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CHARACTERIZATION OF PAIRS OF ADDITIVE FUNCTIONS WITH VALUES IN COMPACT ABELIAN GROUPS

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Dedicated to Professor Karl-Heinz Indlekofer on the occasion of his fiftieth birthday

Abstract. In this paper we give a complete characterization of those pairs of additive functions with values in compact Abelian groups which satisfy some regularity properties. Our result improves some results of [6] and [8] concerning this problem.

1. Introduction

Let G be an additively written, metrically compact Abelian topological group, $\mathbb{N}$ be the set of all positive integers. A function $f : \mathbb{N} \to G$ will be called completely additive, if

$$f(nm) = f(n) + f(m)$$

holds for all n, $m \in \mathbb{N}$. Let $\mathcal{A}_G^*$ denote the class of all completely additive functions $f : \mathbb{N} \to G$.

Let $A > 0$ and $B \neq 0$ be fixed integers. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $D[A, B]$ if for any convergent subsequence $\{x_{\nu_n}\}_{n=1}^\infty$ the sequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ has a limit, too. We say that it has property $E[A, B]$ if for any convergent subsequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ the sequence $\{x_{\nu_n}\}_{n=1}^\infty$ is convergent. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $\Delta[A, B]$ if the sequence $\{x_{A\nu+B} - x_\nu\}_{\nu=1}^\infty$ has a limit.

Let $\mathcal{A}_G^*(D[A, B])$, $\mathcal{A}_G^*(E[A, B])$ and $\mathcal{A}_G^*(\Delta[A, B])$ be the classes of those $f \in \mathcal{A}_G^*$ for which $\{x_\nu = f(\nu)\}_{\nu=1}^\infty$ is of property $D[A, B]$, $E[A, B]$ and $\Delta[A, B]$, respectively.

It is obvious that

$$\mathcal{A}_G^*(\Delta[A,B]) \subseteq \mathcal{A}_G^*(D[A,B]), \quad \mathcal{A}_G^*(\Delta[A,B]) \subseteq \mathcal{A}_G^*(E[A,B]).$$

Z.Daróczy and I.Kátai proved in [1] that

$$\mathcal{A}_G^*(\Delta[1,1]) = \mathcal{A}_G^*(D[1,1]),$$

and by using the result due to E.Wirsing, in [2] they deduced the following assertion: If $f \in \mathcal{A}_G^*(D[1,1])$, then there exists a continuous homomorphism $\Phi : R_* \to G$, where R_* denotes the multiplicative group of the positive reals, such that $f(n) = \Phi(n)$ for all $n \in I\!N$.

For the case $A = 2$ and $B = -1$ the complete characterization of $\mathcal{A}_G^*(D[2,-1])$ and $\mathcal{A}_G^*(\Delta[2,-1])$ has been given by Z.Daróczy and I.Kátai [4], [5]. The basic idea of their proof is to reduce the condition $f \in \mathcal{A}_G^*(D[2,-1])$ to the relation

$$f(2n+1) - f(2n-1) \to 0 \quad \text{as} \quad n \to \infty$$

and apply the modification of Wirsing's theorem.

In [7] and [8] we have given a complete determination of $\mathcal{A}_G^*(E[A,B])$, $\mathcal{A}_G^*(D[A,B])$ and $\mathcal{A}_G^*(\Delta[A,B])$. We proved the following results:

Theorem A. ([7]) *For any fixed integers $A > 0$ and $B \neq 0$, we have*

$$\mathcal{A}_G^*(E[A,B]) = \mathcal{A}_G^*(\Delta[A,B]).$$

If

$$f \in \mathcal{A}_G^*(E[A,B]) = \mathcal{A}_G^*(\Delta[A,B]),$$

then there exists a continuous homomorphism $\Phi : R_* \to G$, R_* denotes the multiplicative group of the positive reals, such that f is a restriction of Φ on the set $I\!N$, i.e.

$$f(n) = \Phi(n)$$

for all $n \in I\!N$.

Conversely, let $\Phi : R_* \to G$ be arbitrary continuous homomorphism. Then the function

$$f(n) := \Phi(n) \quad (\text{for all} \quad n \in I\!N)$$

belongs to $\mathcal{A}_G^*(E[A,B]) = \mathcal{A}_G^*(\Delta[A,B])$.

Theorem B. ([8]) *Let $A > 0$ and $B \neq 0$ be fixed integers for which $(A, B) = 1$. If $f \in \mathcal{A}_G^*(D[A, B])$, then there are $U \in \mathcal{A}_G^*$ and a continuous homomorphism $\Phi : R_* \to G$, where R_* denotes the multiplicative group of the positive reals, such that*

(I) $f(n) = \Phi(n) + U(n)$ *for all* $n \in I\!N$.

(II) $U(n + A) = U(n)$ *for all* $n \in I\!N, (n, A) = 1$.

(III) If X_1, Γ *denote the set of all limit points of* $\{\Phi(n) \mid n \in I\!N\}$ *and* $\{U(n) \mid n \in I\!N\}$, *respectively, then*

$$X_1 \cap \Gamma = \{0\}$$

and Γ *is the smallest closed group generated by*

$$\{U(m) \mid 1 \leq m \leq A, \ (m, A) = 1\} \ \cup \ \{U(p) \mid p \ \text{ is prime}, \quad p|A\}.$$

Conversely, let $\Phi : R_* \to G$ be an arbitrary continuous homomorphism, X_1 be the smallest compact subgroup generated by $\{\Phi(n) \mid n \in I\!N\}$. Let $U \in \mathcal{A}_G^*$ be so chosen that $U(n + A) = U(n)$ for all $n \in I\!N$, $(n, A) = 1$ and the smallest closed group Γ generated by $U(I\!N)$ has the property $X_1 \cap \Gamma = \{0\}$. Then the function

$$f(n) := \Phi(n) + U(n) \quad (n \in I\!N)$$

belongs to $\mathcal{A}_G^*(D[A, B])$.

Let G_1 and G_2 be additively written, metrically compact Abelian topological groups. Let $A > 0$ and $B \neq 0$ be integers. In the following we shall denote by $\mathcal{A}_{G_1,G_2}^*(D[A, B])$ the class of all completely additive functions $\varphi_1 \in \mathcal{A}_{G_1}^*$ and $\varphi_2 \in \mathcal{A}_{G_2}^*$, which have the following property:

If

$$\lim_{\nu \to \infty} \varphi_1(n_\nu) = g \qquad (g \in G_1),$$

then the following limit exists:

$$\lim_{\nu \to \infty} \varphi_2(An_\nu + B) = h \qquad (h \in G_2).$$

In this case we shall write $(\varphi_1, \varphi_2) \in \mathcal{A}_{G_1,G_2}^*(D[A, B])$. It is obvious that in the case $G_1 = G_2 = G$ we have

$$(\varphi, \varphi) \in \mathcal{A}_{G,G}^*(D[A, B]) \quad \text{is equivalent to} \quad \varphi \in \mathcal{A}_G^*(D[A, B]).$$

In [3], [6] Z.Daróczy and I.Kátai considered some problems concerning characterizations of pairs of additive functions with regularity properties. For example, under the condition that G_1 is a T_0 group, the class $\mathcal{A}^*_{G_1,G_2}(D[1,-1])$ was completely characterized in [6].

We can extend Theorem B as follows:

Theorem. *Let G_1 and G_2 be additively written, metrically compact Abelian topological groups and let $A > 0$ and $B \neq 0$ be fixed integers for which $(A, B) = 1$. If*

$$(\varphi_1, \varphi_2) \in \mathcal{A}^*_{G_1,G_2}(D[A,B]) \quad \text{and} \quad (\varphi_2, \varphi_1) \in \mathcal{A}^*_{G_2,G_1}(D[A,B]),$$

then for each $i \in \{1,2\}$ there are $U_i \in \mathcal{A}^*_{G_i}$ and a continuous homomorphism $\Phi_i : R_* \to G_i$, R_* denotes the multiplicative group of the positive reals, such that

(a) $\varphi_i(n) = \Phi_i(n) + U_i(n)$ *for all* $n \in I\!N$.

(b) $U_i(n + A) = U_i(n)$ *for all* $n \in I\!N, (n, A) = 1$.

(c) If X_i^*, Γ_i denote the set of all limit points of $\{\Phi_i(n) \mid n \in I\!N\}$ and $\{U_i(n) \mid n \in I\!N\}$, respectively, then

$$X_i^* \cap \Gamma_i = \{0\}$$

and Γ_i is the smallest closed group generated by

$$\{U_i(m) \mid 1 \leq m \leq A, (m, A) = 1\} \cup \{U_i(p) \mid p \text{ is prime}, \quad p|A\}.$$

(d) There exists a topological isomorphism $\Psi : X_1^* \to X_2^*$ such that $\Psi\Phi_1 = \Phi_2$.

Conversely, for each $i \in \{1,2\}$ let $\Phi_i : R_* \to G_i$ be an arbitrary continuous homomorphism, X_i^* be the smallest compact subgroup generated by $\{\Phi_i(n) \mid n \in I\!N\}$. Let $U_i \in \mathcal{A}^*_{G_i}$ be so chosen that $U_i(n+A) = U_i(n)$ for all $n \in$ $\in I\!N$, $(n, A) = 1$ and the smallest closed group Γ_i generated by $U_i(I\!N)$ has the property $X_i^* \cap \Gamma_i = \{0\}$, furthermore let $\Psi : X_1^* \to X_2^*$ be a topological isomorphism such that $\Psi\Phi_1 = \Phi_2$. Then

$$\varphi_i(n) := \Phi_i(n) + U_i(n) \quad (n \in I\!N)$$

satisfy

$$(\varphi_1, \varphi_2) \in \mathcal{A}^*_{G_1,G_2}(D[A,B]) \quad \text{and} \quad (\varphi_2, \varphi_1) \in \mathcal{A}^*_{G_2,G_1}(D[A,B]).$$

2. Proof of the theorem

In the following we assume that G_1 and G_2 are additively written, metrically compact Abelian topological groups. Let $A > 0$ and $B \neq 0$ be fixed integers for which $(A, B) = 1$. Assume that

$$(\varphi_1, \varphi_2) \in \mathcal{A}^*_{G_1,G_2}(D[A,B]) \quad \text{and} \quad (\varphi_2, \varphi_1) \in \mathcal{A}^*_{G_2,G_1}(D[A,B]).$$

It is easy to show that

$$\text{(1)} \qquad \begin{cases} \mathcal{A}^*_{G_1,G_2}(D[A,B]) \subseteq \mathcal{A}^*_{G_1,G_2}(D[A,1]) \\ \mathcal{A}^*_{G_2,G_1}(D[A,B]) \subseteq \mathcal{A}^*_{G_2,G_1}(D[A,1]). \end{cases}$$

For each $k \in I\!N$ we shall denote by $X_1^k = X_{\varphi_1}^k$ (resp. $X_2^k = X_{\varphi_2}^k$) the set of limit points of $\{\varphi_1(kn+1) \mid n \in I\!N\}$ (resp. $\{\varphi_2(kn+1) \mid n \in I\!N\}$), i.e. $g \in X_i^k$ if there exists a sequence

$$n_1 < \ldots < n_\nu < \ldots \quad (n_\nu \in I\!N),$$

for which

$$\varphi_i(kn_\nu + 1) \to g \quad \text{as} \quad \nu \to \infty.$$

By using Theorem (9.16) of [9], it can be proved as in [8] that for each $k \in I\!N$ and $i \in \{1,2\}$ the set X_i^k is a compact subgroup in G_i and $\varphi_i(kn+1) \in X_i^k$ for all $n \in I\!N$.

For each $i \in \{1,2\}$ let $X_i := X_i^1$ and $X_i^* := X_i^A$. Let $g \in X_1$ and $\varphi_1(n_\nu) \to g$ as $\nu \to \infty$. Then, by using (1), it follows that the sequence $\{\varphi_2(An_\nu + 1)\}_{\nu=1}^{\infty}$ is convergent. Let $\varphi_2(An_\nu + 1) \to g'$, $(g' \in X_2^*)$. It is easily seen that g' is determined by g, and so the correspondence

$$H_1 : g \to g' \quad (g \in X_1, g' \in X_2^*)$$

is a function. Similarly, if $h \in X_2$, $\varphi_2(m_\mu) \to h$ as $\mu \to \infty$, then the following limit exists:

$$\lim_{\mu \to \infty} \varphi_1(Am_\mu + 1) := h'.$$

The correspondence

$$H_2 : h \to h' \quad (h \in X_2, h' \in X_1^*)$$

is a function. The following assertion can be proved easily by using the same method as was used in [1].

Lemma 1. *The functions $H_1 : X_1 \to X_2^*$ and $H_2 : X_2 \to X_1^*$ are continuous, furthermore*

$$H_1(X_1) = X_2^* \quad \textit{and} \quad H_2(X_2) = X_1^*.$$

For each $g \in X_1$ we denote by $X_2(g) \subseteq X_2$ the set of accumulation points of $\{\varphi_2(n_\nu)\}_{\nu=1}^{\infty}$, while $\varphi_1(n_\nu) \to g$ as $\nu \to \infty$, i.e. $h \in X_2(g)$ if there exists a sequence

$$n_{\nu_1} < \ldots < n_{\nu_j} < \ldots \quad (j \in \mathbb{N})$$

for which

$$\varphi_2(n_{\nu_j}) \to h \quad \text{as} \quad j \to \infty.$$

Similarly, we define the set $X_1(h) \subseteq X_1$ for each $h \in X_2$ as the set of limit points of $\{\varphi_1(m_\nu)\}_{\nu=1}^{\infty}$, if $\varphi_2(m_\nu) \to h$.

The following lemma is a key of our proof.

Lemma 2. *We have*

$$(2) \qquad H_1(g + X_1(h) + \varphi_1(A)) + H_1(g) = H_1[g + H_2(h + H_1(g))]$$

and

$$(3) \qquad H_2(h + X_2(g) + \varphi_2(A)) + H_2(h) = H_2[h + H_1(g + H_2(h))]$$

for all $g \in X_1$ and $h \in X_2$, where

$$H_1(g + X_1(h) + \varphi_1(A)) := \{H_1(g + g' + \varphi_1(A)) \mid g' \in X_1(h)\},$$

$$H_2(h + X_1(g) + \varphi_2(A)) := \{H_2(h + h' + \varphi_2(A)) \mid h' \in X_2(g)\}.$$

Proof. Let $g \in X_1$ and $h \in X_2$ be arbitrary elements. Let

$$n_1 < \ldots < n_\nu < \ldots \quad \text{and} \quad m_1 < \ldots < m_\nu < \ldots \quad (n_\nu,\ m_\nu \in \mathbb{N})$$

be such sequences for which $\varphi_1(n_\nu) \to g$ and $\varphi_2(m_\nu) \to h$ as $\nu \to \infty$.

By applying the following relations

$$(A^2 n_\nu m_\nu + 1)(A n_\nu + 1) = A n_\nu [A m_\nu (A n_\nu + 1) + 1] + 1,$$

$$(A^2 m_\nu n_\nu + 1)(A m_\nu + 1) = A m_\nu [A_\nu (A m_\nu + 1) + 1] + 1$$

and using the definitions of H_1, H_2, $X_1(h)$ and $X_2(g)$ we get immediately that (2) and (3) hold. The proof of Lemma 2 is finished.

Lemma 3. *There are $g_0 \in X_1$ and $h_0 \in X_2$ such that*

$$H_1(g_0) = H_2(h_0) = 0 \quad \textit{and} \quad g_0 \in X_1(h_0).$$

Proof. Let S ($\subseteq X_1^*$) denote the set of all limit points of sequences $\{\varphi_1(An_\nu + 1)\}_{\nu=1}^{\infty}$ while $\varphi_2(An_\nu + 1) \to 0$ as $\nu \to \infty$. By using Theorem (9.16) of [9], it can be proved as in [8] that S is a closed semigroup in X_1^*, consequently S is a compact group. Therefore $0 \in S$, i.e. there is a sequence $\{N_\nu\}_{\nu=1}^{\infty}$ such that

$$\varphi_1(AN_{\nu'} + 1) \to 0, \qquad \varphi_2(AN_\nu + 1) \to 0 \quad \text{as} \quad \nu', \nu \to \infty$$

for some subsequence $\{\nu'\}$ of $\{\nu\}$. Let $\{\nu'''\} \subseteq \{\nu''\} \subseteq \{\nu'\}$ be suitable rarefactions of $\{\nu'\}$ such that

$$\varphi_1(N_{\nu'''}) \to g_0 \in X_1, \qquad \varphi_2(N_{\nu''}) \to h_0 \in X_2.$$

Then $H_1(g_0) = H_2(h_0) = 0$ and $g_0 \in X_1(h_0)$. This completes the proof of Lemma 3.

Lemma 4. *We have*

$$H_1(-\varphi_1(A)) = H_2(-\varphi_2(A)) = 0. \tag{4}$$

Proof. Let $i \in \{1, 2\}$ be a fixed integer and let

$$E(\varphi_i) := \{\varrho \in X_i \mid H_i(\varrho) = 0\}.$$

Since X_i^* is a group, therefore $0 \in X_i^*$. Thus, it follows from Lemma 1 that there is at least one ϱ for which $H_i(\varrho) = 0$. Then $E(\varphi_i) \neq \emptyset$.

First we note from (3) that

$$H_1(\varrho_1 + X_1(\varrho_2) + \varphi_1(A)) = 0 \quad \text{if} \quad H_1(\varrho_1) = H_2(\varrho_2) = 0. \tag{5}$$

Let $g_0 \in E(\varphi_1)$ and $h_0 \in E(\varphi_2)$ be the elements determined in Lemma 3. By using (5) and induction on k, one can deduce that

$$H_1(kg_0 + (k-1)\varphi_1(A)) = 0 \tag{6}$$

holds for all $k \in \mathbb{N}$, which with the method used in the proof of Lemma 4 in [8] implies that (6) also holds for all integers k. In the case when $k = 0$ we have $H_1(-\varphi_1(A)) = 0$.

In the same way, we also obtain that $H_2(-\varphi_2(A)) = 0$. Lemma 4 is proved.

Now we prove the Theorem.

We apply (3) with $g = -\varphi_1(A)$ and use Lemma 4, we have

$$H_1[X_1(h)] = H_1[H_2(h) - \varphi_1(A)] \quad \text{for all} \quad h \in X_2.$$

This with the definitions of H_1 and $X_1(h)$ shows that if $\varphi_2(m_\nu) \to h$, then

$$\varphi_2(Am_\nu + 1) \to H_1[H_2(h) - \varphi_1(A)],$$

i.e.

$$\varphi_2 \in \mathcal{A}^*_{G_2}(D[A,1]).$$

Similarly, we have

$$\varphi_1 \in \mathcal{A}^*_{G_1}(D[A,1]).$$

From Theorem B, for each $i \in \{1,2\}$ there is $U_i \in \mathcal{A}^*_{G_i}$ and a continuous homomorphism $\Phi_i : R_* \to G_i$ which satisfy the parts (a), (b) and (c) of our theorem. It is easy to show that the correspondence $\Phi_1(n) \leftrightarrow \Phi_2(n)$ ($n \in I\!N$) generates a topological isomorphism Ψ between X_1^* and X_2^* and $\Psi\Phi_1 = \Phi_2$.

The converse assertion is true as well. The proof of the theorem is finished.

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