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# TRIGONOMETRIC SUMS AND THEIR APPLICATIONS

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*Dedicated to Professor Karl-Heinz Indlekofer on the occasion of his fiftieth birthday*

## 1. Introduction

Some problems of analytic number theory are connected with estimates of sums of type

$$(*)\qquad S = \sum_{(n_1,\dots,n_k)\in D} e^{2\pi i f(n_1,\dots,n_k)},$$

where $D$ is a domain of dimension $k$, $f(x_1,\dots,x_k)$ is a real function which has a certain number of derivatives $\dfrac{\partial^{i_1+\dots+i_k} f}{\partial x_1^{i_1}\dots\partial x_k^{i_k}}$. Estimates for trigonometric sums $(*)$ have been obtained by many authors, for example by van der Corput [1], E.Titchmarsh [2], [3], P.Schmidt [4], B.R.Srinivasan [5], G.Kolesnik [6], [7]. They used the method of exponent pairs which was introduced by van der Corput [1] and developed by P.Phillips [8].

In this paper we shall use this method to construct estimates of sums $(*)$ which we shall apply in some problems of probabilistic number theory.

**Notations.** Throughout the paper we suppose that $D, D_1, \dots$ are domains of dimension $k$, such that their boundaries consist of a bounded number of algebraic curves the degrees of which are bounded.

$X_i$, $i = 1,\dots k$ are sufficiently large numbers;

$f(x) \ll g(x)$ means that $f(x) = O(g(x))$ and $g(x) > 0$;

$n \sim N$ means that $N \le n \le N' \le 2N$;

$N_1 \asymp N_2$ means that $c_1 N_2 < N_1 < c_2 N_2$, $c_1, c_2 > 0$ - constants;

$e(f) = e^{2\pi i f}$.

## 2. Basic definitions and lemmas

**Definition 1.** The real function $g(x_1,\dots,x_k)$ is said to be an approximation of degree $r$ to the real function $f(x_1,\dots,x_k)$ in $D$ if $f$ and $g$ posses partial derivatives up to the order $r$ in $D$ and

$$\left|f_{x_{i_1}\dots x_{i_m}} - g_{x_{i_1}\dots x_{i_m}}\right| < c\left|g_{x_{i_1}\dots x_{i_m}}\right|$$

for all $(x_1,\dots,x_k)\in D$, $m\le r$, and for some sufficiently small number $c\in$ $\in[0,\frac{1}{2}]$. We than write

$$f \underset{D}{\overset{r}{\sim}} g.$$

**Lemma 1 (Lemma of partial summation).** *Let $g(m,n)$ denote arbitrary numbers, real or complex, such that if*

$$G(m,n)=\sum_{\substack{1\le\mu\le m\\ 1\le\nu\le n\\ (\mu,\nu)\in D}} g(\mu,\nu),$$

*then $|G(m,n)|\le G$, $(1\le m\le M,\ 1\le n\le N)$ for any arbitrary region $D$ contained in the rectangle $1\le m\le M,\ 1\le n\le N$. Let $h(m,n)$ denote real numbers $0\le h(m,n)\le H$ such that the following three expressions*

$$h(m,n)-h(m+1,n),\quad h(m,n)-h(m,n+1),$$

$$h(m,n)-h(m+1,n)-h(m,n+1)+h(m+1,n+1)$$

*keep a fixed sign for all values of $m,n$ considered. Then*

$$\left|\sum_{(m,n)\in D} g(m,n)h(m,n)\right| \ll G\cdot H.$$

**Lemma 2.** *If $f(x_1,x_2)\underset{D}{\overset{r}{\sim}} Ax_1^{-s_1}x_2^{-s_2}$ and*

$$F(x_1,x_2,h)=f(x_1+h,x_2)-f(x_1,x_2)$$

*where $h>0$, $(x_1,x_2)\in D$, $(x_1+h,x_2)\in D$. Then*

$$F \underset{D(h)}{\overset{r-1}{\sim}} Ahs_1x_1^{-s_1-1}x_2^{-s_2}$$

*where $D(h)=\{(x_1,x_2)\in D\mid (x_1+h,x_2)\in D\}$.*

**Lemma 3.** *Let $D$ be a region contained in the rectangle $\{a_i \le x_i \le \le 2a_i,\ i = 1,\dots,k\}$ and let*

$$f \underset{D}{\overset{r}{\sim}} Ax_1^{-s_1}\dots x_k^{-s_k}.$$

*Let us denote by $\Delta$ the image of $D$ under the mapping*

$$y_j = f'_{x_j}, \quad j = 1,\dots,\ell;$$
$$y_j = x_j, \quad j = \ell+1,\dots,k.$$

*Let $\varphi(y_1,\dots,y_k) = f(\mu_1,\dots,\mu_\ell,y_{\ell+1},\dots,y_k) - \sum_{i=1}^{\ell}\mu_i y_i$, where $\mu_1,\dots,\mu_\ell$ is the solution of the equations*

$$y_i = \frac{\partial f}{\partial x_i}, \qquad i = 1,\dots,\ell.$$

*Then*

$$\varphi \underset{\Delta}{\overset{r}{\sim}} By_1^{-\sigma_1}\dots y_k^{-\sigma_k}$$

*where*

$$B = \left(\frac{A}{\sigma}\right)^{\sigma}\cdot\sigma_1^{\sigma_1}\dots\sigma_\ell^{\sigma_\ell}, \quad \frac{1}{\sigma} = 1+s_1+\dots+s_\ell = s, \quad \sigma_i = -\frac{s_i}{s}, \quad i = 1,\dots,k.$$

Lemmas 2, 3 have been proved in [5].

**Lemma 4 (Transformation A).** *Let $f(x_1,\dots,x_k)$ be a real function, $1 \le \ell \le k$, $\ell \in \mathbb{N}$,*

$$f_1(x;h) := \int_0^1 \dots \int_0^1 \frac{\partial f(x_1+h_1t_1,\dots,x_\ell+h_\ell t_\ell,x_{\ell+1},\dots,x_k)}{\partial t_1\dots\partial t_\ell}dt_1\dots dt_\ell.$$

*Let $q_1,\dots,q_\ell$ be positive numbers such that*

$$\frac{q_1}{X_1} = \dots = \frac{q_\ell}{X_\ell} = \sqrt[\ell]{\frac{Q}{X}},$$

$$Q = q_1\dots q_\ell, \qquad X = X_1\dots X_\ell.$$

*Then for* $\frac{q_1}{X_1} \le \frac{|D|}{X}$, $|D| = \sum_{(n_1,\dots n_k)\in D} 1$ *we have*

$$|S| = \left|\sum_{x\in D} e(f(x))\right| \ll \frac{|D|}{Q^{1/2}} + \frac{|D|^{1/2}}{Q^{1/2}}\left|\sum_{|h_j|\le q_j}\sum_{x\in D_1} f_1(x;h)\right|^{1/2}$$

*where*

$$D_1 = \{(x_1,\dots,x_k)\in D \mid (x_1+h_1,\dots,x_\ell+h_\ell,x_{\ell+1},\dots,x_k)\in D,$$

$$|h_j|\le q_j;\ j=1,\dots,\ell\}.$$

**Corollary.** Let the conditions of Lemma 4 be satisfied with $q_j^2$ instead of $q_j$ $(j=1,\dots,\ell)$. Then

$$|S| \ll \frac{|D|}{Q^{1/2}} + \frac{|D|^{3/4}}{Q}\left\{\sum_{\substack{0<h_j^{(1)}<q_j \\ h_j^{(1)}\cdot h_j^{(2)}>q_j^{1/2} \\ (j=1,\dots,\ell)}}\left|\sum_{0<h_j^{(2)}<q_j^2}\sum_{x\in D_2} e\left(f_2(x;h^{(1)},h^{(2)})\right)\right|^{1/2}\right\}^{1/2}$$

where

$$D_2 =$$

$$\left\{(x_1,\dots,x_k)\in D \mid (x_1+h_1^{(1)}+h_1^{(2)},\dots,x_\ell+h_\ell^{(1)}+h_\ell^{(2)},x_{\ell+1},\dots,x_k)\in D\right\}.$$

Lemma 4 is a generalization of the Weyl–van der Corput's inequality (see [5]).

**Lemma 5 (Transformation B).** *Let* $f(x_1,\dots,x_k) \overset{\infty}{\underset{D}{\approx}} Ax_1^{\alpha_1}\dots x_k^{\alpha_k}$. *Then for any* $\ell$, $1\le\ell\le k$, *we have*

$$|S| = \left|\sum_{x\in D} e(f(x_1,\dots,x_k))\right| \ll$$

$$\ll X_1\dots X_\ell\,(AX_1^{\alpha_1}\dots X_k^{\alpha_k})^{-\frac{\ell}{2}}\,|S_1| + \frac{X_1\dots X_k}{\sqrt{AX_1^{\alpha_1}\dots X_k^{\alpha_k}}}$$

*where*

$$S_1 = \sum_{y\in D_3} e(f_1(y_1,\dots,y_k)),$$

$$D_3 \subset R = \{y \mid Y_i \le y_i \le 2Y_i, \quad i = 1, \dots, k\},$$

$$Y_i = AX_1^{\alpha_1} \dots X_k^{\alpha_k} \cdot X_i^{-1}, \quad i = 1, \dots, \ell; \qquad Y_i = X_i, \quad i = \ell + 1, \dots, k;$$

$$f_1(y_1, \dots, y_k) = f(\varphi_1, \dots, \varphi_\ell, y_{\ell+1}, \dots, y_k) - \varphi_1 y_1 - \dots - \varphi_\ell y_\ell,$$

$\varphi_1, \dots, \varphi_\ell$ *are the solutions of the system*

$$f'_{x_j}(\varphi_1, \dots, \varphi_\ell, y_{\ell+1}, \dots, y_k) = y_j, \quad (j = 1, \dots, \ell).$$

This lemma is essentially the combination of the Poisson summation formula with the method of stationary phase (see [7]).

**Definition 2 (see [5]).** Let $l_0, l_1$ be nonnegative numbers, $k$ be a positive integer, such that $0 \le l_1 - l_0 \le \dfrac{1}{2(k+1)}$, $l_0 \ge 0$. We say that $(l_0, l_1)$ is an exponent pair of dimension $k$ if the following condition is valid: If $s_1, \dots, s_k$ are arbitrary nonzero real numbers such that

$$\sum_{i=1}^{k} (\mu + \mu_i) s_i + \mu + \mu' + 1 \ne 0$$

holds for every choice of $\mu, \mu' \in I\!N \cup \{0\}$, $\mu_i \in \{0, 1\}$ $(i = 1, \dots, r)$ then there exists an integer $r \ge k + 4$, such that for each domain $D$ contained in the rectangle

$$\{X_j \le n_j \le uX_j, \quad j = 1, \dots, k; \quad u > 1\}$$

and satisfying the properties

$$Z_j := |As_i| X_1^{-s_1} \dots X_k^{-s_k} X_j^{-1} > 1, \quad X_j \gg 1 \quad (j = 1, \dots, k)$$

for each $f$ defined in $D$ under the condition

$$f \underset{D}{\overset{r}{\sim}} AX_1^{-s_1} \dots X_k^{-s_k},$$

the inequality

$$\sum_{(n_1, \dots, n_k)} e\left(f(n_1, \dots, n_k)\right) \ll \prod_{j=1}^{k} Z_j^{l_0} X_j^{1-l_1}$$

holds (here the constant implied by $\ll$ may depend only on $u$ and on $s_1, \dots, s_k$). Obviously $(0, 0)$ is an exponent pair.

Srinivasan [5] gave the following transformation formulas of $k$-dimensional exponent pairs:

$$A(l_0,l_1)=\left(\frac{l_0}{2(1+kl_0)},\quad \frac{l_0+l_1}{2(1+kl_0)}\right),$$

$$AB(l_0,l_1)=\left(\frac{1/2-l_1}{2+k(1-2l_1)},\quad \frac{1-l_0-l_1}{2+k(1-2l_1)}\right).$$

The method of exponent pairs can be applied to the estimate of sums of form

$$\sum_{x\in D}\Psi(g(x)),\quad \Psi(u)\stackrel{\text{def}}{=}u-[u]-1/2,\quad u\in I\!R.$$

**Lemma 6.** *Let $\alpha,\beta,\gamma$ be positive real numbers, $(l_0,l_1)$ be a one-dimensional exponent pair, $l_0>0$. Then*

$$\sum_{n\le x^\alpha}\Psi\left(\frac{x^\beta}{n^\gamma}\right)\ll x^{\alpha-\frac{\beta-\alpha\gamma}{2}}+\begin{cases} x^{\frac{\alpha(1-l_1)+(\beta-\alpha\gamma)l_0}{1+l_0}} & \text{if } \gamma l_0+l_1<1,\\ x^{\frac{\beta l_0}{1+l_0}}\cdot\log x & \text{if } \gamma l_0+l_1=1,\\ x^{\frac{\beta l_0}{(1+\gamma)l_0+l_1}} & \text{if } \gamma l_0+l_1>1.\end{cases}$$

(Proof: see [9], Lemma 8.)

**Lemma 7.** *Let $g(x)$ be a real-valued function in $D$. Then for every $\Delta>0$ we have*

$$\sum_{x\in D}\Psi(g(x))\ll$$

$$\ll\frac{|D|}{\Delta}+\max_{M\le\Delta}\max_{M'\le 2M}\frac{1}{M}\left|\sum_{M\le m\le M'}\sum_{x\in D}e(mg(x))\right|\log|D|.$$

(The proof can be obtained by using the Fourier-series expansion and Lemma 1; see also [4], [6]).

## 3. The estimate of the sum $\sum\limits_m\sum\limits_n e\left(\frac{m^\beta x}{n^\alpha}\right)$

Let

$$S_{\alpha,\beta}(M,N)=\sum_{(m,n)\in D(M,N)}e\left(\frac{m^\beta x}{n^\alpha}\right),$$

where $D(M,N)$ denote a region contained in the rectangle

$$\{M \le m \le 2M, \quad N \le n \le 2N\}.$$

Using Lemma 5 and Abel's summation formula (Lemma 1), we get

$$(1) \qquad S_{\alpha,\beta}(M,N) \ll \left(\frac{xM^\beta}{N^\alpha}\right)^{-1/2} \cdot N \cdot |S_1| + MN(xM^\beta)^{-1/2}N^{\alpha/2},$$

where

$$S_1 = \sum_{(m,n)\in D_1(M,N_1)} e(f_1(m,n)), \qquad f_1(m,n) \underset{D_1}{\overset{\infty}{\sim}} (m^\beta x)^{\frac{1}{1+\alpha}} n^{\frac{\alpha}{1+\alpha}},$$

and the region $D_1(M,N_1) = D_1$ is contained in the rectangle

$$\{M \le m \le 2M, \quad N_1 \le n \le 2N_1\}, \quad N_1 \asymp c_1 M^\beta x N^{-\alpha-1}, \quad c_1 > 0 - \text{const}.$$

Using Corollary (with $\ell = 1,\ x_1 = n$) we obtain

$$(2) \quad S_1 \ll \frac{M^{\beta+1}xN^{-1-\alpha}}{q^{1/2}} + \frac{(M^{\beta+1}xN^{-1-\alpha})^{3/4}}{q}\left(\sum_{h_1=1}^{q}\left(\sum_{h_2=1}^{q^2}|S_2|\right)^{1/2}\right)^{1/2}$$

where

$$S_2 = \sum_{(m,n)\in D_1^h(M,N_1)} e(f_2(m,n)), \quad f_2(m,n) \sim h_1h_2(m^\beta x)^{\frac{1}{1+\alpha}} n^{-\frac{2+\alpha}{1+\alpha}},$$

$$D_1^h(M,N_1) = \{(m,n) \in D_1(M,N) \mid (m, n+h_1+h_2) \in D_1(M,N_1),$$
$$1 \le h_i \le q^i,\ i = 1,2\},$$

$q$ is given in (**), below.

Now we apply Lemma 5 ($k = 1, x_1 = n$) and Lemma 1:

$$(3) \qquad \begin{aligned} S_2 \ll &(h_1h_2)^{-1/2}\left(M^\beta x\right)^{-\frac{1}{2(1+\alpha)}}\left(M^\beta xN^{-1-\alpha}\right)^{\frac{4+3\alpha}{2(1+\alpha)}}|S_3| + \\ &+ M^{1+\beta}xN^{-1-\alpha}\left[h_1h_2\left(M^\beta x\right)^{\frac{1}{1+\alpha}}\left(M^\beta xN^{-1-\alpha}\right)^{-\frac{2+\alpha}{1+\alpha}}\right]^{-1/2}\end{aligned}$$

where

$$S_3=\sum_{m\sim M}\sum_{n\sim N_2} e(f_3(m,n)),\quad f_3(m,n)\underset{D_2^*(M,N_2)}{\overset{\infty}{\approx}}\left(h_1h_2(m^\beta x)^{\frac{1}{1+\alpha}}\right)^{\frac{1+\alpha}{3+2\alpha}} n^{\frac{2+\alpha}{3+2\alpha}},$$

(4)
$$N_2\asymp c_2h_1h_2\left(M^\beta x\right)^{-2}N^{3+2\alpha},\qquad c_2>0\ -\ \text{const}.$$

The sum $S_3$ can be estimated by applying the definition of one-dimensional exponent pairs, van der Corput's theorem (see [1]) for summing over $m$, and trivially over the variable $n$:

$$S_3\ll$$

$$h_1h_2\frac{N^{3+2\alpha}}{(M^\beta x)^2}\left[\left(h_1h_2x^{\frac{1}{1+\alpha}}\right)^{\frac{1+\alpha}{3+2\alpha}}\left(h_1h_2\frac{N^{3+2\alpha}}{(M^\beta x)^2}\right)^{\frac{2+\alpha}{3+2\alpha}}M^{-\frac{2(1+\alpha)}{3+2\alpha}\beta}\right]^{l_0}M^{1-l_1}+$$

$$+h_1h_2\left(M^\beta x\right)^{-2}N^{3+2\alpha}M^{1/2}\left(h_1h_2x^{-1}N^{2+\alpha}M^{-2-\beta}\right)^{-1/2}\ll$$

(5)
$$\ll(h_1h_2)^{1+l_0}x^{-2-l_0}N^{3+2\alpha+(2+\alpha)l_0}M^{1-2\beta-2\beta l_0-l_1}+$$

$$+(h_1h_2)^{\frac{1}{2}}M^{\frac{5\beta}{2}+\frac{3}{2}}x^{-\frac{3}{2}}N^{2+\frac{3}{2}\alpha}.$$

Put

$$M_0=\left(x^{-3-2l_0}N^{4+3\alpha+2l_0(2+\alpha)}\right)^\theta,\qquad \theta=\frac{1}{3\beta+2(1+\beta)+2l_1},$$

(**)
$$q=\left[x^{3+2l_0}M^{3\beta+2l_0(1+\beta)+2l_1}N^{-(4+3\alpha+2l_0(2+\alpha))}\right]^{\frac{1}{7+6l_0}}.$$

If $M<M_0$, then $S_1$ is estimated trivially, while for $M\ge M_0$ the sum $S_1$ is estimated by making use of (2)-(5).

So we obtain

**Theorem 1.** *Let $x$, $M$, $N>1$ be real numbers, $(l_0,\ l_1)$ be a one-dimensional exponent pair, $\alpha\ne 0,1$; $\beta\ge 1$ be real numbers. If the condition*

$$x^{3+2l_0}M^{3\beta+2l_0(1+\beta)+2l_1}N^{-4-3\alpha-2l_0(2+\alpha)}\le\left(M^\beta xN^{-1-\alpha}\right)^{\frac{7+6l_0}{2}}$$

*is satisfied, then*

$$S_{\alpha,\beta}(M,N) \ll \begin{cases} \left(\frac{xM^{\beta+2}}{N^\alpha q}\right)^{1/2} + \left(q^{-3}M^{9\beta+8}x^9N^{-10-9\alpha}\right)^{1/8} & \text{if } M \geq M_0, \\ x^{1/2}M^{1+\beta/2}N^{-\alpha/2} & \text{if } M < M_0. \end{cases}$$

If we apply Lemma 4 over $m$ to the sum $S_3$, then we obtain

(6) $$S_3 \ll$$

$$\ll \frac{h_1h_2\left(M^\beta x\right)^{-2}N^{3+2\alpha}M}{\rho^{1/2}} + \left(\frac{h_1h_2M^{1-2\beta}x^{-2}N^{3+2\alpha}}{\rho}\right)^{1/2}\left(\sum_{p=1}^{\rho-1}|S_4|\right)^{1/2},$$

$$S_4 = \sum_{m\sim M_1}\sum_{n\sim N_2} e((f_4(m,n)),$$

$$f_4(m,n) \underset{D_4^{h,p}}{\overset{\infty}{\approx}} p(h_1h_2)^{\frac{1+\alpha}{3+2\alpha}}x^{\frac{1}{3+2\alpha}}n^{\frac{2+\alpha}{3+2\alpha}}m^{-\frac{3+2\alpha-\beta}{3+2\alpha}},$$

where $M_1 \asymp M$, $D_4^{h,p} \subset D_2^h(M,N_2)$.

We have

$$\frac{\partial f_4}{\partial m} \underset{D_4^{h,p}}{\overset{\infty}{\approx}} p(h_1h_2)^{\frac{1+\alpha}{3+2\alpha}}x^{\frac{1}{3+2\alpha}}n^{\frac{2+\alpha}{3+2\alpha}}m^{\frac{\beta}{3+2\alpha}-2} \asymp$$
$$\asymp c_3p(h_1h_2)x^{-1}N^{2+\alpha}M^{-2-\beta} = z_1,$$

$$\frac{\partial f_4}{\partial n} \underset{D_4^{h,p}}{\overset{\infty}{\approx}} p(h_1h_2)^{\frac{1+\alpha}{3+2\alpha}}x^{\frac{1}{3+2\alpha}}n^{-\frac{1+\alpha}{3+2\alpha}}m^{\frac{\beta}{3+2\alpha}-1} \asymp$$
$$\asymp c_4pxN^{-1-\alpha}M^{\beta-1} = z_2.$$

Obviously, $z_2 \gg 1$.

If $z_1 \gg 1$ then for $f_4(m,n)$ the conditions in Definition 2 (for $k=2$) are satisfied and we have for every exponent pair $(l_0,l_1)$ of dimension 2:

(7) $$S_4 \ll (z_1z_2)^{l_0}(M_1N_2)^{1-l_1} \ll$$

$$\ll p^{2l_0}(h_1h_2)^{1+l_0-l_1}x^{-2+2l_1}N^{(3+2\alpha)+l_0-(3+2\alpha)l_1}M^{(1-2\beta)-3l_0-(1-2\beta)l_1}.$$

If $z_1 < 1$, then we apply the lemma of van der Corput to the inner sum $S_4$ over $m$:

$$S_4 \ll \left(1 + p^{-\frac{1}{2}}(h_1h_2)^{-\frac{1}{2}}x^{\frac{1}{2}}N^{-1-\frac{\alpha}{2}}M^{\frac{\beta}{2}+\frac{3}{2}}\right)h_1h_2\left(M^\beta x\right)^{-2}N^{3+2\alpha} \ll$$

$$\ll p^{-\frac{1}{2}}(h_1h_2)^{\frac{1}{2}}x^{-\frac{3}{2}}N^{2+\frac{3}{2}\alpha}M^{\frac{3-3\beta}{2}}. \tag{8}$$

Hence, from (5)–(6)

$$S_4 \ll p^{2l_0}(h_1h_2)^{1+l_0-l_1}x^{-2+2l_1}N^{(3+2\alpha)(1-l_1)+l_0}M^{(1-2\beta)(1-l_1)-3l_0}+$$

$$+p^{-\frac{1}{2}}(h_1h_2)^{\frac{1}{2}}x^{-\frac{3}{2}}N^{\frac{4+3\alpha}{2}}M^{\frac{3-3\beta}{2}}. \tag{9}$$

Put

$$\rho = \left((h_1h_2)^{l_1-l_0}x^{-2l_1}N^{(3+2\alpha)l_1-l_0}M^{(1-2\beta)l_1+3l_0}\right)^{\frac{1}{1+2l_0}}.$$

We must guarantee the fulfilment of the inequality $\rho \le M$. Since $h_1 \le q$, $h_2 \le q^2$, the condition $\rho \le M$ is satisfied whenever

$$q \le \left(x^{2l_1}N^{-(3+2\alpha)l_1-l_0}M^{(2\beta-1)l_1+1-l_0}\right)^{\frac{1}{3(l_1-l_0)}}. \tag{10}$$

Put

$$q_1 = \left(x^{2l_1}N^{-(3+2\alpha)l_1+l_0}M^{1+(2\beta-1)l_1-l_0}\right)^{\frac{1}{3(l_1-l_0)}}, \tag{11}$$

$$q_2 = \left(x^{3+6l_0-2l_1}N^{-\alpha(1+2l_0)-(3+2\alpha)l_1+l_0}M^{3\beta(1+2l_0)+(2\beta-1)l_1+3l_0}\right)^{\frac{1}{7+17l_0-3l_1}},$$

$$q_3 =$$

$$\left(x^{5+10l_0-2l_1}N^{-(6+5\alpha)(1+2l_0)-(3+2\alpha)l_1+l_0}M^{(5\beta-1)(1+2l_0)+(1-2\beta)l_1+3l_0}\right)^{\frac{1}{11+25l_0-3l_1}}$$

$$q = \min\left(q_1, q_2, q_3, (xMN^{-3})^{1/2}\right).$$

Now from (1)–(6), (9) we obtain

$$S_1 \ll \frac{xN^{-1-\alpha}M^{1+\beta}}{q^{1/2}} + x^{3/4}M^{1+3\beta/4}N^{-\frac{3(1+\alpha)}{4}}\log(MN).$$

Thus we have

**Theorem 2.** *Let $x, M, N$ be positive numbers, $M < N \le x^{1/3}$, and $(l_0, l_1)$ be an exponent pair of dimension 2. Then*

$$S_{\alpha,\beta}(M,N) \ll$$

$$\ll \frac{x^{\frac{1}{2}} N^{-\frac{\alpha}{2}} M^{1+\frac{\beta}{2}}}{q^{1/2}} + x^{\frac{1}{4}} M^{\frac{\beta}{4}} N^{\frac{3}{4}-\frac{\alpha}{4}} \log(MN) + x^{-\frac{1}{2}} M^{1-\frac{\beta}{2}} N^{1+\frac{\alpha}{2}},$$

*where $q$ is defined in (11).*

We shall prove the following

**Theorem 3.** *Let $(l_0, l_1)$, $(L_0, L_1)$ be exponent pairs of dimension 2 and 3, respectively. Then*

$$S_{2,1}(M,N) \ll M^{\frac{15}{8}} x^{\frac{9}{8}} N^{-\frac{19}{8}} q^{-1} \min\left(q^{\frac{1}{4}},\ M^{\frac{1}{2}} x^{\frac{1}{2}} N^{-\frac{7}{4}}\right) + \\ + M^{\frac{3}{2}} x^{\frac{1}{2}} N^{-1} q^{-\frac{1}{2}} + \left(M^{a_1} x^{a_2} N^{-a_3} q^{a_4}\right)^{a_0},$$

*where*

$$\begin{aligned}
q &= \min\left(q_1, q_2, q_3, \left(xMN^{-3}\right)^{1/2}\right),\\
q_1 &= \left(M^{\frac{5+18L_1}{2}} x^{7+8L_0-10L_1} N^{-22-26L_0+34L_1}\right)^{\frac{1}{15+30L_0-21L_1}},\\
q_2 &= \left(M^{13+14L_0-16L_1} x^{10+12L_0-12L_1} N^{-19-22L_0+32L_1}\right)^{\frac{1}{24+30L_0-21L_1}},\\
q_3 &= \left(M^{b_1} x^{b_2} N^{-b_3}\right)^{b_0},\\
b_1 &= 4 - 9l_0 - 3l_1 + (2 - 12l_0 - 4l_1)L_0 - (10 - 6l_0 - 2l_1)L_1,\\
b_2 &= 2(1 - l_0 - l_1)(3 + 4L_0 - 2L_1) + (1 - 6L_1),\\
b_3 &= (6 - 7l_0 - 7l_1)(3 + 4L_0 - 2L_1) + (4 + 2L_0 - 22L_1),\\
b_0^{-1} &= (5 + l_0 - 5l_1)(3 + 4L_0 - 2L_1) + (3 + 2L_0 + 7L_1),\\
a_1 &= 44 + (62 + 12l_0 + 4l_1)L_0 - (22 + 6l_0 + 2l_1)L_1,\\
a_2 &= 17 + (24 + 8l_0 + 2l_1)L_0 - (6 + 4l_0 + 4l_1)L_1,\\
a_3 &= 50 + (70 + 28l_0 + 28l_1)L_0 - 14(1 + l_0 + l_1)L_1,\\
a_4 &= 6 + (6 + 4l_0 - 20l_1)L_0 + (5 - 2l_0 + 10l_1)L_1,\\
a_0^{-1} &= 8(3 + 4L_0 - 2L_1).
\end{aligned}$$

**Proof.** For the sake of convenience we shall introduce special notations:
($\alpha$) The sum

$$\sum_{m\sim M} \sum_{n\sim N} e(f(m,n))$$

will be written as

$$\sum e(f), \qquad P\{f(m,n);\ M,N\}.$$

($\beta$) Let

$$\underset{y}{A^{k_j}}\ \underset{x}{B} \dots \underset{x}{A^{k_2}}\ \underset{y}{B}\ \underset{y}{A^{k_1}}\ \underset{x}{B}\ \underset{x}{A^{k_0}}$$

be the result of transforming $S$ by $k_0$ times applying $A$ with respect to the variable $x$, then applying $B$ with respect to $x$, then applying $k_1$ times the transformation $A$ with respect to $y$, and so on, according to the notation proceeding from right to left.

Now consider the trigonometric sum

$$S_{2,1}(M,N) = \sum e(f), \qquad P\left\{f(m,n) = \frac{mx}{n^2};\ M, N\right\}.$$

Then

$$\underset{n}{B}\ S_{2,1}(M,N)$$

gives

$$S_{2,1}(M,N) \ll (Mx)^{-\frac{1}{2}} N^2 (|S_1| + M \log x) \tag{12}$$

where

$$S_1 = \sum e(f_1), \qquad P\left\{f_1(m,n) \underset{D_1}{\overset{\infty}{\approx}} (mx)^{1/3} n^{2/3};\ M_1, N_1\right\},$$

$D_1 \subset \{M_1 \le m \le 2M_1;\ N_1 \le n \le 2N_1\}$, $M_1 = M$, $N_1 = \dfrac{c_1 Mx}{N^3}$, $c_1$ – const.

Next, applying $\underset{n}{A^2}\ S_1$ we obtain

$$S_1 \ll \frac{M^2 x N^{-3}}{q^{1/2}} + \frac{(M^2 x N^{-3})^{3/4}}{q} \left( \sum_{\substack{h_1 < q \\ h_1 H_2 \ge q^{1/2}}} \max_{H_2 < q^2} |S_2|^{1/2} \right)^{1/2}, \tag{13}$$

where

$$S_2 = \sum e(f_2), \quad P\left\{f_2(h_2, m, n) \underset{D_2^h}{\overset{\infty}{\approx}} h_1 h_2 (mx)^{1/2} n^{-4/3};\ M_2, N_2\right\},$$

$$D_2^h \subset D_1, \quad M_2 = M_1, \quad N_2 = N_1; \quad q \le (MxN^{-3})^{1/2}$$

($q$ is a parameter to be chosen to our advantage). After application

$$\underset{n}{B}\, S_2$$

we have

(14) $$S_2 \ll (h_1H_2)^{-\frac{1}{2}}(Mx)^{\frac{3}{2}}N^{-5}(|S_3| + MH_2\log x)$$

where

$$S_3 = \sum e(f_3), \quad P\left\{f_3(h_2,m,n) \underset{D_3}{\overset{\infty}{\sim}} (h_1h_2)^{\frac{3}{7}}(mx)^{\frac{1}{7}}n^{\frac{4}{7}};\ M_3, N_3\right\},$$

$D_3 \subset \{M_3 \le m \le 2M_3,\ N_3 \le n \le 2N_3\}$, $M_3 = M_2$, $N_3 = c_3h_1H_2(Mx)^{-2}N^7$.
Now $\underset{m}{A}\, S_3$ gives

(15) $$S_3 \ll \frac{h_1H_2^2M^{-1}x^{-2}N^7}{\rho^{1/2}} + \frac{(h_1H_2^2M^{-1}x^{-2}N^7)^{1/2}}{\rho^{1/2}}\left(\sum_{p\le\rho}|S_4|\right)^{1/2},$$

where

$$S_4 = \sum e(f_4), \quad P\left\{f_4(h_2,m,n) \underset{D_3^p}{\overset{\infty}{\sim}} p(h_1h_2)^{3/7}x^{1/7}m^{-6/7}n^{4/7}; M_4, N_4\right\},$$

$$M_4 = M_3, \quad N_4 = N_3, \quad \rho \le M \text{ (the value of } \rho \text{ is given later).}$$

Next, $\underset{m}{B}\, S_4$ gives

(16) $$S_4 \ll (ph_1H_2)^{-\frac{1}{2}}M^2x^{\frac{1}{2}}N^{-2}\left(|S_5| + h_1H_2^2(Mx)^{-2}N^7\log x\right)$$

where

$$S_5 = \sum e(f_5), \quad P\left\{f_5(h_2,m,n) \underset{D_4}{\overset{\infty}{\sim}} (p^7h_1^3h_2^3m^6xn^4)^{\frac{1}{13}};\ M_5, N_5\right\},$$

$$D_4 \subset \{M_5 \le m \le 2M_5,\ N_5 \le n \le 2N_5\},$$

$$M_5 \asymp c_4ph_1H_2x^{-1}M^{-3}N^4,\ N_5 = N_4.$$

Now if $h_1$ and $H_2$ are such that

($I$) $$h_1H_2 \ll (Mx)^2N^{-7},$$

then $N_5 \ll 1$, and we can estimate the sum $S_5$ trivially

$$S_5 \ll H_2 M, \tag{17}$$

whence

$$S_1 \ll h_1^{-\frac{1}{2}} H_2^{\frac{1}{2}} M^{\frac{5}{2}} x^{\frac{3}{2}} N^{-5}$$

follows. So the contribution to $S_1$ of that $h_1$, $H_2$ for which the condition (I) is satisfied, is less than

$$\ll \frac{(M^2 x N^{-3})^{3/4}}{q} \left( \sum_{\substack{h_1 \leq q \\ q^{1/2} \leq h_1 H_2 \leq (Mx)^2 N^{-7}}} \max_{H_2 \leq q^2} \left( h_1^{-1/4} H_2^{1/4} M^{5/4} x^{3/4} N^{5/2} \right) \right)^{1/2} \ll$$

$$\ll \frac{M^{17/8} x^{9/8} N^{-7/2}}{q} \max_{\substack{H_2 \leq q^2 \\ q^{1/2} \leq h_1 H_2 \leq M^2 x^2 N^{-7}}} \left( \sum_{h_1 \leq q} h_1^{-1/4} H_2^{1/4} \right)^{1/2} \ll \tag{18}$$

$$\ll \frac{M^{19/8} x^{11/8} N^{-35/8}}{q} \min \left( q^{1/4},\ (M^2 x^2 N^{-7})^{1/4} \right).$$

Now suppose

$$h_1 H_2 \gg M^2 x^2 N^{-7}. \tag{$II$}$$

If $M_5 = p h_1 H_2 x^{-1} M^{-3} N^4 \ll 1$, then

$$S_5 \ll H_2 N_5 \ll h_1 H_2^2 (Mx)^{-2} N^7$$

holds obviously.

The contribution in $S_4$ of such $p, h_1, H_2$, for which $M_5 \ll 1$ is satisfied is less than

$$\ll M h_1^{1/2} H_2^{3/2} x^{-3/2} N^5 p^{-1/2}.$$

The contribution in $S_3$ of them is

$$\ll \left( \frac{h_1 H_2^2 M^{-1} x^{-2} N^7}{p} \right)^{1/2} \times$$

$$(19)\qquad \times\left(\sum_{p\le\min(\rho,\ h_1^{-1}H_2^{-1}xM^3N^{-4})} h_1^{1/2}H_2^{3/2}Mx^{-3/2}N^5p^{-1/2}\right)^{1/2} \ll$$

$$\ll \frac{h_1^{3/4}H_2^{7/4}x^{-7/4}N^6}{\rho^{1/2}}\min\left(\rho^{1/4},\ h_1^{-1/4}H_2^{-1}x^{1/4}M^{3/4}N^{-1}\right)$$

and hence the contribution in $S_2$

$$(20)\qquad \ll \frac{h_1^{1/4}H_2^{5/4}M^{3/2}x^{-1/4}N}{\rho^{1/2}}\min\left(\rho^{1/4},\ h_1^{-1/4}H_2^{-1}x^{1/4}M^{3/4}N^{-1}\right).$$

Since $\rho$ may depend on $h_1$, $H_2$, the contribution in $S_1$, $S_{2,1}(M,N)$ will be defined after the choice of $\rho$.

If $M_4 \gg 1$ we have

$$(III)\qquad p \gg xM^{-3}N^{-4}h_1^{-1}H_2^{-1}.$$

The conditions (II), (III) imply $N_5 \gg 1$, $M_5 \gg 1$. We have

$$(21)\qquad \begin{aligned} &\frac{\partial f_5}{\partial n} \underset{D_4}{\overset{\infty}{\sim}} \left((h_1h_2)^3xn^{-9}m^6p^7\right)^{1/13} \gg xN^{-3}p \gg 1,\\ &\frac{\partial f_5}{\partial m} \underset{D_4}{\overset{\infty}{\sim}} \left((h_1h_2)^3xn^4m^{-7}p^7\right)^{1/13} \gg M \gg 1,\\ &\frac{\partial f_5}{\partial m} \underset{D_4}{\overset{\infty}{\sim}} \left((h_1h_2)^3h_2^{-1}xn^4m^6p^7\right)^{1/13} \asymp cM^{-2}N^4x^{-1}h_1p. \end{aligned}$$

Hence if

$$(IV)\qquad p \gg h_1^{-1}xM^2N^{-4},$$

then the sum $S_5$ is estimated by a method of 3-dimensional exponent pairs, and in the opposite case by a method of 2-dimensional exponent pairs over $m$, $n$ and trivially over $h_2$. Then the contribution in $S_4$ is less than

$$(22)\qquad \begin{aligned} &(h_1M^{-1}Np^2)^{L_0}(h_1^2H_2M^{-5}x^{-3}N^{11}p)^{1-L_1} \quad \text{if} \quad p \gg h_1^{-1}xM^2N^{-4},\\ &(xMpN^{-2})^{l_0}(h_1^2H_2^2M^{-3}x^{-3}N^{11}p)^{1-l_1} \quad \text{if} \quad p \ll h_1^{-1}xM^2N^{-4}. \end{aligned}$$

Next, the contribution in $S_3$

$$(23)\qquad \ll \frac{h_1H_2^2M^{-1}x^{-2}N^7}{\rho^{1/2}} +$$

$$+\left(\frac{h_1H_2^2M^{-1}x^{-2}N^7}{\rho}\right)^{1/2}\left(\sum_{h_1^{-1}xM^2N^{-4}<p\le\rho}|S_4'|+\sum_{p\le h_1^{-1}xM^2N^{-4}}|S_4''|\right)^{1/2}\ll$$

$$\ll\left(\frac{h_1H_2^2M^{-1}x^{-2}N^7}{\rho}\right)^{1/2}\left\{h_1^{\frac{3}{4}-l_1}H_2^{\frac{5}{4}-l_1}M^{-\frac{1}{2}+\frac{l_0}{2}+\frac{3}{2}l_1}N^{\frac{9-3l_0-11l_1}{2}}\times\right.$$

$$\times\, x^{\frac{-5+2l_0+6l_1}{4}}\left(h_1^{-1}xM^2N^{-4}\right)^{\frac{3+2l_0-2l_1}{4}}+\left(h_1H_2^2M^{-1}x^{-2}N^7\right)^{1/2}+$$

$$\left.+h_1^{\frac{3+2L_0-4L_1}{4}}H_2^{\frac{5-6L_1}{4}}M^{\frac{-3-L_0+5L_1}{2}}N^{\frac{9+L_0-11L_1}{2}}x^{\frac{-5+6L_1}{4}}\rho^{\frac{3+4L_0-2L_1}{4}}\right\}.$$

Let

$$\rho=\left(h_1^{-1-2L_0+4L_1}H_2^{-1+6L_1}M^{4+2L_0-10L_1}x^{1-6L_1}N^{-4-2L_0+22L_1}\right)^{\frac{1}{3+4L_0-2L_1}}.$$

Then, the contribution in $S_3$ can be estimated by

(24)
$$\ll\frac{h_1H_2^2M^{-1}x^{-2}N^7}{\rho^{1/2}}\left[1+h_1^{\frac{-1+l_0-l_1}{2}}H_2^{-\frac{1+4l_1}{4}}M^{\frac{3+3l_0+l_1}{2}}N^{-\frac{4+7l_0+7l_1}{2}}x^{\frac{1+2l_0+2l_1}{2}}\right].$$

The contribution in $S_2$ is less than

(25)
$$\frac{h_1^{1/2}H_2^{3/2}M^{1/2}x^{-1/2}N^2}{\rho^{1/2}}\cdot$$

$$\cdot\left[1+h_1^{-1+l_0-l_1}H_2^{\frac{-1+4l_1}{2}}M^{3+3l_0+l_1}N^{-4-7l_0-7l_1}x^{1+2l_0+2l_1}\right]^{\frac{1}{2}}$$

The general contribution in $S_1$, provided that the condition (II) is satisfied

$$\ll\frac{M^2xN^{-3}}{q^{1/2}}+\frac{(M^2xN^{-3})^{3/4}}{q}\left\{\sum_{\substack{h_1<q\\ h_1H_2\gg\max(q_1^{1/2},\ M^2x^2N^{-7})}}\max_{H_2<q^2}\left[\frac{h_1^{\frac{1}{4}}H_2^{\frac{5}{4}}M^{\frac{3}{2}}x^{-\frac{1}{4}}N}{\rho^{1/2}}\cdot\right.\right.$$

$$\cdot\min\left(\rho^{1/4},\left(\frac{xM^3}{h_1H_2N^4}\right)^{1/4}\right)+\frac{h_1^{1/2}H_2^{3/2}M^{1/2}x^{-1/2}N^2}{\rho^{1/2}}\cdot$$

$$\left.\left.\cdot\left(1+h_1^{\frac{-1+l_0-l_1}{2}}H_2^{-\frac{1+4l_1}{4}}M^{\frac{3+3l_0-l_1}{2}}N^{-\frac{4+7l_0+7l_1}{2}}x^{\frac{1+2l_0+2l_1}{2}}\right)\right]^{1/2}\right\}^{1/2}\ll$$

$$\ll \frac{M^2xN^{-3}}{q^{1/2}} + \frac{(M^2xN^{-3})^{3/4}}{q} \times \tag{26}$$

$$\left\{ \sum_{h_1 \le q} \left( qM^{9/8} + h_1^{1/4} q^{3/2} M^{1/4} x^{-1/4} N + h_1^{\frac{l_0-l_1}{4}} q^{\frac{5-4l_1}{4}} M^{\frac{4+3l_0+l_1}{4}} x^{\frac{l_0+l_1}{2}} N^{-\frac{7l_0+7l_1}{4}} \right) \right.$$

$$\left. \cdot \left( \frac{M^{2+L_0-5L_1} x^{1-6L_1} N^{-4-2L_0+22L_1}}{h_1^{1+2L_0-4L_1} q^{2-2L_1}} \right)^{\frac{-1}{4(3+4L_0-2L_1)}} \right\}^{1/2} \ll \frac{M^2xN^{-3}}{q^{1/2}} +$$

$$+ \left( q^{3+2L_0-7L_1} M^{\frac{25-4L_0+20L_1}{2}} x^{17+24L_0-6L_1} N^{-50-70L_0+14L_1} \right)^{\frac{1}{8(3+4L_0-2L_1)}} +$$

$$+ \left( q^{12+14L_0-13L_1} M^{35+50L_0-16L_1} x^{14+20L_0-4L_1} N^{-38-54L_0+6L_1} \right)^{\frac{1}{8(3+4L_0-2L_1)}} +$$

$$+ q^{\frac{1+l_0-5l_1}{8} + \frac{3+2L_0+7L_1}{8(3+4L_0-2L_1)}} M^{\frac{4+3l_0+l_1}{8} - \frac{4+2L_0-5L_1}{8(3+4L_0-2L_1)}} x^{\frac{2l_0+2l_1}{8} - \frac{1-6L_1}{8(3+4L_0-2L_1)}} \times$$

$$\times N^{-\frac{7l_0+7l_1}{8} + \frac{-4-2L_0+22L_1}{8(3+4L_0-2L_1)}}.$$

Now successively equating summand (I) (the right in (26)) with summands (II), (III), (IV), respectively, we define $q_1,\ q_2,\ q_3$ (see the statement of the theorem).

Let

$$q = \min(q_1,\ q_2,\ q_3,\ x^{1/2}M^{1/2}N^{-3/2}).$$

Then from (12), (18) and (26) we find

$$S_{2,1}(M,N) \ll \left(M^{15}x^9N^{-19}q^{-8}\min(q^2,\ M^4x^4N^{-14})\right)^{\frac{1}{8}} + $$
$$+ (M^3xN^{-2}q^{-1})^{1/2} + (M^{a_1}x^{a_2}N^{-a_3}q^{a_4})^{a_0}.$$

This completes the proof of our theorem.

By using similar arguments, we can prove

**Theorem 4.** *Let $\alpha_1,\ \alpha_2 > 1$ be positive numbers and let $(l_0, l_1)$ be an exponent pair of dimension 3. Then for*

$$M \ll Q^{1/5} x^{-3/5} \prod_{j=1}^{2} N_j^{\frac{2-3\alpha_j}{5}}$$

*we have*

$$\sum_{m \sim M} \sum_{\substack{n_j \sim N_j \\ j=1,2}} e\left(\frac{mx}{n_1^{\alpha_1} n_2^{\alpha_2}}\right) \ll TN_1^{1-\alpha_1} N_2^{1-\alpha_2} (xM)^{-1} +$$

$$+N_1^{1-\frac{\alpha_1}{2}}N_2^{1-\frac{\alpha_2}{2}}(xM)^{-1/2}+x^{-1/4}M^{1/8}Q^{3/4}\prod_{j=1}^{2}N_j^{\frac{3-\alpha_j}{4}}$$

*where*

$$T=\left(x^{6+17l_0+8l_1}M^{10+25l_0+7l_1}Q^{6+21l_0-6l_1}\prod_{j=1}^{2}N_j^{\beta_j}\right)^{\frac{1}{8(1+3l_0)}},$$

$$\beta_j=(6\alpha_j-2)(1+3l_0)+(8\alpha_j-5)l_1+(1-\alpha_j)l_0,\quad j=1,2;$$

$$Q=\min\left(xM\prod_{j=1}^{2}N_j^{\alpha_j-\frac{1}{2}},\ \max(Q_1,Q_2),\ \max(Q_3,Q_4)\right),$$

$$Q_1=\left(xM^{\frac{15}{2}}\prod_{j=1}^{2}N_j^{5\alpha_j-3}\right)^{\frac{1}{7}},$$

$$Q_2=\left(x^{11}M^{12}\prod_{j=1}^{2}N_j^{13\alpha_j-8}\right)^{\frac{1}{9}},$$

$$Q_3=\left(x^{10+31l_0-8l_1}M^{14+47l_0-7l_1}\prod_{j=1}^{2}N_j^{\gamma_j}\right)^{\frac{1}{10+33l_0-6l_1}},$$

$$\gamma_j=(2\alpha_j-1)(6+18l_0)+(5-8\alpha_j)l_1-(1-\alpha_j)l_0,\quad (j=1,2),$$

$$Q_4=\left(x^{13+40l_0-8l_1}M^{11+38l_0-7l_1}\prod_{j=1}^{2}N_j^{\delta_j}\right)^{\frac{1}{9+30l_0-6l_1}},$$

$$\delta_j=(13\alpha_j-8)(1+3l_0)+(5-8\alpha_j)l_1-(1-\alpha_j)l_0,\quad (j=1,2).$$

## 4. Some applications

We shall say that the multiplicative function belongs to the class $M_2$ if $f(n)=\sum_{d^2|n}\Phi(d)$ holds with a suitable multiplicative function $\Phi(n)$.

**Theorem.** *Let $f(n) \in M_2$, such that $\Phi(n) \ll n^{1-\varepsilon}$, $\varepsilon > 0$ be arbitrarily small. Then*

$$\sum_{x<n\leq x+h} f(n) = h\sum_{n=1}^{\infty}\frac{\Phi(n)}{N^2} + O\left((h^{1/2}+x^{0,2196})\max_{1\leq n\leq 2x}|\Phi(n)|\right).$$

*In particular, for $f(n) = \mu^2(n)$ we have*

$$\sum_{x<n\leq x+h} \mu^2(n) = \frac{6}{\pi^2}h + O(h^{1/2}) + O(x^{0,2196}).$$

This result is an improvement of estimate of an error term in asymptotic formula for the number of the squarefree integers in a short interval $(x, x+h]$ (see [4], [6], [10]). It should be mentioned here that in 1991 M. Filaseta and O. Trifonov proved that there exists a constant $c > 0$ such that the interval $(x,\ x + cx^{\frac{1}{5}}\log x]$ contains a squarefree number for every large $x$.

**Proof of the theorem.** From [10] we have

$$\sum_{x<n\leq x+h} f(n) = h\sum_{n=1}^{\infty}\frac{\Phi(n)}{n^2} + O\left(h^{\frac{1}{2}}F(x^{\frac{1}{2}})\right) + \tag{27}$$

$$+O\left(F(x^{\frac{1}{2}})\left|\sum_{n\leq x^{1/3}}\Psi\left(\sqrt{\frac{x}{n}}\right)\right|\right) + O\left(F(x^{\frac{1}{2}})\left|\sum_{n\leq x^{1/3}}\Psi\left(\frac{x}{n^2}\right)\right|\right),$$

where

$$F(u) = \max_{n\leq u}|\Phi(n)|.$$

Lemma 6 with the exponent pair $(l_0, l_1) = \left(\frac{11}{30}, \frac{14}{30}\right)$ implies

$$\sum_{n\leq x^{1/3}}\Psi\left(\sqrt{\frac{x}{n}}\right) \ll x^{9/41} \ll x^{0,2196}. \tag{28}$$

Next

$$\sum_{n\leq x^{1/3}}\Psi\left(\frac{x}{n^2}\right) \ll \left(\max_{1\leq N\leq X_1}\left|\sum_{n\sim N}\Psi\left(\frac{x}{n^2}\right)\right| +\right.$$

$$\left. + \max_{X_1<N\leq X_2}\left|\sum_{n\sim N}\Psi\left(\frac{x}{n^2}\right)\right| + \max_{X_2<N\leq x^{1/3}}\left|\sum_{n\sim N}\Psi\left(\frac{x}{n^2}\right)\right|\right)\log x.$$

For $N \leq X_1$ we apply Lemma 6:

$$\sum_{n\sim N} \Psi\left(\frac{x}{n^2}\right) \ll N^2 x^{-\frac{1}{2}} + N^{\frac{1-l_1-2l_0}{1+l_0}} x^{\frac{l_0}{1+l_0}} \ll N^2 x^{-\frac{1}{2}} + N^{\frac{19}{65}} x^{\frac{9}{65}}$$

(we take $(l_0, l_1) = \left(\frac{9}{56}, \frac{19}{56}\right)$ which can be found in [11]). For $N \in [X_1, X_2]$ we apply Theorem 3 with

$$(l_0, l_1) = (0,0), \quad (L_0, L_1) = ABAB(0,0) = \left(\frac{3}{38}, \frac{7}{38}\right).$$

We have

$$q = M^{\frac{248}{491}} x^{\frac{220}{491}} N^{-\frac{676}{491}}.$$

Hence

$$S_{2,1}(M,N) \ll$$

$$\ll (Mx)^{-\frac{1}{2}} N^2 \left(M^{\frac{23}{8}-\frac{248}{491}} x^{\frac{15}{8}-\frac{220}{491}} N^{-\frac{49}{8}+\frac{676}{491}} + M^{\frac{858}{491}} x^{\frac{381}{491}} N^{-\frac{1135}{491}}\right) \ll$$

$$\ll \begin{cases} M^{1,8699} x^{0,9269} N^{-2,7482} & \text{if } M \geq x^{-1,0458} N^{3,9145} = M_0, \\ M^{1,2475} x^{0,2760} N^{-0,3116} & \text{if } M < M_0. \end{cases}$$

Then applying Lemma 7 we find

$$\sum_{n\sim N} \Psi\left(\frac{x}{n^2}\right) \ll \left\{ \max_{M \leq M_0} \left(\frac{1}{M} \left|\sum_{m\sim M} \sum_{n\sim N} e\left(\frac{mx}{n^2}\right)\right|\right) + \frac{N}{\Delta} + \right.$$

$$\left. + \max_{M_0 \leq M \leq \Delta} \left(\frac{1}{M} \left|\sum_{m\sim M} \sum_{n\sim N} e\left(\frac{mx}{n^2}\right)\right|\right)\right\} \log^2 x \ll \tag{29}$$

$$\ll \left(M_0^{0,2475} x^{0,2760} N^{-0,3116} + \frac{N}{\Delta} + \Delta^{0,8699} x^{0,9269} N^{-2,7482}\right) \log^2 x \ll$$

$$\ll x^{0,0172} N^{0,6571} + x^{0,4957} N^{-1,0045}.$$

For $N \in [X_2, x^{1/3}]$ we apply Theorem 2. We take $(l_0, l_1) = \left(\frac{23}{250}, \frac{56}{250}\right)$ as it was chosen by Srinivasan [5]. Then we have

$$\sum_{n\sim N} \Psi\left(\frac{x}{n^2}\right) \ll x^{0,32934} N^{-0,35685} \qquad (\text{for } N \gg x^{0,3}). \tag{30}$$

Now we choose

$$X_1 = x^{0,27547}, \qquad X_2 = x^{0,30785}.$$

From (28)-(30) we obtain

$$\sum_{n \le x^{1/3}} \Psi\left(\frac{x}{n^2}\right) \ll x^{0,2196}. \tag{31}$$

Now by (28) and (31) we finish the proof of the theorem.

The Theorems 1–4 can be used for studying other problems of statistical number theory.

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