \frac{x\log x}{q^M} \sum_{\nu < x/\log x} \frac{\rho(\nu)}{\nu} \ll \frac{x(\log x)^2}{q^M} = o(x).$$

To find an upper bound for $\sum_2$ we replace $\beta(p)$ by M so that

$$\sum\nolimits_2 \ll M \sum_{\substack{x\log x < p \leq x^2+a \\ p\equiv -1(\text{mod } q^m)}} \rho(p).$$

Hence an upper bound is obtained when we can find an upper estimate for the number of integers $n \leq x$ with the property

$$n^2 + a = \nu P, \qquad P \equiv -1 \text{ (mod } q^m) \text{ and } \nu < x/\log x. \tag{4.6}$$

For such n

$$n^2 + a \equiv -\nu \text{ (mod } \nu q^m). \tag{4.7}$$

For a fixed ν let n_0 be a solution to (4.7). Since $(n_0 + t\nu q^m)$ is also a solution to (4.7) it follows that

$$f(t) = (n_0 + t\nu q^m)^2 + a = \nu^2 q^{2m} t^2 + 2n_0\nu q^m t + n_0^2 + a \equiv -\nu \text{ (mod } \nu q^m).$$

From (4.7) it can be seen that the coefficients of the polynomial $F(t) = f(t)/\nu$ are integers. When $F(t) = P$, a prime, then $P \equiv -1$ (mod q^m) and there is some n, $n = n_0 + t\nu q^m$, such that $n^2 + a = \nu P$ with $P \equiv -1$ (mod q^m).

Let $N_\nu = \sharp\left\{t \leq \frac{x}{\nu q^m} \;\middle|\; F(t) \text{ is a prime}\right\}$. Then

$$\sum\nolimits_3 \leq M \cdot \sum_{\nu \leq x/\log x} \rho(\nu q^m) \cdot N_\nu. \tag{4.8}$$

An upper bound for N_ν can be given by applying Theorem 5.4 in [4] from which it follows that

$$N_\nu \leq C_F \frac{y}{\log y}\left\{1 + O_F\left(\frac{\log_2 3y}{\log y}\right)\right\} \quad \text{where } y = \frac{x}{\nu q^m}. \tag{4.9}$$

Before the estimate (4.9) can be used in (4.8) it is necessary to investigate the dependence of the constants upon $F(t)$.

The constant $C_F = \prod_p 1 - \frac{\rho(p)-1}{p-1}$, where $\rho(p) = \rho_F(p)$. It is not hard to show that $\rho_F(p) = \rho_h(p)$ except when p is a divisor of ν and $\rho_h(p) = 2$, in which case $\rho_F(p) = 1$. Hence it follows that

$$C_F = C \cdot \prod_{2<p|\nu} \left(1 - \frac{1}{p-1}\right)^{-1} \leq C \log_2 x,$$

where C is independent of the polynomial F. The implied O constant also depends upon $F(t)$ through $\rho_F(p)$. From the proof of (4.9) it can be seen that replacing ρ_F with ρ_h will introduce at most a finite power of $\log_2 x$. Hence it follows that

$$\sum\nolimits_3 \ll M \cdot \sum_{\nu < x/\log x} \rho(\nu q^m) \frac{x \log_2 x}{\nu q^m \log(x/\nu q^m)} \ll$$

$$\ll \frac{x(\log_2 x)^2}{q^m} \sum_{\nu \leq x/\log x} \frac{\rho(\nu)}{\nu \log(x/\nu q^m)}. \tag{4.10}$$

To estimate the inner sum let

$$I_j := \{\nu \mid e^j < x/\nu q^m \leq e^{j+1}\} = \left\{\nu \,\middle|\, \frac{x}{q^m e^{j+1}} \leq \nu < \frac{x}{q^m e^j}\right\} =$$

$$= \{\nu \mid T \leq \nu < eT\} \text{ with } T = \frac{x}{q^m e^j} \text{ and } 1 \leq j \leq \log x.$$

Then

$$\sum\nolimits_3 \ll \frac{x(\log_2 x)^2}{q^m} \sum_{j=1}^{\log x} \frac{1}{j} \sum_{T \leq \nu < eT} \frac{\rho(\nu)}{\nu} \ll \frac{x(\log_2 x)^3}{q^m} = o(x). \tag{4.11}$$

From (4.4), and (4.11) it follows that $S_4 \ll x(\log_3 x)^2$ from which Theorem 4 now follows.

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