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ON FAST FOURIER ALGORITHMS

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Dedicated to Professor Karl-Heinz Indlekofer on his fiftieth birthday

Abstract. In this paper we synthesize the various known *FFT* methods. We show for a number of orthonormed systems, that the *Fourier coefficients*, similar to the *Fast Fourier Transform*, can be computed from a more general algorithm. These orthonormed systems can be represented as *product systems* of other systems which have a certain measurability property. *Fourier synthesis* with respect to such systems can be made by a fast algorithm. The various known *FFT* methods with respect to the one- and multidimensional *trigonometric* and *Walsh* systems are special cases of the method presented here. Moreover fast algorithms for certain biorthogonal expansion are investigated.

1. Introduction

Sequences of numbers and functions are usually indexed by natural numbers or integers. In many questions connected with Walsh series and dyadic harmonic analysis or with the *FFT algorithm* it is convenient to use the set of *p - adic intervals* as an index set [15]. That is the set of intervals of the form

$$\mathcal{J}^p := \left\{ \left[\frac{k}{p^n}, \frac{k+1}{p^n} \right) : k = 0, 1, ..., p^n - 1, n \in \mathbf{N} \right\},$$

where $\mathbf{N}$ is the set of non-negative integers and $p \in \mathbf{N}^! := \mathbf{N} \setminus \{0, 1\}$. In the case $p = 2$ the elements of $\mathcal{J}^p$ are called dyadic intervals and the set in question is simply denoted by $\mathcal{J}$. The length of an interval $I \in \mathcal{J}^p$ is denoted by $|I|$ and the p-adic subintervals of I of the length $|I|/p$ are denoted by $I_0, I_1, \ldots, I_{p-1}$. Let I^+ denote the interval in $\mathcal{J}^p$ with the length $p|I|$ containing I and set

$$\mathcal{J}_n^p := \{I \in \mathcal{J}^p : |I| = p^{-n}\} \qquad (n \in \mathbf{N}).$$

Obviously $(J_j)^+ = J$. To simplify notation, set $J_j^+ := (J^+)_j$ for $J \in \mathcal{J}^p$ and $j \in \mathbf{P}$, where

$$\mathbf{P} := \{0, 1, \ldots, p-1\} \quad (p \in \mathbf{N}^!). \tag{1.0}$$

The set of complex-valued sequences indexed by $\mathcal{J}^p$ is denoted by $\mathcal{S}^p$:

$$\mathcal{S}^p := \{\mathbf{a} = (a_I, I \in \mathcal{J}^p) : a_I \in \mathbf{C} \qquad (I \in \mathcal{J}^p)\}.$$

There exist two types of first order difference equations for sequences in $\mathcal{S}^p$. The *decreasing type (D-type)*, can be given by a sequence of functions $F_I : \mathbf{C} \to \mathbf{C} \quad (I \in \mathcal{J}^p)$ as follows:

$$a_I = F_I(a_{I^+}) \qquad (I \in \mathcal{J}^p, |I| \le p^{-1}).$$

Starting from the value $a_{[0,1)}$ the sequence $(a_I, I \in \mathcal{J}^p)$ is uniquely defined by these recurrence formulas.

To get *increasing difference equations* let us be given a sequence of functions $G_I : \mathbf{C}^p \to \mathbf{C} \quad (I \in \mathcal{J}^p)$. The system of equations

$$a_I = G_I(a_{I_0}, a_{I_1}, \ldots, a_{I_{p-1}}) \qquad (I \in \mathcal{J}^p)$$

is called a *first order difference equation of inceasing type (I-type)*. Let $N \in \mathbf{N}$ be a fixed number. Starting from the initial values a_I $(I \in \mathcal{J}_N^p)$ we obtain the a_I's for $|I| < p^{-N}$ in $(p^N - 1)/(p-1)$ steps.

In algorithms connected with *FFT* we use special double sequences indexed by p-adic intervals of the form $(a_{IJ}, (I, J) \in \mathcal{J}^p \times \mathcal{J}^p)$ and recurrence formulas increasing in I and decreasing in J. More exactly, for a fixed $N \in \mathbf{N}$ let us be given a sequence of functions

$$F_{IJ} : \mathbf{C}^p \to \mathbf{C} \qquad (I \in \mathcal{J}_{N-n}^p, J \in \mathcal{J}_n^p, n = 1, 2, \ldots, N).$$

The system of equations

(1.1)
$$a_{IJ} = F_{IJ}(a_{I_0 J^+}, a_{I_1 J^+}, \ldots, a_{I_{p-1} J^+}) \quad (I \in \mathcal{J}_{N-n}^p, J \in \mathcal{J}_n^p, n = 1, 2, \ldots, N)$$

is called a *first order partial difference equation of ID-type*.

Starting from the initial values

$$a_{I[0,1)} \qquad (I \in \mathcal{J}_N^p) \tag{1.2}$$

and applying the recurrence formulas (1.1) we get the values

$$a_{[0,1)J} \qquad (J \in \mathcal{J}_N^p) \tag{1.3}$$

in Np^N steps. If the number of necessary operations (the cost) of calculating the functions F_{IJ} is the same for every I, J (say α) then computing all the values

$$a_{IJ} \quad (I \in \mathcal{J}_{N-n}^p, \ \ J \in \mathcal{J}_n^p, \ \ n = 1, 2, \ldots, N)$$

requires $Np^N\alpha$ operations.

It is convenient to identify the set $\mathbf{P}^m$ with the set of numbers $\mathcal{P}_m := \{0, 1, \ldots, p^m - 1\}$ via one of the next two maps. For $k \in \mathcal{P}_m$ with the p-adic expansion

$$k = k_0 + k_1 p + \ldots + k_{m-1}p^{m-1} \qquad (k_j \in \mathbf{P})$$

define the map $\pi_m : \mathcal{P}_m \to \mathbf{P}^m$ by

$$\pi_m(k) := (k_0, k_1, \ldots, k_{m-1})$$

and the inverse of π_m by

$$\pi_{-m}(k) := (k_{m-1}, \ldots, k_1, k_0).$$

We identify the index sets in (1.1) with the set $\mathbf{P}^N$. To this end for each fixed $N \in \mathbf{N} \setminus \{0\}$ define the map

$$\theta : \bigcup_{n=0}^{N} \mathcal{J}_{N-n}^p \times \mathcal{J}_n^p \to \mathbf{N}^N$$

as follows: For $I = \left[\frac{k}{p^{N-n}}, \frac{k+1}{p^{N-n}}\right)$ and $J = \left[\frac{\ell}{p^n}, \frac{\ell+1}{p^n}\right)$ let

$$\theta(I, J) := (\pi_{-n}(\ell), \pi_{N-n}(k)) = (\ell_{n-1}, \ldots, \ell_1, \ell_0, k_0, k_1, \ldots, k_{N-n-1}), \tag{1.4}$$

and for $n = N$ and $n = 0$ set

$$\theta([0,1), J) := (\pi_{-N}(\ell)) \quad , \quad \theta(I, [0,1)) := (\pi_N(k)). \tag{1.5}$$

It is easy to check for all $I \in \mathcal{J}^p_{N-n}$, $J \in \mathcal{J}^p_n$ and $n = 1, 2, \dots, N$ that

$$\theta(I_j, J^+) = \theta(I, J_j^+). \tag{1.6}$$

If $J \in \mathcal{J}^n$ then $J = J_j^+$ for some $j \in \mathbf{P}$ and so via the map θ and (1.6) we can store the values of $a_{IJ} = a_{IJ_j^+}$ obtained from (1.1) into the place of the values $a_{I_jJ^+}$ which are used on the right-hand side of (1.1). To store the initial values (1.2) we need p^N places. Using this method of storage one needs no more space than that alloted for the initial values.

We show that a large class of function systems can be constructed with algorithms of Fourier analysis and Fourier synthesis of the above type [1]-[9].

2. Product systems

For the description of the abovementioned algorithm, we use the notion of *conditional expectation* [10], [15].

We begin with the definition and some inportant properties of conditional expectation. Let $(X, \mathcal{A}, \mu)$ be a probability measure space and $\mathcal{B}$ a sub-σ - algebra of $\mathcal{A}$. For an arbitrary function $f \in L^1 := L^1(X, \mathcal{A}, \mu)$ we denote by $E(f|\mathcal{B})$ the *conditional expectation of* f *with respect to* $\mathcal{B}$. The conditional expectation can be characterized by the following two properties: $E(f|\mathcal{B})$ is integrable and $\mathcal{B}$-measurable, i.e.

$$E(f|\mathcal{B}) \in L^1(X, \mathcal{B}, \mu) \tag{2.1}$$

and for every $\mathcal{B}$ -measurable set B

$$\int_B f \, d\mu = \int_B E(f|\mathcal{B}) \, d\mu \tag{2.2}$$

holds.

By the *Radon-Nikodym theorem*, the function $E(f|\mathcal{B})$ satisfying the above two properties exists and is unique up to a set of zero μ-measure, for any integrable function f. It is known that the *conditional expectation operator* $L^1 \ni f \to E(f|\mathcal{B}) \in L^1(X, \mathcal{B}, \mu)$ is bounded and linear, moreover for any pair of functions $\lambda \in L^1(X, \mathcal{B}, \mu)$ and $f \in L^1$ such that $\lambda f \in L^1$, we have

$$E(\lambda f|\mathcal{B}) = \lambda E(f|\mathcal{B}). \tag{2.3}$$

Furthermore, for arbitrary σ -algebras $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ the following equalities hold:

$$(2.4) \qquad E(E(f|\mathcal{C})|\mathcal{B}) = E(E(f|\mathcal{B})|\mathcal{C}) = E(f|\mathcal{C}) \quad (f \in L^1).$$

On the basis of (2.1) and (2.2) it is obvious that if $\mathcal{B} := \{X, \emptyset\}$ is the trivial σ -algebra, then

$$(2.5) \qquad E(f|\mathcal{B}) = \int_X f \, d\mu \qquad (\mathcal{B} := \{X, \emptyset\}),$$

i.e. the conditional expectation is a ***generalization of the notion of integral*** [10]. In the other special case if $\mathcal{B} = \mathcal{A}$ then $E(f|\mathcal{B}) = f$.

Conditional expectation can be used to generalize the concept of orthogonality, biorthogonality, and Fourier coefficients [12]. A system $\Phi := \{\phi_n : n \in \in \mathcal{N}\}$ with function in $L^2 := L^2(X, \mathcal{A}, \mu)$ is called a $\mathcal{B}$ - *orthonormal system* if for every $n, m \in \mathcal{N}, m \neq n$

$$E(\phi_m \overline{\phi}_n|\mathcal{B}) = \delta_{mn},$$

where $\delta_{mn} = 0$ if $m \neq n$ and $\delta_{mn} = 1$ if $m = n$.

More generally, the systems Φ and $\Upsilon := \{v_n, n \in \mathcal{N}\}$ in L^2 are called ***$\mathcal{B}$-biorthogonal*** if for every $m, n \in \mathcal{N}$

$$(2.6) \qquad E(\phi_m \overline{v}_n|\mathcal{B}) = \delta_{mn}.$$

For any $f \in L^2$ the $\mathcal{B}$-measurable functions

$$(2.7) \qquad E(f\overline{\phi}_n|\mathcal{B}) \quad (n \in \mathcal{N})$$

are called the $\mathcal{B}$ *-Fourier coefficients* of f with respect to the system Φ. If $\mathcal{B} = \{X, \emptyset\}$ then the above definitions reduce to that of usual orthogonality, biorthogonality and Fourier coefficients. Furthermore, by (2.4) and (2.5)

$$\int_X \phi_m \overline{v}_n \, d\mu = E(E(\phi_m \overline{v}_n|\mathcal{B})|\{X, \emptyset\}) = \delta_{mn},$$

i.e. $\mathcal{B}$-biorthogonal systems are biorthogonal in the usual sense.

A generalization of ***Bessel's identity*** and the ***minimum property*** of Fourier coefficients hold for $\mathcal{B}$ -orthogonal systems (see [12]).

In order to give the general form of *FFT* algorithms we consider systems which are ***product systems*** of collections with certain measurability properties (with P and M properties) [13]-[15]. Let us be given a finite collection of function systems

(2.8) $$\Phi_n := \{\phi_n^j : j \in \mathbf{P}\} \quad (n \in \mathbf{N}, n < N),$$

where the complex-valued functions ϕ_n^j are defined on X and $N = 1, 2, \ldots$.

We define the *product system*

$$\Psi := \{\psi_m : m \in \mathcal{P}_N\}$$

as follows: for any natural numbers $m \in \mathcal{P}_N$ represented in the form

$$m = m_0 + m_1 p + \ldots + m_{N-1} p^{N-1} \quad (m_n \in \mathbf{P})$$

let

(P) $$\psi_m := \prod_{n=0}^{N-1} \phi_n^{m_n}.$$

If the system Ψ is of the form (P) we say that Ψ has the *P-property*.

From (P) it follows that

(2.9) $$\Psi = \Phi_0 \Phi_1 \ldots \Phi_{N-1},$$

where the product of the function sets on the right hand side is defined in the usual way.

To define the mentioned *measurability property* we fix a monotone increasing sequence of σ-algebras

$$\mathcal{A}_0 := \{X, \emptyset\} \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_{N-1} \subseteq \mathcal{A}_N = \mathcal{A}$$

and denote the conditional expectation operator with respect to $\mathcal{A}_n$ by E_n. We shall say that a collection of systems $(\Phi_n, n = 0, 1, \ldots, N-1)$ has the *M_1-property* if the functions in Φ_n are in L^2 and are $\mathcal{A}_{n+1}$- measurable, i.e.

(M_1) $$\Phi_n \subseteq L^2(X, \mathcal{A}_{n+1}, \mu) \quad (n = 0, 1, \ldots, N-1).$$

If for $n = 0, 1, \ldots, N-1$ the systems Φ_n and Υ_n are $\mathcal{A}_n$-biorthogonal, i.e.

(M_2) $$E_n(\phi_n^k \overline{v}_n^\ell) = \delta_{k\ell} \quad (k, \ell \in \mathbf{P}, n = 0, 1, \ldots, N-1)$$

then we shall say that the systems in question have the *M_2-property*. The next theorem gives a possibility to construct biorthogonal systems.

Theorem 1. *The product system of systems having properties M_1 and M_2 is a biorthogonal system.*

Proof. Denote by Ψ and Γ the product system of the systems Φ_n and Υ_n $(n = 0, 1, \dots, N-1)$, respectively. If $k \neq \ell$ then there exists an index $j \in \{0, 1, \dots, N-1\}$ such that $k_j \neq \ell_j$ and $k_i = \ell_i$ if $j < i < N$. Write

$$\psi_k \gamma_\ell = \prod_{i=0}^{j-1} \psi_i^{k_i} \overline{\gamma}_i^{\ell_i} \cdot \psi_j^{k_j} \overline{\gamma}_j^{\ell_j} \cdot \prod_{i=j+1}^{N-1} \psi_i^{k_i} \overline{\gamma}_i^{\ell_i} =: \alpha \, \beta \, \gamma$$

and observe by (M_1) that α is $\mathcal{A}_j$-measurable , β is $\mathcal{A}_{j+1}$-measurable and by (M_2) that $E_j(\beta) = 0$. On the basis of (M_1), (M_2), (2.3) and (2.4) we get

$$E_{j+1}(\gamma) = E_{j+1}(\phi_{j+1}^{k_{j+1}} \gamma_{j+1}^{k_{j+1}} E_{j+2}(\dots E_{N-1}(\phi_{N-1}^{k_{N-1}} \gamma_{N-1}^{k_{N-1}}) \dots)) = 1.$$

Consequently using (2.3) and (2.4) we get

$$\int_X \psi_k \gamma_\ell \, d\mu = E_0(\alpha E_j(\beta E_{j+1}(\gamma))) = 0.$$

A similar argument shows that the integral in question is 1 if $k = \ell$.

It is important that *properties P and M_i $(i = 1, 2)$ are invariant with respect the Kronecker product.* Indeed, denote by

$$(f \times g)(x, y) := f(x) \, g(y) \quad (x \in X, y \in Y)$$

the Kronecker product of the function $f : X \to \mathbf{C}$ and $g : Y \to \mathbf{C}$. For the two collections of functions $F := \{f_n : n \in \mathcal{N}\}$ and $G := \{g_m : m \in \mathcal{M}\}$ defined on X and Y, respectively, we define the Kronecker product by

$$F \times G := \{f_n \times g_m : n \in \mathcal{N}, m \in \mathcal{M}\}.$$

Suppose that for $j = 1, 2$ the systems Ψ^j are the product systems of the collections Φ_n^j $(n = 0, 1, \dots, N-1)$, i.e. via (2.9) we have

$$\Psi^j = \Phi_0^j \, \Phi_1^j \, \dots \Phi_{N-1}^j \quad (j = 1, 2).$$

It is easy to check that

$$\Psi^1 \times \Psi^2 = (\Phi_0^1 \times \Phi_0^2) \, (\Phi_1^1 \times \Phi_1^2) \dots (\Phi_{N-1}^1 \times \Phi_{N-1}^2)$$

and consequently $\Psi^1 \times \Psi^2$ is the product system of the systems $\Phi_n^1 \times \Phi_n^2$ $(n = 0, 1, \ldots, N-1)$, i.e. *the Kronecker product of systems having the P-property, has also this property.*

To prove the invariance of the M-properties, suppose that they are satisfied for the systems Φ_n^j $(n = 0, 1, \ldots, N-1)$ with the σ-algebras $\mathcal{A}_n^j$ $(n = = 0, 1, \ldots, N-1)$. Then it is easy to see that for the collection $\Phi_n^1 \times \Phi_n^2$ $(n = = 0, 1, \ldots, N-1)$, $M1$ and $M2$ is satisfied with respect the collection of σ-algebras

$$\mathcal{A}_0^1 \times \mathcal{A}_0^2 \subseteq \mathcal{A}_1^1 \times \mathcal{A}_1^2 \subseteq \ldots \subseteq \mathcal{A}_{N-1}^1 \times \mathcal{A}_{N-1}^2.$$

3. Fourier analysis and synthesis

The Fourier coefficients of $f \in L^2$ with respect to the product system Ψ can be written in the form

(3.1)

$$\hat{f}(m) := \int_X f\overline{\psi}_m \, d\mu = E_0(\overline{\phi}_0^{m_0} E_1(\overline{\phi}_1^{m_1} \ldots E_{N-2}(\overline{\phi}_{N-2}^{m_{N-2}} E_{N-1}(\overline{\phi}_{N-1}^{m_{N-1}} f)) \ldots)).$$

That is a consequence of the properties P and M_1 of the systems Φ_n's (compare (2.3) and (2.4)) [13].

Using (3.1), we can compute the Fourier coefficients of the functions $f \in L^2$ with respect to Ψ in the following way: first calculate the $\mathcal{A}_{N-1}$ - Fourier coefficients of the $\mathcal{A}_n$-measurable f with respect to Φ_{N-1} (there are p of them), then the $\mathcal{A}_{N-2}$-Fourier coffecients of each $\mathcal{A}_{N-1}$-measurable function just obtained. Writing down the $\mathcal{A}_{N-3}$-Fourier coefficients of the ($\mathcal{A}_{N-2}$-measurable, p^2) functions with respect to the system Φ_{N-3}, obviously we get p^3 functions, each being $\mathcal{A}_{N-2}$-measurable. Continuing this procedure, we obtain the p^N Fourier coefficients $\hat{f}(m)$ at the N-th step.

For every $n = 1, 2, \ldots, N$ the set of functions

$$\Phi_{N-n}\,\Phi_{N-n+1} \cdots \Phi_{N-1}$$

has p^n elements which can be indexed by the intervals in $\mathcal{J}_n^p$. It is convenient to do this in the following way: For each interval $J = \left[\frac{m}{p^n}, \frac{m+1}{p^n}\right) \in \mathcal{J}_n^p$ we set

$$\psi_J := \phi_{N-n}^{m_{n-1}} \phi_{N-n+1}^{m_{n-2}} \cdots \phi_{N-1}^{m_0},$$

where the m_j's are the reverse digits of m, i.e. $m = m_{n-1} + m_{n-2}p + \ldots +$ $+m_0 p^{n-1}$. The M_1-property, (2.3) and (2.4) imply

$$E_{N-n}(f\overline{\psi}_J) = E_{N-n}(\overline{\phi}_{N-n}^{m_{n-1}}\, E_{N-n+1}(f\overline{\psi}_{J^+}))$$

and

$$\hat{f}(m) = E_0(f\overline{\psi}_J) \tag{3.2}$$

$$\left(J = \left[\frac{m}{p^N}, \frac{m+1}{p^N}\right),\ m = m_{N-1} + m_{N-2}p + \ldots + m_0 p^{N-1}\right).$$

Obviously $m_{n-1} = \ell$ if and only if $J_\ell^+ = J$ and consequently

$$E_{N-n}(f\overline{\psi}_J) = E_{N-n}(\overline{\phi}_{N-n}^{\ell}\, E_{N-n+1}(f\overline{\psi}_{J^+})) \tag{3.3}$$

$$(J \in \mathcal{J}_n^p,\ J_\ell^+ = J,\ n = 1, 2, \ldots, N).$$

To get an algorithm of the form (1.1) we choose a special kind of σ-algebras. We suppose that the σ-algebras in question *are atomic and every atom of* $\mathcal{A}_n$ *is the union of* p *atoms, belonging to* $\mathcal{A}_{n+1}$. In this case the Boolean structure of the collection of σ-algebras is the same as that of p-adic intevals, which will be called *p-atomic* structure. Therefore it is convenient to index the atoms of $\mathcal{A}_n$ by the intervals of $\mathcal{J}_n^p$. Using this identification, the conditional expectation of an $\mathcal{A}_{n+1}$-measurable function g with respect to $\mathcal{A}_n$ can be expressed in the form

$$(E_n(g))_J = \sum_{I \in \mathcal{J}_{n+1}^p, I \subset J} g_I\, \rho_I \quad (J \in \mathcal{A}_n^p), \tag{3.4}$$

where g_I is the value of g on the atom indexed by I and ρ_I is the μ-measure of this atom in question.

According to (1.1) it is convenient to denote the value of the function $E_{N-n}(f\overline{\psi}_J)$ at the atom I by $\hat{f}_{IJ}$. Then (3.4) implies that (3.3) is equivalent to

$$\hat{f}_{IJ} = \sum_{j=0}^{p-1} \hat{f}_{I_j J^+}\, \overline{\phi}_{N-n}^{\ell}(I_j)\rho_{I_j} \quad (I \in \mathcal{J}_{N-n}^p, J \in \mathcal{J}_n^p, n = 1, 2, \ldots, N) \tag{3.5}$$

where $\overline{\phi}_{N-n}^{\ell}(I_j)$ denotes the (common) value of the function ϕ_{N-n}^{ℓ} at the points of the atom indexed bi I_j. Thus we get

Theorem 2. *Suppose that the collection of σ-algebras $\mathcal{A}_n, n = 0,1,\ldots,N$ has a p-atomic structure and for the systems Φ_n, $n = 0,1,\ldots,N-1$ (M_1) satisfies. Then the Fourier coefficients with respect to the product system Ψ can be obtained as the solution of the following initial value problem with respect to the linear partial difference equation of ID-type:*

$$\hat{f}_{IJ} = \sum_{j=0}^{p-1} \alpha_{IJ}^j \, \hat{f}_{I_j J^+} \quad (I \in \mathcal{J}_{N-n}^p, \ J \in \mathcal{J}_n^p, \ n = 1,2,\ldots,N)$$

where

$$\alpha_{IJ}^j := \overline{\phi}_{N-n}^{\ell}(I_j)\, \rho_{I_j} \quad (J = J_\ell^+).$$

Starting with the function values

$$\hat{f}_{I[0,1)} := f_I \quad (I \in \mathcal{J}_N^p)$$

we get the Fourier coefficients in (3.4):

$$\hat{f}(m) = \hat{f}_{J[0,1)} \quad (J \in \mathcal{J}_N^p).$$

To compute the sum

$$S = \sum_{k \in \mathcal{P}_N} a_k \psi_k, \tag{3.6}$$

i.e. make *Fourier synthesis*, we introduce the notations

$$S_{I[0,1)} := a_i \quad \left(I = \left[\frac{i}{p^N}, \frac{i+1}{p^N}\right), \ i \in \mathcal{P}_N \right). \tag{3.7}$$

Obviously the $S_{I[0,1)}$'s are $\mathcal{A}_0$-measurable. Using recursion, we define

$$S_I := \sum_{j=0}^{p-1} S_{I_j} \phi_{n-1}^j \quad (I \in \mathcal{J}_{N-n}^p, \ n = 1,2,\ldots,N).$$

If the collection of systems Φ_n $(n = 0,1,\ldots,N-1)$ has the M_1-property then $I \in \mathcal{A}_{N-n}^p$ implies the $\mathcal{A}_n$-measurability of S_I for $n = 1,2,\ldots,N$. If we denote the value of S_I at the atom $J \in \mathcal{J}_n^p$ by S_{IJ} then we get the following recurrence of ID-type:

$$S_{IJ} = \sum_{j=0}^{p-1} S_{I_j J^+} \phi_{n-1}^j(J) \quad (I \in \mathcal{A}_{N-n}^p, \ \ J \in \mathcal{A}_n^p, \ \ n = 1,2,\ldots,N). \tag{3.8}$$

It is easy to see that $S_{[0,1)J}$ is the value of S at the atom corresponding to $J \in \mathcal{J}_N^p$.

Theorem 3. *Suppose that for the systems* $\Phi_n, n = 0, 1, \ldots, N-1$ (M_1) *is satisfied. Then the partial sum* S *in* (3.6) *with respect to the product system* Ψ *can be obtained as the solution of the an initial value problem with respect the linear partial difference equation of ID-type* (3.8). *The initial values are given by* (3.7) *and the value of* S *at the atom corresponding to* J *is* $S_{[0,1)J}$.

4. Examples

By a suitable choice of the system Φ_n and the σ-algebras, we can obtain every known *FFT method.* In what follows, we present some of them.

4.1. Independent systems. Suppose that the systems $\Phi_n \subset L^2$ $(n = 0, 1, \ldots, N-1)$ are independent and orthonormed in the usual sense in L^2. Let $\mathcal{A}_0 = \{X, \emptyset\}$ and let $\mathcal{A}_n$ denote the σ-algebra generated by the systems Φ_j $(j = 0, 1, \ldots, n-1)$. Since in this case

$$E_n(\phi_n^k \overline{\phi}_n^\ell) = \int\limits_X \phi_n^k \overline{\phi}_n^\ell \, d\mu = \delta_{k\ell},$$

(M_1) and (M_2) are satisfied and (1.1) can be applied to the product system, provided that $\mathcal{A}$ has a p-atomic structure [13] .

Let us examine the following special cases.

4.2. Walsh-Paley system. Let $X := [0, 1)$ and let r_n denote the n-th *Rademacher function* $(n = 0, 1, \ldots)$. If $\mathcal{A}_n$ is the σ -algebra generated by the dyadic intervals of $\mathcal{J}_n$, then conditons (M_1) and (M_2) are satisfied for the systems $\Phi_n := \{1, r_n\}$. The product system of these systems is *the Walsh system in Paley's ordering* [11]. Thus algorithm (1.1) can be applied for the Walsh-Paley system and it is known as the *Fast Walsh Transform* algorithm [1].

4.3. Walsh system. The *original Walsh system* (see [15]) can be obtained as product system of the systems

$$\Phi_0 := \{1, r_0\}, \quad \Phi_n := \{1, r_n r_{n-1}\} \quad (n = 1, 2, \ldots).$$

Obviously, if $\mathcal{A}_n$ is the same as before, than (M_1) and (M_2) is satisfied and the method (1.1) is usable.

4.4. Walsh-Kaczmarz system, Hadamard transform. Fix $N \in \mathbf{N}^{!}$ and denote $\Phi_n := \{1, r_{N-n-1}\}$ $(n = 0, 1, \ldots, N-1)$. These systems are independent, the σ-algebra $\mathcal{A}_N$ is generated by the intervals in $\mathcal{J}_N$ and we get a special case of the example in 4.1. The Fourier transform with respect to this system is the same as the *Hadamard transform* [15].

4.5. Multiple Walsh systems. On the basis of the previous sections the multiple Walsh systems (corresponding to the Paley, the original or the Kaczmarz ordering) are product systems for which (M_1) and (M_2) are satisfied. Consequently, (1.1) can be used in multiple Walsh analysis and synthesis.

4.6. Chrestenson systems. It is easy to check that *Chrestenson systems* are special cases of systems considered in 4.1 with σ-algebras, having p-adic structure [15].

4.7. The trigonometric systems. Let

$$e_n(x) := \exp(2\pi i n x) \quad (x \in [0,1),\ n \in \mathbf{N},\ i := \sqrt{-1})$$

denote the complex trigonometric system. This system is orthonormed with respect to the Lebesgue measure on $[0, 1)$. The discrete trigonometric system can be obtained as the restriction of the functions e_n $(n \in \mathcal{P}_N)$ to the set

$$X := \left\{ \frac{k}{2^N} : k \in \mathcal{P}_N \right\}$$

and it is an orthonormed system with respect the measure μ defined by $\mu(\{x\}) := 2^{-N}$ $(x \in X)$. The product system of the systems

$$\Phi_n := \{1, e_{2^{N-n-1}}\} \quad (n = 0, 1, \ldots, N-1) \tag{4.1}$$

is an arrangement of the system $(e_n, n \in \mathcal{P}_N)$.

Indeed, let $\overline{m} := m_{N-1} + m_{N-2}2 + \ldots + m_0 2^{N-1}$ denote the reverse of $m := m_0 + m_1 2 + \ldots + m_{N-1}2^{N-1} \in \mathcal{P}_N$. Since

$$e_{2^{N-1}}^{m_0} e_{2^{N-2}}^{m_1} \ldots e_{2^0}^{m_{N-1}} = e^{m_0 2^{N-1} + m_1 2^{N-2} + \ldots + m_{N-1}} = e_{\overline{m}},$$

therefore the product system of the systems in (4.1) is $(e_{\overline{m}}, m \in \mathcal{P}_N)$.

We show that (M_1) and (M_2) are satisfied. For $n = 0, 1, \ldots, N$ let $\mathcal{A}_n$ denote the σ-algebra generated by the atoms A_n^k $(k \in \mathcal{P}_n)$, where

$$A_n^k := \left\{ \frac{k}{2^N} + \frac{\ell}{2^{N-n}} : \ \ell \in \mathcal{P}_{N-n} \right\}.$$

It is easy to see that

$$A_{n+1}^{k} \cup A_{n+1}^{2^n+k} = A_n^k \quad (k \in \mathcal{P}_n,\ n = 0, 1, \dots, N-1),$$

i.e. the collection $(\mathcal{A}_n, n = 0, 1, \dots, N)$ has a dyadic atomic structure. Since $e_{2^{N-n-1}}$ is constant on the atoms of $\mathcal{A}_{n+1}$ the condition (M_1) is satisfied. Moreover, from

$$e_{2^{N-n-1}}(A_{n+1}^{k}) = -e_{2^{N-n-1}}(A_{n+1}^{2^n+k}) \quad (n = 0, 1, \dots, N-1)$$

it follows that (M_2) holds. Thus algorithm (1.1) can be applied for the discrete trigonometric system and it is known as the *FFT algorithm of Cooley and Tukey* [5].

4.8. The multiple trigonometric systems. On the basis of Section 2 the multiple trigonometric system (corresponding to the reverse ordering) is a product system for which (M_1) and (M_2) are satisfied. Consequently (1.1) gives a fast algorithm for trigonometric multiple Fourier analysis and synthesis.

Other examples can be found in [13] and [14].

References

[1] **Beauchamp K.G.**, *Walsh Functions and Their Application*, Academic Press, London-New York-San Francisco, 1975.

[2] **Bloomfield P.**, *Fourier Analysis of Time Series: an Introduction*, John Wiley, London-New York-Sydney-Toronto, 1976.

[3] **Cooley J.W., Lewis P.D. and Welch P.D.**, Historical notes on the fast Fourier transform, *Proc. IEEE,* **55** (1967), 1675-1677.

[4] **Cooley J.W., Lewis P.D. and Welch P.D.**, The fast Fourier transform and its application to time series analysis, *Stat.Methods for Digital Computers Vol.III*, 1977, 377-423.

[5] **Cooley J.W. and Tukey J.W.**, An algorithm for the machine calculation of complex Fourier series, *Math.Comp.*, **19** (1965), 297-301.

[6] **Gentleman W.M. and Sande G.**, Fast Fourier transforms for fun and profit, *Proc. AFIPS Fall Joint Computer Conference,* **29** (1958), 361-372.

[7] **Good I.J.**, The interaction algorithm and practical Fourier analysis, *J.Roy.Stat.Soc., Ser.B*, **20** (1958), 361-372.

[8] **Good I.J.**, The relationship between two fast Fourier transforms, *IEEE Trans.Comput.,* **C-20** (1971), 310-317.

[9] **Henrici P.**, Einige Anwendungen der schnellen Fouriertransformation, *Moderne Methoden der Numerischen Mathematik,* Birkhäuser Verlag, 1976, 111-124.

[10] **Neveu J.**, *Discrete-parameter martingales,* North-Holland, 1975.

[11] **Paley R.E.A.C.**, A remarkable system of orthogonal functions, *Proc. London Math. Soc.,* **34** (1932), 241-279.

[12] **Schipp F.**, On a generalization of the concept of orthogonality, *Acta Sci.Math.,* **37** (1975), 279-285.

[13] **Schipp F.**, Fast Fourier transform and conditional expectation, *Coll.Math. Soc.J.Bolyai 22. Numerical Methods,* Keszthely, Hungary, 1977, 565-576.

[14] **Schipp F.**, Fast algorithm to compute Fourier coefficients with respect to spherical function, *Math. Models in Physics and Chemistry and Numerical Methods of Their Realization*, Teubner-Texte Band 61, 1982, 79-87.

[15] **Schipp F., Wade W.R., Simon P. and Pál J.**, *Walsh series: an introduction to dyadic harmonic analysis,* Akadémiai Kiadó, Budapest, 1990.

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