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ON POWER PROCESSES DEFINED BY INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES HAVING REGULARLY VARYING DISTRIBUTION FUNCTION AT THE INFINITY

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Dedicated to Professor Karl-Heinz Indlekofer on his fiftieth birthday

Abstract. In case of sequences of independent identically distributed random variables having regularly varying distribution function at infinity with exponent of variation $-\gamma < 0$ we give a representation for the limit of power processes as with probability 1 piecewise continuous stochastic process with simple structure and prove the invariance principle. These results, which are valid on the interval $(\gamma/2, \infty)$ when the limit process has stable (non-gaussian) marginal distribution, generalize the ones concerning the empirical moment process (see M.Csörgő, S.Csörgő, L.Horváth and D.M.Mason [3]) and provide possibility to represent the limit distribution in a plausible form for a wide class of symmetric statistics.

1. Preliminaries

Let $X, X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with common distribution function $F(x)$, for which the relations

$$F(-y) = (C_- + o(1))y^{-\gamma} L(y), \qquad 1 - F(y) = (C_+ + o(1))y^{-\gamma} L(y),$$

$$(1) \qquad y \to \infty$$

hold. Here γ, C_-, C_+ are real constants and $\gamma > 0$, $C_- \geq 0$, $C_+ \geq 0$, $C_- + C_+ > 0$, $L(y)$ is slowly varying function at the infinity.

Let in case of arbitrary $x \in \mathbb{R}$ be $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, $h_\pm(x,t) = (x^\pm)^t I(x > 0)$, $0 \leq t < \infty$ (I is the indicator function, $0^0 = 1$) and let us define for arbitrary $n \geq 1$ the twodimensional vector-process

$$s^{(n)}(t) = \left(s_+^{(n)}(t),\ s_-^{(n)}(t)\right), \qquad 0 \leq t < \infty$$

as a power process determined for the positive and negative parts of random variables X_j, where

$$s_\pm^{(n)}(t) = \sum_{j=1}^{n} h_\pm(X_j^\pm, t).$$

Having chosen the nonrandom centering and normalizing values (see (4) and (5))

$$a^{(n)}(t) = \left(a_+^{(n)}(t),\ a_-^{(n)}(t)\right), \qquad b^{(n)}(t) \neq 0$$

and introducing the normalized power process

$$Z_\pm^{(n)}(t) = \left(b^{(n)}(t)\right)^{-1} \left(s_\pm^{(n)}(t) - a^{(n)}(t)\right) \tag{2}$$

on the basis of results [18] (see also [7]) we can obtain the following

Theorem 1. *If condition (1) is fulfilled then there exists such stochastic process* $\{Z(t) = (Z_+(t),\ Z_-(t)),\ 0 \leq t < \infty\}$ *that the finite dimensional distributions of stochastic processes* $\{Z^{(n)}(t),\ 0 \leq t < \infty\}$, $n = 1, 2, \ldots$ *converge to the finite dimensional distributions of stochastic process* $\{Z(t),\ 0 \leq t < \infty\}$.

With the convergence of the sequence of stochastic processes $\{Z^{(n)}(t),\ 0 \leq t < \infty\}$, with determination of limit process $\{Z(t),\ 0 \leq t < \infty\}$ and the investigation of its properties we will deal later. It is important because of the following reasons.

It can be seen that the sequence of twodimensional power process $s^{(n)}(t)$ defined by the positive and negative parts of random variables X_j and the empirical power (moment) process

$$S^{(n)}(t) = (S_1^{(n)}(t),\ S_2^{(n)}(t)), \quad 0 \leq t < \infty, \quad n = 1, 2, \ldots,$$

$$S_1^{(n)}(t) = \sum_{j=1}^{n} I(X_j > 0)|X_j|^t \operatorname{sgn} X_j, \qquad S_2^{(n)}(t) = \sum_{j=1}^{n} I(X_j > 0)|X_j|^t$$

(investigated in [21] and [22]) mutually determine each other since a simple linear relation exists between them:
(3)

$$S_1^{(n)}(t) = \frac{1}{2}\left(s_+^{(n)}(t) - s_-^{(n)}(t)\right), \quad S_2^{(n)}(t) = \frac{1}{2}\left(s_+^{(n)}(t) + s_-^{(n)}(t)\right), \quad t \geq 0.$$

At the same time a wide class of symmetric statistics T_{kn} formed from the sequence of random variables $X_1, X_2, \ldots$ can be represented as a function g_{kn} of values of stochastic process $S^{(n)}(t)$ in suitable (fixed) time moments $t_1, t_2, \ldots, t_k$, namely

$$T_{kn} = g_{kn}\left(S^{(n)}(t_1), \ldots, S^{(n)}(t_k)\right).$$

So in consequence of relations (2) and (3) it can be represented in the form $T_{kn} = \tilde{g}_{kn}\left(Z^{(n)}(t_1), \ldots, Z^{(n)}(t_k)\right)$, as a function of values of stochastic process $Z^{(n)}(t)$, where the functions $\tilde{g}_{kn}$ are determined by g_{kn}, and by the values $b^{(n)}(t_j)$, $a^{(n)}(t_j)$. If the sequence of functions $\tilde{g}_{kn}$ are continuous and converge to a continuous function g as $n \to \infty$, then the sequence of statistics T_{kn} has limit distribution and it is equal to the distribution of random variable $g(Z(t_1), \ldots, Z(t_k))$.

The statistics of this type are, for example, symmetrical polynomials, symmetrical rational functions, U- and V-statistics with power function kernel formed from the random variables $X_1, X_2, \ldots$ were investigated by several authors, the closest to our article are the papers [4], [6],[12], [14], [16], [18-23], [26-28], see also [13]. This means that the investigation of power processes $Z^{(n)}(t)$ and the limit process $Z(t)$ plays essential role in limit theorems for an important class of symmetric statistics and in addition it can be used for nonlinear modelling ([24]). We note that the limit distribution of general symmetric statistics and weighted symmetric statistics in the case, when there exists bounded second moment, was investigated in the papers [8] and [15], respectively.

Remark 1. In the case $P(X > 0) = 1$, $EX^{2a} + EX^{2b} < \infty$, $a < 0 < b$ the power process $Z^{(n)}(t)$, $a \leq t \leq b$ was studied in the paper of M.Csörgő, S.Csörgő, L.Horváth and D.M.Mason [3] (in our case $2b \geq \gamma$ and $s_-^{(n)}(t) \equiv 0$).

Remark 2. Limit theorems for the sums of positive and negative parts of random variables $X_1, X_2, \ldots$ (i.e. $s_+^{(n)}(1)$ and $s_-^{(n)}(1)$) were investigated in [25] by Tucker.

2. Main results

For the description of results let us introduce the following notations: $p_+ = P(X > 0)$, $p_- = P(X < 0)$, $p_0 = P(X = 0)$, $L_0(s) = \int\limits_1^s \frac{L(y)}{y} dy$, $s \geq 1$. Let us denote for sufficient large natural n

(4)
$$b^{(n)}(t) = \begin{cases} n^{1/2} & \text{if } t \leq \frac{\gamma}{2} \text{ and } E|X|^{2t} < \infty, \\ \sup\{s^t : \ n \geq s^2 L_0(s), \ s > 1\} & \text{if } t = \frac{\gamma}{2} \text{ and } E|X|^{\gamma} = \infty, \\ D_n^t, \text{ where} & \\ \quad D_n = \sup\{s : \ n \geq s^{\gamma} L^{-1}(s), \ s \geq 1\} & \text{if } \gamma/2 < t \end{cases}$$

and

$$a_{\pm}^{(n)}(t) = \begin{cases} nEh_{\pm}(X,t) & \text{if } \ 0 \leq t \leq \gamma/2, \\ nEh_{\pm}(X,t) I(h_{\pm}(X,t) < b^{(n)}(t)) & \text{if } \ \gamma/2 < t < \infty. \end{cases} \tag{5}$$

Let us denote for arbitrary natural number M, positive number C and arbitrary real vector $t = (t_1, \ldots, t_M) \in \mathbb{R}^M$ with the property $\gamma/2 < t_1 < t_2 < \ldots < t_M < \infty$ the function

(6)
$$f_*(\lambda; t, \gamma, C) =$$
$$= \exp\left\{ C \int\limits_0^{\infty} \left[\exp\left\{ i \sum_1^M \lambda_j u^{t_j/\gamma} \right\} - 1 - iI(0 < u < 1) \sum_1^M \lambda_j u^{t_j/\gamma} \right] u^{-2} du \right\},$$

where $\lambda = (\lambda_1, \ldots, \lambda_M) \in \mathbb{R}^M$. Let us denote the sets

$$T_1 = [0, \gamma/2] \text{ and } T_2 = \{\emptyset\} \quad \text{if } E|X|^{\gamma} < \infty,$$
$$T_1 = [0, \gamma/2) \text{ and } T_2 = \{\gamma/2\} \quad \text{if } E|X|^{\gamma} = \infty, \quad \text{and}$$
$$T_3 = (\gamma/2, \infty).$$

Theorem 2. *If condition (1) is fulfilled, then the limit process Z(t) in Theorem 1 possesses the following properties:*

(A) The parts $\{Z(t),\ t \in T_1\}$, $\{Z(t),\ t \in T_2\}$ *and* $\{Z(t),\ t \in T_3\}$ *of stochastic process* $\{Z(t),\ 0 \leq t < \infty\}$ *are independent; the process*

$\{Z(t),\ t \in T_1 \cup T_2\}$ is gaussian process; the onedimensional distributions of process $\{Z(t),\ t \in T_3\}$ are stable.

(B) The process $\{Z(t),\ t \in T_1\}$ can be represented in the form

$$Z(t) = V(t) + W(t),$$

where

$$V(t) = (V_+(t),\ V_-(t)),$$
$$W(t) = (W_+(t),\ W_-(t)) = ([Eh_+(X,t)/p_+]\,W_+,\ [Eh_-(X,t)/p_-]\,W_-)\,,$$

for which the following relations hold

(a) the vector-valued gaussian process $\{V(t),\ t \in T_1\}$ does not depend on the gaussian random variable $W = (W_+, W_-)$;

(b) the components $V_+(t)$ and $V_-(t)$ of process $V(t)$ are independent with zero expectation and with covariance functions $(t,\ s \in T_1)$

$$(7)\quad R_\pm(t,s) = \operatorname{cov}(V_\pm(t),\ V_\pm(s)) = Eh_\pm(X,t+s) - Eh_\pm(X,t)Eh_\pm(X,s)/p_\pm,$$

$$(if\ \ p_\pm = 0,\quad then\ we\ define\ R_\pm(t,s) = 0);$$

(c) the random vector W has zero expectation and its covariance matrix C_W is

$$(8)\qquad C_W = \operatorname{cov}(W) = \begin{pmatrix} p_+(1-p_+) & -p_+p_- \\ -p_+p_- & p_-(1-p_-) \end{pmatrix}.$$

(C) If the set T_2 is non-empty, then the gaussian random variables $Z_+(\gamma/2)$ and $Z_-(\gamma/2)$ are independent with zero expectations and with variances C_+ and C_-.

(D) The components $\{Z_+(t),\ t \in T_3\}$ and $\{Z_-(t),\ t \in T_3\}$ of process $\{Z(t),\ t \in T_3\}$ are independent and the characteristic functions of their finite dimensional distributions are (see (4))

$$f_+(\lambda,t) = f_*(\lambda;t,\gamma,C_+) \quad \text{and} \quad f_-(\lambda,t) = f_*(\lambda;t,\gamma,C_-).$$

Remark 3. The statements of Theorem 2 remain valid in all interval $T_1 = [0, a/2]$, $a > 0$ if condition (1) is replaced by condition $E|X|^a < \infty$.

Remark 4. In the case $p_{\pm} > 0$ we define the distribution functions

$$F_+(y) = (F(y) - F(+0))/p_+ \quad \text{if } y > 0 \text{ and } F_+(0) = 0,$$

(9) and

$$F_-(y) = (F(0) - F(-y))/p_-, \quad \text{if } y \geq 0,$$

then the covariance functions of processes $V_{\pm}(t)$ have the forms

$$R_{\pm}(t,s) = p_{\pm} \left(\int_0^{\infty} y^{t+s} dF_{\pm}(y) - \int_0^{\infty} y^t dF_{\pm}(y) \int_0^{\infty} y^s dF_{\pm}(y) \right). \tag{10}$$

Remark 5. The determinant of covariance matrix C_W is $\det(C_W) = p_+p_-p_0$, therefore the distribution of gaussian random vector is nondegenerate iff $p_+p_-p_0 \neq 0$.

In Theorem 2 the limit process $Z(t)$ is given in weak sense, e.g. with the finite dimensional distributions. We determine it in the following theorem as piecewise continuous with probability 1 process.

Theorem 3. *Assume that condition (1) holds. Then there exists a stochastic process $\{Z(t),\ 0 \leq t < \infty\}$ on some probability space with finite dimensional distributions given in Theorem 2, continuous with probability 1 in the intervals $(0, \gamma/2)$, $(\gamma/2, \infty)$. Moreover, if the condition $E|\log|X|| < \infty$ (resp. $E|X|^{\gamma}|\log|X|| < \infty$) holds, the process $Z(t)$ is continuous from right with probability 1 at t=0 (resp. from left at $t = \gamma/2$).*

The existence of such process on the interval $[0, \gamma/2]$ is guaranteed by the behaviour of expectation and covariance function. In the interval $(\gamma/2, \infty)$ we construct it on basis of the following theorem using the representation of random variables having stable distribution with parameter α, $0 < \alpha < 2$ by means of Poisson integrals from [6] (see also [5], Theorem 3).

Let be $\psi_1, \psi_2, \ldots$ a sequnce of independent exponentially distributed with parameter 1 random variables. Let us introduce the left continuous standard Poisson process $\{N(t),\ 0 \leq t < \infty\}$ with the equation

$$N(t) = \sum_{j=1}^{\infty} I(\psi_1 + \ldots \psi_j < t), \quad 0 \leq t < \infty,$$

and define the stochastic process $\{\zeta(t),\ 1/2 \leq t < \infty\}$ on the following way

$$\zeta(t) = \int_0^1 N(u) t u^{-1-t} du + \int_1^{\infty} (N(u) - u) t u^{-1-t} du + 1.$$

We note that $\zeta(t)$ is well defined, finite with probability 1 (see Theorem 3 [5]) and its characteristic function is equal to

$$E\exp\{i\lambda\zeta(t)\} = \exp\left\{\int_0^\infty \left[\exp\{i\lambda u^t\} - 1 - i\lambda I(0 < u < 1)u^t\right] u^{-2}du\right\}.$$

By means of $\zeta(t)$ we can immediately construct the processes $Z_+(t)$ and $Z_-(t)$ satisfying the conditions of Theorem 3, so in the future we will deal with the most important properties of $\zeta(t)$ formulated in the following statement.

Let M be arbitrary natural number, $\lambda \in \mathbb{R}^M$ and $t = (t_1, \ldots, t_M)$ arbitrary vector satisfying condition

$$1/2 < t_1 < \ldots < t_M < \infty.$$

Theorem 4. *The stochastic process has the following properties:*

a) the characteristic function of finite dimensional distributions of $\zeta(t)$ *is* $f_*(\lambda; t, 1, 1)$*;*

b) in case $\gamma > 0,\ C > 0$ *for the process*

$$\zeta_{\gamma,C}(s) = C^{s/\gamma}\zeta(s/\gamma) + r(s), \qquad \gamma/2 < s < \omega \tag{11}$$

derived from the process $\zeta(t)$*, the characteristic function of finite dimensional distributions is* $f_*(\lambda; t, \gamma, C)$*, where*

$$r(s) = I(s = \gamma)C\log C + I(s \neq \gamma)\gamma(s-\gamma)^{-1}\left(C^{(s-\gamma)/\gamma} - 1\right),$$

$$\gamma/2 < s < \infty,$$

is arbitrary many times differentiable;

c) $\zeta(t)$ *is arbitrary many times differentiable with probability 1 process on the interval* $(1/2, \infty)$ *and for the k-th* ($k \geq 1$) *derivative process* $\{\zeta^{(k)}(t),\ 1/2 < t < \infty\}$ *the representation*

$$\begin{aligned}\zeta^{(k)}(t) = &\int_0^1 N(u)u^{-1-t}(-\log u)^{k-1}(k - t\log u)du + \\ &+ \int_1^\infty (N(u) - u)u^{-1-t}(-\log u)^{k-1}(k - t\log u)du\end{aligned} \tag{12}$$

takes place and the characteristic function of its finite dimensional distributions is

$$f_*^{(k)}(\lambda;t,1,1) = \exp\left\{\int_0^\infty \left[\exp\left[i\sum_{j=1}^{M}\lambda_j u^{t_j}(\log u)^k\right] - \right.\right.$$

$$\left.\left. - 1 - iI(0 < u < 1)\Big(\sum_{j=1}^{M}\lambda_j u^{t_j}(\log u)^k\Big)\right] u^{-2}du\right\}.$$

Corollary 1. Starting from the process $\{\zeta(t), 1/2 < t < \infty\}$ in case of arbitrary $\gamma > 0$, $C > 0$ the finite dimensional distributions of $\{\zeta_{\gamma,C}(t),\ \gamma/2 < t < \infty\}$ defined by (11) coincide with the finite dimensional distributions of processes $Z_+(t)$, $Z_-(t)$ $(\gamma/2 < t < \infty)$ when the parameter C takes on the values C_+ and C_- respectively. From another side it is obvious that the mapping (11) preserves the continuity with probability 1 and the differentiability with probability 1 of the process (on the corresponding interval). So the proof of Theorem 4 completes at the same time the proof of Theorem 3 on the interval $(\gamma/2, \infty)$.

The processes $Z^{(n)}(t)$, $n = 1, 2, \ldots$ and the limit process $Z(t)$ are piecewise continuous with probability 1. This fact makes easier the proof of invariance principle.

Let $[a,b]$ be an arbitrary closed interval of the open interval $(0, \gamma/2)$ or $(\gamma/2, \infty)$. Let us define $(\mathcal{C}[a,b] \times \mathcal{C}[a,b])$ as the space of $\mathbb{R}^2$–valued continuous functions on the interval $[a,b]$ and let $\mathcal{B}[a,b]$ be the minimal σ-algebra which contains all cylinder sets from $(\mathcal{C}[a,b] \times \mathcal{C}[a,b])$. Note that this space with the supremum norm is a complete separable metric space and $\mathcal{B}[a,b]$ is Borel σ-algebra.

The sequence of processes $Z^{(n)}(t)$ and the limit process $Z(t)$ determine on the measurable space $\{(\mathcal{C}[a,b] \times \mathcal{C}[a,b]),\ \mathcal{B}[a,b]\}$ the measures $\mathcal{P}_{[a,b]}^{(n)}$ and $\mathcal{P}_{[a,b]}$, respectively. Then there is valid the following

Theorem 5. *If (1) is fulfilled then the weak convergence of measures*

$$\mathcal{P}_{[a,b]}^{(n)} \xrightarrow{w} \mathcal{P}_{[a,b]}, \qquad n \to \infty$$

holds.

Note that one can consider the invariance principle in like manner in the case, when the processes $Z^{(n)}(t)$ and $Z(t)$ are replaced with their derivative processes.

Remark 6. Notice that from Theorem 1 immediately follows that for arbitrary natural k, $t = (t_1, \dots, t_k) \in \mathbb{R}_+^k$ and arbitrary real valued Borel-measurable function g the convergence (in distribution)

$$g(Z^{(n)}(t_1), \dots, Z^{(n)}(t_k)) \xrightarrow{d} g(Z(t_1), \dots, Z(t_k)), \quad n \to \infty$$

holds. Theorem 5 shows that by the Prohorov theorem for every bounded $\mathcal{B}[a,b]$-measurable functional G given on $(\mathcal{C}[a,b] \times \mathcal{C}[a,b])$, which is $\mathcal{P}_{[a,b]}$ almost surely continuous, the following convergence is satisfied

$$G\left(Z^{(n)}(\cdot)\right) \xrightarrow{d} G(Z(\cdot)), \qquad n \to \infty.$$

3. Proof of theorems

Let us denote for sufficient large natural n the sequence of normalizing and centralizing values playing role in [18]

$$B^{(n)}(t) = \begin{cases} n^{1/2} & \text{if } t \le \frac{\gamma}{2} \text{ and } E|X|^{2t} < \infty, \\ \sup\{s^t : n \ge s^2 E|X|^\gamma I(|X| < s)\}, & \text{if } t = \frac{\gamma}{2} \text{ and } E|X|^\gamma = \infty, \\ D_n^t, \text{ where} & \\ D_n = \sup\{s : n \ge s^\gamma L^{-1}(s),\ s \ge 1\} & \text{if } \gamma/2 < t \end{cases}$$

and

$$A_\pm^{(n)}(t) = \begin{cases} nEh_\pm(X,t) & \text{if } \ 0 \le t < \gamma, \\ nEh_\pm(X,\gamma), & \text{if } \ \gamma \le t < \infty, \\ 0, & \text{if } \ \gamma < t < \infty, \end{cases}$$

where

$$\widetilde{h}_\pm(x,\gamma) = \left((B^{(n)}(\gamma))^2 h_\pm(X,\gamma)\right)\left((B^{(n)}(\gamma))^2 + (h_\pm(X,\gamma))^2\right)^{-1}.$$

With the help of results concerning slowly varying functions it is not difficult to verify that the following relations hold

$$\lim_{n \to \infty} B^{(n)}(\gamma/2)\left(b^{(n)}(\gamma/2)\right)^{-1} = C_+ + C_-,$$

$$\lim_{n\to\infty}\left(b^{(n)}(\gamma)\right)^{-1}\left(a_{\pm}^{(n)}(\gamma)-A_{\pm}^{(n)}(\gamma)\right)=0,$$

and

$$\lim_{n\to\infty}\left(b^{(n)}(t)\right)^{-1}\left(a_{\pm}^{(n)}(t)-A_{\pm}^{(n)}(t)\right)=$$

$$=\begin{cases} C_{\pm}\int\limits_{1}^{\infty}u^{t/\gamma}u^{-2}du=C_{\pm}\dfrac{t}{\gamma-t}, & \text{if } \gamma/2<t<\gamma,\\ C_{\pm}\int\limits_{0}^{1}u^{t/\gamma}u^{-2}du=C_{\pm}\dfrac{t}{t-\gamma}, & \text{if } \gamma<t<\infty,\end{cases}$$

so Theorem 1 and the parts (A), (C), (D) of Theorem 2 are simple consequences of paper [18].

Proof of part (B) of Theorem 2. We carry out the proof by investigation of the process $Z^{(n)}(t)$ and give a clear notion on the nature of given representation of the limit process $Z(t)$.

By the definition for every $t\in T_1$ and $n\geq 1$ hold

$$b^{(n)}(t)=n^{1/2},\quad a_{\pm}^{(n)}(t)=nEh_{\pm}(X,t)\quad\text{and}\quad a_{\pm}^{(n)}(0)=\lim_{t\to+0}nEh_{\pm}(X,t)=np_{\pm}.$$

Let us represent the process $Z^{(n)}(t)$ as the sum of two processes as follows

$$Z^{(n)}(t)=V^{(n)}(t)+W^{(n)}(t),\qquad t\in T_1,$$

where

$$V^{(n)}(t)=\left(V_{+}^{(n)}(t),V_{-}^{(n)}(t)\right),\quad W^{(n)}(t)=\left(W_{+}^{(n)}(t),W_{-}^{(n)}(t)\right),\quad t\in T_1,$$

and the individual components are defined by the equations

$$V_{\pm}^{(n)}(t)=n^{-1/2}\sum_{j=1}^{n}I(X_j^{\pm}>0)\left[h_{\pm}(X_j,t)-Eh_{\pm}(X_j,t)/p_{\pm}\right],$$

and

$$W_{\pm}^{(n)}(t)=\left[Eh_{\pm}(X,t)/p_{\pm}\right]n^{-1/2}\sum_{j=1}^{n}\left(I(X_j^{\pm}>0)-p_{\pm}\right).$$

Here the components indexed by signs "+" and "−" we equalize with zero if the conditions $p_{+}=0$ and $p_{-}=0$ hold, respectively.

We prove the weak convergence (convergence of finite dimensional distributions)

$$(V^{(n)}(t), W^{(n)}(t)) \xrightarrow{d} (V(t), W(t)), \qquad n \to \infty, \tag{13}$$

where the gaussian process $\{(V(t), W(t)),\ \ t \in T_1\}$ has expectation and covariance function given in Theorem 2. Then from the relation (13) follows the weak convergence

$$Z^{(n)}(t) \xrightarrow{d} Z(t) = \big(V_+(t) + W_+(t),\ V_-(t) + W_-(t)\big), \quad n \to \infty, \tag{14}$$

which means the verification of the part (B) of Theorem 2.

Since the random variables $X_1, X_2, \ldots$ are independent and identically distributed, therefore it is clear that for all $n \geq 1$ the processes $\{V^{(n)}(t),\ t \in T_1\}$ and $\{W^{(n)}(t),\ t \in T_1\}$ have zero expectation functions, their covariance functions are equal to (7) and (8), respectively, they do not depend on n and by the central limit theorem the limits of finite dimensional distributions are gaussian.

Proof of Theorem 3. The fact that the processes $\{Z(t),\ t \in T_1\}$, $\{Z(t),\ t \in T_2\}$ and $\{Z(t),\ t \in T_3\}$ (given in Theorem 2 in weak sense) are independent, makes possible the representation of them independently of others.

First we note that in the case $T_2 \neq \{\emptyset\}$ the representation of $\{Z(t),\ t \in T_2\}$ means to give a probability space with independent gaussian random variables $Z_+(\gamma/2)$ and $Z_-(\gamma/2)$ having zero expectations and variances C_+ and C_-, for this reason it is enough to deal with the representation of processes $\{Z(t),\ t \in T_1\}$ and $\{Z(t),\ t \in T_3\}$, which we give in parts A) and B) respectively.

Since the processes $\{W(t),\ t \in T_1\}$ and $\{V(t),\ t \in T_1\}$ are independent, therefore we can carry out their representation one by one. In both cases the nature of representation depends on the smoothness of function $E|X|^t$, $t \in T_1$, thus we need the following statement (we define $0 \cdot \log 0 = 0$).

Lemma 1. *Let U be a random variable. If for some natural number $k \geq 1$ and for some positive value b the condition*

$$E|U|^b|\log^k|U||I(U \neq 0) < \infty \tag{15}$$

is satisfied, then the function $u(t) = E|U|^t$ is k times continuously differentiable on the interval $(0,b]$ (at the point b it has left continuous derivative). If, in addition, the condition

$$E|\log^k|U||I(U \neq 0) < \infty \tag{16}$$

holds, then u(t) is k times continuously differentiable from right at 0. In both cases we have (in the corresponding interval)

$$(u(t))^{(j)} = E|U|^t \log^j |U|, \quad 1 \le j \le k. \tag{17}$$

Corollary 2. We get in special case analogous to (17) formulae for the derivatives (on corresponding intervals) of functions $E|X|^t$, $E(X^+)^t$ and $E(X^-)^t$, if the random variable U is replaced with random variables $|X|$, X^+ and X^- respectively.

Proof of Lemma 1. Let a, c_1 and c_2 be positive constants, for which the conditions $0 < a < b$ and $0 < c_1 < 1 < c_2$ are satisfied. Let us form the function $E|U|^t \log^j |U|$, $t \in [a, b]$, $0 \le j < k$, as a sum of functions

$$u_1(t; j, c_1, c_2) = E|U|^t \log^j |U| I(c_1 \le |U| \le c_2)$$

and

$$u_2(t; j, c_1, c_2) = E|U|^t \log^j |U| \big(I(0 < |U| < c_1) + I(c_2 < |U|)\big).$$

Since the function $y^t \log^j y$, $c_1 \le y \le c_2$, $a \le t \le b$ is continuous in the variables t and y and its derivative with respect to t is uniformly continuous, the function $u_1(t; j, c_1, c_2)$ is differentiable and

$$\frac{d}{dt} u_1(t; j, c_1, c_2) = E|U|^t \log^{j+1} |U| I(c_1 < |U| < c_2). \tag{18}$$

Let us define the difference quotient for arbitrary t, $t + h \in [a, b]$, $h \ne 0$

$$\Delta_h u_2(t; j, c_1, c_2) =$$

$$= E\left(\frac{|U|^{t+h} \log^j |U| - E|U|^t \log^j |U|}{h} [I(0 < |U| < c_1) + I(c_2 < |U|)]\right).$$

Let us denote $t' = \min(t, t + h)$, $t'' = \max(t, t + h)$. By the Lagrange theorem ($c_1 \le y \le c_2$) for some h', $0 < h' < |h|$ it is valid the equation $\left|\frac{y^{t''-t'} - 1}{t'' - t'}\right| = y^{h'} \log y$ (here the value h' depends on y), so we find the following inequality

$$|\Delta_h u_2(t; j, c_1, c_2)| \le$$

$$
(19)\quad
\begin{aligned}
&\le E|U|^{t'} \left|\frac{|U|^{t''-t'}}{t''-t'} \log^j |U|\right| \left(I(0<|U|\le c_1)+I(c_2<|U|)\right) \le \\
&\le E|U|^{t'+h'} \left|\log^{j+1}|U|\right| \left(I(0<|U|\le c_1)+I(c_2<|U|)\right) \le \\
&\le E|U|^a \left|\log^{j+1}|U|\right| I(0<|U|<c_1) + E|U|^b \log^{j+1}|U| I(c_2<|U|).
\end{aligned}
$$

It is clear that the condition (15) implies the relation ($0 < a < b$)

$$E(|U|^a + |U|^b) \left|\log^k |U|\right| < \infty,$$

because of it the function $\Delta_h u(t; j, c_1, c_2)$ converges uniformly to zero as $|h| \to 0$, $c_1 \to 0$ and $c_2 \to \infty$. On the other hand, the j-th derivative of function $u_1(t; 0, c_1, c_2)$ tends to the right side of (17) as $c_1 \to 0$, $c_2 \to \infty$, in consequence of this the relation (17) holds on the interval $[a, b]$ and it is also true on the interval $(0, b]$.

It is obvious that in the case, when condition (16) holds, the estimate (19) remains valid with value $a = 0$ and in a similar manner we can obtain the relation (17) (for right sided derivative) at the point 0.

Let us return to the representation of processes $W(t)$ and $V(t)$, which we can derive independently of each other.

a) It is clear that there exists a gaussian random vector $W = (W_+, W_-)$ on some probability space with zero expectation and covariance matrix (8), therefore it suffices to consider the deterministic functions $E(X^+)^t$ and $E(X^-)^t$ in the interval $[0, \gamma/2]$. By assumption (1) we can get that for all natural numbers k and arbitrary $t \in (0, \gamma)$ it is true $E|X|^t \left|\log^k |X|\right| < \infty$, so the continuity and differentiability of functions $E(X^+)^t$ and $E(X^-)^t$ on the corresponding interval follow immediately by Lemma 1.

b) By making use of the independence of processes $V_+(t)$ and $V_-(t)$ it is sufficient to consider one of them, say $V_+(t)$ (we assume in this case that the condition $p_+ > 0$ holds).

First we note that by the Kolmogorov consistency theorem and by [10] (Theorems 2 and 5 of Chapter III.2) there exists a separable gaussian process $\{V_+(t),\ t \in T_1\}$ on some probability space, it has zero expectation and covariance function $R_+(t, s)$. Here in consequence of smoothness of function $R_+(t, s)$ the process $V_+(t)$ is stochastically continuous and we can choose any countable dense set in T_1 as a separability set.

It remains to prove that the process $\{V_+(t),\ t \in (0, \gamma/2)\}$ has continuous with probability 1 version, it is right continuous at 0 (resp. left at $\gamma/2$) with probability 1, when the additional condition holds. For this purpose it suffices to prove (see [10], Theorem 7, Chapter III.5) that for arbitrary t, $t+h \in T_1$ the variance

$$\sigma_+^2(t,h) =$$
$$= E(V_+(t+h) - V_+(t))^2 = R_+(t+h,t+h) - 2R_+(t,t+h) + R_+(t,t) =$$
$$= \left[E\left(X^+\right)^{2t+2h} - E\left(X^+\right)^{t+h} E\left(X^+\right)^{t+h}\right] -$$
$$-2\left[E\left(X^+\right)^{2t+h} - E\left(X^+\right)^{t+h} E\left(X^+\right)^{t}\right] + \left[E\left(X^+\right)^{2t} - E\left(X^+\right)^{t} E\left(X^+\right)^{t}\right]$$

has upper bound $c|h|$ for some $c > 0$. It is obvious that one can write $\sigma_+^2(t,t+h)$ in the form

(20)

$$\sigma_+^2(t,t+h) \leq \left|E\left(X^+\right)^{2t+2h} - E\left(X^+\right)^{2t+h}\right| + \left|E\left(X^+\right)^{2t+h} - E\left(X^+\right)^{2t}\right| +$$
$$+E\left(X^+\right)^{t+h}\left|E\left(X^+\right)^{t+h} - E\left(X^+\right)^{t}\right| + E\left(X^+\right)^{t}\left|E\left(X^+\right)^{t+h} - E\left(X^+\right)^{t}\right|.$$

Since in consequence of assumption (1) for all β, $0 < \beta < \gamma$ the expectation $E|X|^\beta$ is finite, so by Lemma 1 follows that the derivative of function $E(X^+)^t$ is continuous in arbitrary closed interval $[a, 2b] \subset (0,\gamma)$ $(a < b)$. The continuity of derivative remains valid in the cases $a = 0$ and $b = \gamma/2$, respectively, if the additional conditions $E|\log|X|| < \infty$ and $E|X|^\gamma|\log|X|| < \infty$ hold. From this we get that the derivative is bounded by some constant $c > 0$ in the interval $[a, b]$, and with help of the Lagrange theorem one can find for arbitrary t, $t+h \in [a,b]$ the inequality $\sigma_+^2(t,t+h) \leq c|h|$, which remains true in the cases $a = 0$ and $b = \gamma/2$, if the additional conditions hold.

We mention here that under conditions (1) and $E\log^2 X_+ < \infty$ one can define by Lemma 1 a continuous with probability 1 gaussian stochastic process $\{V_0(t),\ 0 < t \leq b\}$ $(0 < b < \gamma/2)$ with zero expectation and with covariance function

$$\int_0^\infty y^{t+s}\log^2 y\,dF_+(y) - \int_0^\infty y^t \log y\,dF_+(y)\int_0^\infty y^s\log y\,dF_+(y).$$

In this case the process $V_+(t) = p_+^{1/2}\int_0^t V_0(s)ds$ is gaussian with zero expectation and with covariance function $R_+(t,s)$, it is continuously differentiable with probability 1 and the derivative process is $p_+^{1/2}V_0(t)$.

Remark 7. By means of the above classical method (i.e. with the help of investigation of expectation and covariance function) one can construct the process $\{V_+(t),\ 0 < t < \gamma/2\}$ in another way (see [3]):

Let $\{B(s),\ 0 \le s \le 1\}$ be Brown-bridge on some probability space, then the process $\{V_+(t),\ 0 < t < \gamma/2\}$ can be represented in the form

$$V_+(t) = p_+^{1/2} \int_0^\infty y^t dB(F_+(y)).$$

To consider the trajectories in this way does not seem easier (in particular at the points 0 and $\gamma/2$) than in the classical way.

Proof of part a) of Theorem 4. We show that the joint characteristic function of random variables $\zeta(t_1), \ldots, \zeta(t_M)$, $1/2 < t_1 < \ldots < t_M < \infty$ can be represented in the form $\big(\lambda = (\lambda_1, \ldots, \lambda_M) \in \mathbb{R}^M\big)$:

$$E\exp\left\{ i\sum_{j=1}^{M} \lambda_j \zeta(t_j) \right\} = f_*(\lambda; t, 1, 1) =$$

$$= \exp\left\{ \int_0^\infty (\exp\{i\Phi(u,\lambda,t)\} - 1 - iI(0 < u < 1)\Phi(u,\lambda,t))\, u^{-2} du \right\},$$

where $\Phi(u,\lambda,t) = \sum_{j=1}^{M} \lambda_j u^{t_j}$.

Let us introduce the modified sequence of processes $\{\zeta(t,n),\ 1/2 < t < < \infty\}$, $n \ge 1$:

$$\zeta(t,n) = \int_{1/n}^{1} N(u) t u^{-1-t} du + \int_1^n (N(u) - u) t u^{-1-t} du + 1.$$

It is clear that for each $t \in (1/2, \infty)$

$$P\left(\int_0^{1/n} N(u) t u^{-1-t} du > 0 \right) = P(\psi_1 < 1/n) = 1 - e^{-1/n} \to 0, \quad n \to \infty.$$

Furthermore, for any $\varepsilon > 0$ by the Markov inequality

$$P\left(\int_n^\infty |N(u)-u| t u^{-1-t} du > \varepsilon\right) \le \varepsilon^{-1} E \int_n^\infty |N(u)-u| t u^{-1-t} du.$$

By making use of the Fubini theorem and Cauchy-Schwarz inequality

$$\begin{aligned} E\int_n^\infty |N(u)-u| t u^{-1-t} du &= \int_n^\infty E|N(u)-u| t u^{-1-t} du \le \\ &\le \int_n^\infty \left(D^2(N(u)-u)\right)^{1/2} t u^{-1-t} du = \\ &= \int_n^\infty t u^{-1/2-t} du = t(1/2-t)^{-1} n^{1/2-t} \to 0, \\ & n \to \infty, \end{aligned}$$

there is valid the $\zeta(t) - \zeta(t,n) \xrightarrow{P} 0$, $n \to \infty$, convergence in measure, so by the Slutsky theorem (see [2], p.249) we obtain

$$\sum_{j=1}^M \lambda_j \zeta(t_j, n) \xrightarrow{d} \sum_{j=1}^M \lambda_j \zeta(t_j), \qquad n \to \infty.$$

Let us denote the corresponding characteristic functions by

$$\tau(\lambda, t, n) = E \exp\left\{ i \sum_{j=1}^M \lambda_j \zeta(t_j, n) \right\}$$

and

$$\tau(\lambda, t) = E \exp\left\{ i \sum_{j=1}^M \lambda_j \zeta(t_j) \right\},$$

then one can write

$$\lim_{n\to\infty} \tau(\lambda, t, n) = \tau(\lambda, t). \tag{21}$$

On the interval $[1/n, n]$

$$\frac{\partial}{\partial u}\Phi(u^{-1},\lambda,t) = -\sum_{j=1}^{M}\lambda_j t_j u^{-1-t_j},$$

from which as consequence of (21) we obtain the convergence

$$\tau(\lambda,t,n) = E\exp\left\{-i\int_{1/n}^{n} N(u)\frac{\partial}{\partial u}\left(\Phi(u^{-1},\lambda,t)\right)du + \right. \tag{22}$$
$$\left. +\,i\left(\int_{1}^{n} u\frac{\partial}{\partial u}\left(\Phi(u^{-1},\lambda,t)\right)du + \sum_{j=1}^{M}\lambda_j\right)\right\} \to \tau(\lambda,t),$$

$$n \to \infty.$$

Let us consider the expressions appearing in the exponent. Since $N(n)$ is finite with probability 1, $N(t)$ $(t \geq 0)$ is monotone increasing and Φ continuously differentiable on the interval $[1/n, n]$, so by using the rules of partial integration of Riemann and Riemann-Stieltjes integrals we come to the identities

$$-i\int_{1/n}^{n} N(u)\left(\frac{\partial}{\partial u}\Phi(u^{-1},\lambda,t)\right)du =$$

$$= i\int_{1/n}^{n}\Phi(u^{-1},\lambda,t)dN(u) - i\left(N(n)\Phi(n^{-1},\lambda,t) - N(n^{-1})\Phi(n,\lambda,t)\right),$$

$$i\int_{1}^{n} u\frac{\partial}{\partial u}(\Phi(u^{-1},\lambda,t))du + \sum_{j=1}^{M}\lambda_j = -i\int_{1}^{n}\Phi(u^{-1},\lambda,t)du + in\Phi(n^{-1},\lambda,t).$$

Since

$$P(N(n^{-1}) > 0) = P(\psi_1 < 1/n) = 1 - e^{-1/n} \to 0, \qquad n \to \infty$$

and for arbitrary $\varepsilon > 0$ as $n \to \infty$

$$P(|(N(n) - n)\Phi(n^{-1},\lambda,t)| > \varepsilon) \leq \varepsilon^{-1}E|(N(n) - n)\Phi(n^{-1},\lambda,t)| \leq$$

$$\leq \varepsilon^{-1}\left(E(N(n) - n)^2\right)^{1/2}\Phi(n^{-1},\lambda,t) = \varepsilon^{-1}n^{1/2}\Phi(n^{-1},\lambda,t) \to 0,$$

using once more the Slutsky theorem we get that it is enough to investigate the limit of function

$$\tilde{r}(\lambda,t,n) = E\exp\left\{ i\int_{1/n}^{n} \Phi(u^{-1},\lambda,t)dN(u) - i\int_{1}^{n}\Phi(u^{-1},\lambda,t)du \right\}$$

as $n \to \infty$.

By definition $N(t)$ is homogeneous Poisson process with intensity 1, and $\Phi(u^{-1},\lambda,s)$ is continuous function for the variable u in the interval $[1/n,n]$, so by [1] (Chapter V., Theorem II (p.175)) we have

$$i\int_{1/n}^{n} \Phi(u^{-1},\lambda,s)dN(u) = \exp\left\{ \int_{1/n}^{n} \left[\exp(\Phi(u^{-1},\lambda,s)-1)\right] du \right\}.$$

On the basis of these facts

$$\tilde{r}(\lambda,s,n) =$$

$$= \exp\left\{ \int_{1/n}^{n} \left(\exp\{i\Phi(u^{-1},\lambda,s)\}-1\right) du - i\int_{1}^{n}\Phi(u^{-1},\lambda,s)du \right\} =$$

$$= \exp\left\{ \int_{1/n}^{n} \left[\exp\{i\Phi(u,\lambda,s)\} - 1 - iI(0<u<1)\Phi(u,\lambda,s)\right] u^{-2}du \right\}.$$

Since the expression in the exponent is absolutely integrable on $(0,\infty)$, it is true the convergence

$$\lim_{n\to\infty} \tilde{r}(\lambda,s,n) = \lim_{n\to\infty} r(\lambda,s,n) =$$

$$= \exp\left\{ \int_{0}^{\infty} \left(\exp\{i\Phi(u,\lambda,s)\} - 1 - iI(0<u<1)\Phi(u,\lambda,s)\right) u^{-2}du \right\}.$$

Proof of part b) of Theorem 4. Let $\gamma > 0$ and $C > 0$ be arbitrary. Using the characteristic function of finite dimensional distributions of process $\zeta(t)$ on the basis of formula (6) (the values t_j, $1 \le j \le M$ satisfying the condition $1/2 < t_1 < \ldots < t_M < \infty$ and $\lambda \in \mathbb{R}^M$ are chosen on arbitrary way)

$$r(\lambda,t;\gamma,C) =$$

$$= E\exp\left\{i\sum_{j=1}^{M}\lambda_j\zeta_{\gamma,C}(t_j)\right\} = E\exp\left\{i\sum_{j=1}^{M}\lambda_j C^{t_j/\gamma}\zeta(t_j/\gamma)\right\} =$$

$$= \exp\left\{\int_0^\infty\left[\exp\left\{i\sum_{j=1}^{M}\lambda_j(Cu)^{t_j/\gamma}\right\} - 1 - iI(0<u<1)\sum_{j=1}^{M}\lambda_j(Cu)^{t_j/\gamma}\right]\times\right.$$

$$\left.\times u^{-2}du + iC\int_1^C\sum_{j=1}^{M}\lambda_j u^{t/\gamma}u^{-2}du\right\} = f_*(\lambda;t,\gamma,C).$$

In order to prove the differentiability of process $\zeta(t)$ we need the following lemma. Let us introduce the notations in case of arbitrary $1/2 < t < \infty$

$$\zeta_{k,1}(t) = \int_0^1 N(u)u^{-1-t}(1+|\log u|)^k du,$$

$$\zeta_{k,2}(t) = \int_1^\infty |N(u)-u|u^{-1-t}(1+\log u)^k du.$$

Lemma 3. *For arbitrary fixed $t\in(1/2,\infty)$ and natural number k $\zeta_{k,1}(t)$ and $\zeta_{k,2}(t)$ are finite with probability 1, furthermore $\zeta_{k,1}(t)$ is monotonely increasing, $\zeta_{k,2}(t)$ is monotonely decreasing with probablity 1.*

Proof of Lemma 3. Since $N(u)$ is monotonely increasing and on the interval $[0,\psi_1)$ takes on the value 0, $N(1)$ and ψ_1 are with probability 1 finite, because of

$$\zeta_{k,1}(t) \le I(\psi_1<1)N(1)(1+|\log\psi_1|)^k\int_{\psi_1}^1 u^{-1-t}du \le \tag{23}$$
$$\le I(\psi_1<1)N(1)(1+|\log\psi_1|)^k t^{-1}\psi_1^{-t}.$$

$\zeta_{k,1}(t)$ is with probability 1 finite.

Now we will deal with $\zeta_{k,2}(t)$ and show that its expectation is finite. Since the variance of homogeneous Poisson process $N(u)$ with intensity 1 for arbitrary $u\ge 0$ is equal to u, by making use of Fubini theorem and Cauchy-Schwarz inequality we obtain

$$E\zeta_{k,2}(t) = E\int_1^\infty |N(u)-u|u^{-1-t}(1+\log u)^k \le$$

$$\leq \int_1^\infty \left(E(N(u)-u)^2\right)^{1/2} u^{-1-t}(1+\log u)^k du = \int_1^\infty u^{-t-1/2}(1+\log u)^k du < \infty.$$

Thus Lemma 3 is proved.

Remark. From inequality (23) one can easily derive that for arbitrary $1/2 < t < 1$ the expectation of random variable $\zeta_{k,1}(t)$ is also finite.

Proof of part c) of Theorem 4. By using the monotonicity of $\zeta_{k,1}(t)$ and $\zeta_{k,2}(t)$ from Lemma 3 follows that for the process $\zeta^{(k)}(t)$, defined by (12), in arbitrary $[a,b] \subset (1/2,\infty)$ the inequality

$$|\zeta^{(k)}(t)| \leq (k+b)\left(\zeta_{k,1}(b) + \zeta_{k,2}(a)\right), \qquad t \in [a,b]$$

takes place with probability 1. The first derivative of function tu^{-t} by t is

$$\frac{\partial}{\partial t}\left(tu^{-t}\right) = u^{-t}(1 - t\log u),$$

and for $k = 1,2,\ldots$ there is true the relation

$$\frac{\partial}{\partial t}\left(u^{-t}(-\log u)^{k-1}(k - t\log u)\right) = u^{-t}(-\log u)^k((k+1) - t\log u),$$

so formula (12) can be obtained by induction. Consequently the process $\zeta(t)$ on the interval $(1/2,\infty)$ is arbitrary many times differentiable by t and the k-th derivative process is $\zeta^{(k)}(t)$.

To determine the characteristic function of finite dimensional distributions we consider the formula (12) the k-th ($k \geq 1$) derivative process. In case $1/2 <$ $< t < \infty$ $\zeta^{(k)}(t)$ can be written in the form

$$\zeta^{(k)}(t) = \int_0^1 N(u)u^{-1-t}(-\log u)^{k-1}(k - t\log u)du +$$

$$+ \int_1^\infty (N(u)-u)u^{-1-t}(-\log u)^{k-1}(k - t\log u)du =$$

$$= \int_0^1 N(u)u^{-1-t}\frac{\partial}{\partial u}\left(u^{-t}(\log u^{-1})^k\right)du + \int_1^\infty (N(u)-u)\frac{\partial}{\partial u}\left(u^{-t}(\log u^{-1})^k\right)du.$$

This formula allows us to repeat the line of reasoning given by the proof of part a) of Theorem 4 and we can obtain the characteristic function of finite dimensional distributions of k-th derivative process $\zeta^{(k)}(t)$.

Proof of Theorem 5. By virtue of Theorems 1 and 3 the processes $Z^{(n)}(t)$ and $Z(t)$ are continuous with probability 1 on the given interval $[a,b]$, furthermore the weak convergence $Z^{(n)}(t) \xrightarrow{d} Z(t)$, as $n \to \infty$ is true, then by [10] (Theorem 1 of Chapter VI.4) it is sufficient to prove for arbitrary $\varepsilon > 0$ the relation

$$\lim_{h\to 0} \sup_n P\Big(\sup_{\substack{|t'-t''|<h \\ t',t''\in[a,b]}} \Big\{ |Z_+^{(n)}(t') - Z_+^{(n)}(t'')| + \tag{24}$$
$$+ |Z_-^{(n)}(t') - Z_-^{(n)}(t'')| \Big\} > \varepsilon \Big) = 0,$$

which trivially holds when for arbitrary $\varepsilon > 0$

$$\lim_{h\to 0} \sup_n P\left(\sup_{\substack{|t'-t''|<h \\ t',t''\in[a,b]}} |Z_\pm^{(n)}(t') - Z_\pm^{(n)}(t'')| > \varepsilon \right) = 0. \tag{25}$$

Since the processes $Z_+^{(n)}(t)$ and $Z_-^{(n)}(t)$ have the same structure, it suffices to consider one of them, say $Z_+^{(n)}(t)$ (in this case we assume that $C_+ > 0$, i.e. $p_+ > 0$).

First of all we note that in the case, when the supremum in parentheses of (25) has finite expectation, then by the Markov inequality it is sufficient to prove instead of (25) the relation

$$\lim_{h\to 0} \sup_n E\left(\sup_{\substack{|t'-t''|<h \\ t',t''\in[a,b]}} |Z_\pm^{(n)}(t') - Z_\pm^{(n)}(t'')| \right) = 0. \tag{26}$$

In the case $[a,b] \subset (0,\gamma/2)$ we will choose this way.

The case $[a,b] \subset (0,\gamma/2)$

It can be easily verified that for all, on the interval $[a,b]$ twice continuously differentiable functions $g(t)$ the following inequality holds

$$\sup_{\substack{|t'-t''|<h \\ t',t''\in[a,b]}} |g(t')-g(t'')| \le h\cdot \max_{t\in[a,b]} |g'(t)| \le \tag{27}$$
$$\le h\cdot \max_{t\in[a,b]} \left| g'(a) + \int_a^t g''(u)du \right| \le h\cdot \left(|g'(a)| + \int_a^b |g''(u)|du \right).$$

Since $Z_+^{(n)}(t)$ is infinitely many times differentiable with respect to t on the interval $[a,b]$, then by (27)

$$E\left(\sup_{\substack{|t'-t''|<h \\ t',t''\in[a,b]}} |Z_+^{(n)}(t') - Z_+^{(n)}(t'')| \right) \le$$

$$\le h\cdot E\left\{ \left| \left(\frac{d}{dt} Z_+^{(n)}(t) \right)_{t=a} \right| + \int_a^b \left| \frac{d^2}{du^2} Z_+^{(n)}(u) \right| du \right\}.$$

For arbitrary $t \in [a,b]$ and for arbitrary natural k the random variables $\left\{ (X_j^+)^t \log^k (X_j^+) - E\,(X_j^+)^t \log^k (X_j^+) \right\}$, $n \ge 1$ are independent identically distributed with zero expectation and finite variance, by virtue of the Cauchy-Schwarz inequality and Lemma 1 we get independently of n the inequality

$$E\left| \frac{d^k}{dt^k} Z_+^{(n)}(t) \right| \le \left\{ E\left(\frac{d^k}{dt^k} Z_+^{(n)}(t) \right)^2 \right\}^{1/2} \le$$

$$\le \left\{ \left[E\,(X^+)^t \log^k (X_+) - E\,(X^+)^t \log^k (X^+) \right]^2 \right\}^{1/2} \le$$

$$\le \left\{ E\,(X^+)^{2t} \log^{2k} (X^+) \right\}^{1/2} \le \left\{ E\left[(X^+)^{2a} + (X^+)^{2b} \right] \log^{2k} (X^+) \right\}^{1/2}$$

takes place, hence it follows that independently of n we have

$$hE\left(\sup_{\substack{|t'-t''|<h \\ t',t''\in[a,b]}} |Z_+^{(n)}(t') - Z_+^{(n)}(t'')| \right) \le$$

$$\leq h\left\{E\left[(X^+)^{2a}+(X^+)^{2b}\right]\left[\log^2(X^+)+\log^4(X^+)\right]\right\}^{1/2}, \qquad \text{as } h\to 0.$$

The case $[a,b]\subset(\gamma/2,\infty)$

Let us use the notations

$$X_{*n}=X^+D_n^{-1},\quad X_{jn}=X_j^+D_n^{-1},\quad n\geq 1,\quad j=1,2,\dots,n$$

(here D_n is defined by formula (4)), and let us form the process $Z_+^{(n)}(t)$ as sum of two processes

$$Z_+^{(n)}(t)=\xi_1^{(n)}(t)+\xi_2^{(n)}(t),$$

where

$$\xi_1^{(n)}(t)=\xi_{10}^{(n)}(t)=\sum_{j=1}^n\left(X_{jn}^tI(X_{jn}<1)-EX_{jn}^tI(X_{jn}<1)\right),$$

$$\xi_2^{(n)}(t)=\xi_{20}^{(n)}(t)=\sum_{j=1}^nX_{jn}^tI(X_{jn}\geq 1).$$

Define

$$\xi_{1k}^{(n)}(t)=$$

$$=\frac{d^k}{dt^k}\xi_1^{(n)}(t)=\sum_{j=1}^n\left(X_{jn}^t\log^kX_{jn}I(X_{jn}<1)-EX_{jn}^t\log^kX_{jn}I(X_{jn}<1)\right),$$

$$\xi_{2k}^{(n)}(t)=\frac{d^k}{dt^k}\xi_2^{(n)}(t)=\sum_{j=1}^nX_{jn}^t\log^kX_{jn}I(X_{jn}\geq 1).$$

It is evident that the process $\xi_{2k}^{(n)}(t)$ is monotone increasing and for all $\gamma/2<$ $<t<\infty$, $k\geq 0$ the inequality $\xi_{2k}^{(n)}(t)\leq\xi_2^{(n)}(k+t)$ trivially holds. Hence, by (27) follows

$$\sup_{\substack{|t'-t''|<h\\ t',t''\in[a,b]}}|Z_+^{(n)}(t')-Z_+^{(n)}(t'')|\leq$$

$$\leq h \cdot \left(|\xi_{11}^{(n)}(a)| + \int_a^b |\xi_{12}^{(n)}(t)| dt \right) + h \cdot \left(|\xi_{21}^{(n)}(a)| + \int_a^b |\xi_{22}^{(n)}(t)| dt \right) \leq$$

$$\leq h \cdot \left(|\xi_{11}^{(n)}(a)| + \int_a^b |\xi_{12}^{(n)}(t)| dt \right) + h \cdot \left(|\xi_{21}^{(n)}(a)| + (b-a) |\xi_{22}^{(n)}(b)| \right) \leq$$

$$\leq h \cdot \left(|\xi_{11}^{(n)}(a)| + \int_a^b |\xi_{12}^{(n)}(t)| dt \right) + h \cdot [(1 + (b-a)) \xi_2^{(n)}(2+b)].$$

In order to prove relation (25) it is sufficient to establish the next two statements:

a) for arbitrary fixed natural k there exists a constant K, for which uniformly in n and $t \in [a,b]$

$$E \left| \xi_{1k}^{(n)}(t) \right| \leq K;$$

b) for arbitrary $\varepsilon > 0$ the relation

$$\lim_{h \to 0} \sup_n P \left(h \cdot [(1 + (b-a)) \xi_2^{(n)}(2+b)] > \varepsilon \right) = 0$$

holds.

Proof of statement a). Let us estimate the value $E|\xi_{1k}^{(n)}(t)|$. Since the sequence of random variables

$$\left(X_{jn}^t \log^k X_{jn} I(X_{jn} < 1) - E X_{jn}^t \log^k X_{jn} I(X_{jn} < 1) \right),$$

$$n = 1, 2, \ldots, \qquad k \geq 0$$

are independent identically distributed with zero expectation and finite variance, by the Cauchy-Schwarz inequality we can write

$$\left(E|\xi_{1k}^{(n)}(t)| \right)^2 \leq E \left(\xi_{1k}^{(n)}(t) \right)^2 =$$

$$= nE \left(X_{*n}^t \log^k X_{*n} I(X_{*n} < 1) - E X_{*n}^t \log^k X_{*n} I(X_{*n} < 1) \right)^2 \leq$$

$$\leq nE \left(X_{*n}^{2t} \log^{2k} X_{*n} I(X_{*n} < 1) \right)^2 = -n \int_0^{D_n} \left(\frac{y}{D_n} \right)^{2t} \log^{2k} \left(\frac{y}{D_n} \right) d(1 - F(y))$$

$$\leq -n \int_0^{D_n} \left(\frac{y}{D_n}\right)^{2a} \log^{2k}\left(\frac{y}{D_n}\right) d(1-F(y)) = -n \int_0^1 y^{2a} \log^{2k} y d(1-F(D_n y)).$$

The function $y^{2a}\log^{2k} y$ is differentiable on the interval $(0,\infty)$ and tends to zero as $y \longrightarrow 0$, with integration by part we get

$$\left(E|\xi_{1k}^{(n)}(t)|\right)^2 \leq n \int_0^1 (1-F(D_n y)) dy^{2a} \log^{2k} y =$$

$$= \int_0^1 n(1-F(D_n y)) y^{2a-1} \log^{2k-1} 2(a \cdot \log y + k) dy.$$

In consequence of condition (1) and the definition D_n it is valid $D_n L^{-1}(D_n) \sim$ $\sim n$, as $n \longrightarrow \infty$, hence for some constant K' we can establish

$$\left(E|\xi_{1k}^{(n)}(t)|\right)^2 \leq 2C_+ \int_0^1 n(1-F(D_n y)) y^{2a-1} |\log^{2k-1} y(a \cdot \log y + k)| dy \leq$$

$$\leq 2 \int_0^1 K' \frac{L(D_n y)}{L(D_n)} y^{2a-\gamma-1} |\log^{2k-1} y(a \cdot \log y + k)| dy.$$

By the inequality $2a > \gamma$ and [17] (Theorem 2.7) the last integral asymptotically (as $n \longrightarrow \infty$) is equal to

$$2 \int_0^1 K' y^{2a-\gamma-1} |\log^{k-1} y(a \cdot \log y + k)| dy,$$

so there exists a constant $K = K(k)$, for which uniformly in n

$$E|\xi_{1k}^{(n)}(t)| \leq K. \tag{28}$$

Proof of statement b). The fact that the sequence of random variables $\xi_2^{(n)}(2+b)$, $n = 1, 2, \ldots$ has non-degenerate (infinitely divisible) limit distribution as $n \longrightarrow \infty$, follows from condition (1), it is an immediate consequence of the classical limit theorem (see [11], Theorem 1 of par.25). Hence it is known (see [9], Chapter VIII.2) that in this case the sequence of random variables $\xi_2^{(n)}(2+b)$ is bounded in probability, which is established in statement b). This completes the proof of Theorem 5.

4. Example

Let $w(t)$ be a continuous function on the interval $[a,b] \subset (\gamma/2, \infty)$. Assume that condition (1) holds with the parameter $C_+ > 0$. In consequence of Theorem 5 the sequences of random variables

$$\int_a^b w(t) Z_+^{(n)}(t)dt, \qquad n = 1, 2, \ldots$$

converge in distribution to the random variable represented in the form

$$\int_a^b w(t) Z_+(t)dt.$$

It is not difficult to prove that its characteristic function has the form

$$E\exp\left\{ i\lambda \int_a^b w(t) Z_+(t)dt \right\} =$$

$$= \exp\left\{ \gamma C_+ \int_0^\infty \left[\exp\left(i\lambda \int_a^b w(t)u^t dt \right) - 1 - i\lambda I(0 < u < 1) \int_a^b w(t)u^t dt \right] \frac{du}{u^{\gamma+1}} \right\}.$$

Let us see now the special case, when $w(t) > 0$ holds for all $t \in [a,b]$. Denote by $q^{-1}(y)$ the inverse function of $q(y) = \int_a^b w(t)y^t dt$ then we can write the characteristic function in the form

$$E\exp\left\{ i\lambda \int_a^b w(t) Z_+(t)dt \right\} = \exp\left\{ \gamma C_+ \int_0^\infty [\exp\{i\lambda v\} - 1 - i\lambda v] dQ(v) \right\},$$

where

$$Q(v) = -\left(q^{-1}(v)\right)^{-\gamma}, \qquad v > 0.$$

It is clear that this distribution is infinitely divisible, but it is not stable.

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References

[1] **Blanc-Lapierre A. and Fortet R.**, *Theory of random functions, vol.1,* Gordon and Breach, 1967.

[2] **Chow Y.S. and Teicher H.**, *Probability theory,* Springer, New York-Heidelberg-Berlin, 1978.

[3] **Csörgő M., Csörgő S., Horváth L. and Mason M.D.**, Sup-norm convergence of the empirical process indexed by functions and applications, *Probab. Math. Stat.,* **7** (1) (1986), 13-26.

[4] **Csörgő M. and Horváth L.**, Asymptotic representations of self-normalised sums, *Probab. Math. Stat.,* **9** (1) (1988), 15-24.

[5] **Csörgő S., Haeusler E. and Mason D.M.**, A probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables, *Adv. Appl. Math.,* **9** (3) (1988), 259-333.

[6] **Csörgő S.**, Notes on extreme and self-normalised sums from the domain of attraction of a stable law, *J. London Math. Soc.,* **39** (2) (1989), 369-384.

[7] **Csörgő S. and Viharos L.**, Asymptotic distributions for vectors of power sums, *Third Hungarian Colloquium on Limit Theorems in Probability and Statistics, Pécs, 1989,* eds. I. Berkes, E. Csáki and P. Révész, North-Holland, Amsterdam, 137-156.

[8] **Dynkin E.B. and Mandelbaum A.**, Symmetric statistics. Poisson point processes and multiple Wiener integrals, *Ann. Statist.,* **11** (3) (1983), 739-745.

[9] **Feller W.**, *An introduction to probability theory and its applications, Volume II.,* John Wiley and Sons, New York-London-Sydney-Toronto, 1971.

[10] **Gihman I.I. and Skorohod A.V.**, *The theory of stochastic processes I.,* Springer, Berlin-Heidelberg-New York, 1974.

[11] **Gnedenko B.V. and Kolmogorov A.N.**, *Limit distribution for sums of independent random variables,* Addison-Wesley, Cambridge, 1954.

[12] **Halász G. and Székely G.J.**, On the elementary symmetric polynomials of independent random variables, *Acta Math. Acad. Sci. Hungar.,* **28** (1976), 397-400.

[13] **Королюк В.С. и Боровских Ю.В.**, *Теория U-статистик* (Korolyuk V.S. and Borovskih Yu.V., The theory of U-statistics), Наукова думка, Киев, 1989.

[14] **Logan B.F., Mallows C.L., Rice S.O. and Shepp L.A.**, Limit distribution of self-normalized sums, *Ann. Probab.*, **9** (1973), 624-632.

[15] **Major P.**, Asymptotic distributions for weighted U-statistics, Universität Bielefeld, Preprint-073 (1992), 1-22.

[16] **Móri T.F.**, On a Hoeffding-type problem, *Lecture Notes in Statistics*, **8** (1981), 174-181.

[17] **Seneta E.**, *Regularly varying functions, Lecture Notes in Mathematics* **508**, Springer, Berlin-Heidelberg-New York, 1976.

[18] **Szeidl L.**, On the limit distribution of random symmetric functions, *Theory Probab. Appl.*, **31** (3) (1986), 590-603.

[19] **Szeidl L.**, Limit distribution of random symmetric polynomials, *Theory Probab. Appl.*, **33** (2) (1988), 248-259.

[20] **Szeidl L.**, On the limit distributions of multi-sample von Mises functionals with variable polynomial kernels, *J. Soviet Math.*, **59** (4) (1992), 997-998.

[21] **Сейдль Л.**, О пределе эмпирического степенного процесса, *Теория вероятн. и ее применен.*, (Szeidl L., On the limit of empirical power process) (в печати).

[22] **Szeidl L.**, Über die Grenzverteilung von empirischen Potenzprozessen und zufälligen symmetrischen Polynomen, *Vortragauszüge, Jahrestagung der DMV, Bielefeld,* 1991, 288.

[23] **Székely G.J.**, A limit theorem for elementary symmetric polynomials of independent random variables, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, **59** (1982), 355-360.

[24] **Terdik Gy.**, Validation and verification of the error data by stochastic bilinear systems: a step toward nonlinear modeling, *Verification and Validitation of Human-Machine Systems,* eds. J.A.Wise, V.D.Hopkin and P.Stage, Springer, 1994.

[25] **Tucker H.G.**, On the asymptotic independence of the partial sums of positive and negative parts of independent random variables, *Adv. Appl. Probab.*, **3** (1971), 404-425.

[26] **Yanushkevichiene O.L.**, On the convergence of random symmetric polynomials, *Lecture Notes in Mathematics,* **1546** (1993), 184-188.

[27] **Zolotarev V.M.**, Limit theorems for random symmetric polynomials, *Theory Probab. Appl.*, **30** (1985), 598-599.

[28] **Золотарев В.М.,** О случайных симметрических полиномах, в сб. *"Вероятностные распределения и математицеская статистика"*, (Zolotarev V.M., On random symmetric polynomials) Ташкент, Фан, 1986, 170-188.

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