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NUMBER SYSTEMS IN IMAGINARY QUADRATIC FIELDS

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Dedicated to Professor Karl-Heinz Indlekofer on his fiftieth birthday

1. Introduction

Let $Q(i\sqrt{D})$ be an imaginary quadratic extension of Q, I be the set of integers in $Q(i\sqrt{D})$. Let $\alpha \in I$, $\alpha \neq 0$, and $\alpha \neq$ unit. Let $F = \{f_0 = 0, f_1, \dots, f_{t-1}\}$, $t = |\alpha|^2$ be a complete residue system *mod* α.

Then, for each $\beta \in I$ there exists a unique $a_0 \in F$ and a unique $\beta_1 \in I$ such that

$$\beta = a_0 + \alpha\beta_1. \tag{1.1}$$

The function $J : I \to I$ is defined by $J(\beta) = \beta_1$. Observe that for $K = \max_{f \in F} |f|$ we have

$$|\beta_1| \leq \frac{K}{|\alpha|} + \frac{|\beta|}{|\alpha|}. \tag{1.2}$$

The inequality (1.2) implies that for every $\beta \in I$ the path, defined by iterating J:

$$\beta,\ \beta_1 = J(\beta),\ \beta_2 = j(\beta_1), \dots$$

is eventually periodic.

Some $\beta \in I$ is said to be periodic (with respect to this expansion) if there is some integer $k > 0$ for which $\beta = J^k(\beta)$ holds.

Let P be the set of periodic elements. The following assertions are obvious:

(1) $0 \in P$;

(2) (F, α) is a number-system (NS) if and only if P is singleton, $P = \{0\}$;

(3) If $\Pi \in P$, then

$$|\Pi| \leq \frac{K}{|\alpha| - 1}; \tag{1.3}$$

(4) If $\Pi \in P$, then $J(\Pi) \in P$. Let $G(P)$ be the directed graph defined by $\Pi \to J(\Pi)$ for every $\Pi \in P$. Then $G(P)$ is a disjoint union of circles;

(5) If $\alpha - 1$ is a unit in I, then no NS with base α exists, since for an arbitrary choice of F the elements $x_f = \overline{(1-\alpha)f}$, $f \in F$ are periodic with period 1.

To prove (1.3), assume that $\Pi \in P_1$, and $\Pi = \Pi_0 \to \Pi_1 \to \ldots \to \Pi_k (= \Pi_0)$. Assume that $\max_{\nu=0,\ldots,k-1} |\Pi_\nu| = |\Pi_0|$. Apply (1.2) with $\beta = \Pi_{k-1}$, $\beta_1 = \Pi_0$. (1.3) it follows immediately.

In our paper [1] written jointly with B. Kovács we determined all possible bases for which (F^*, α), $F^* = \{0, 1, \ldots, |a|^2 - 1\}$ is a NS.

The problem to determine all the possible coefficient systems F for which (F, A) is a NS, seems to be very hard. In the other hand, if F is given, to decide whether (F, α) is a NS or not, due to (1.3) is a simple task.

G. Steidl [2] proved that in the ring $Z[i]$ of the Gaussian integers for every $|\alpha| > 1$ except $\alpha = 2,\ 1+i,\ 1-i$ always exists a suitable coefficient set ζ_α by which (ζ_α, α) is a NS. She effectively constructed ζ_α. We shall extend her result to arbitrary imaginary quadratic fields.

2. Construction of the coefficient system

Lemma 1. *Let $e, b, c, a \in Z$ be arbitrary integers, $d = ae - bc$, S be the matrix*

$$S = \begin{bmatrix} e & -b \\ -c & a \end{bmatrix}.$$

Assume that $d \neq 0$. Then there exists a unique set $F = \{\underline{f}_\nu : \nu = 0, 1, \ldots, |d| - 1\}$ of integer vectorials in Z_2, such that

$$\begin{bmatrix} r_\nu \\ s_\nu \end{bmatrix} := S\underline{f_\nu}$$

satisfies the following conditions:

(1) $r_\nu, \quad s_\nu \in \left(-\frac{|d|}{2}, \frac{|d|}{2}\right]$;

(2) $r_\nu \equiv r_\mu \pmod d$, $\quad s_\nu \equiv s_\mu \pmod d$ *cannot hold simultaneously for* $\nu \neq \mu$.

Proof. This assertion is well known in number theory.

Remarks.

(1) If d is odd, then $F = -F$.

(2) If $b = c = 0$, then F is of simple shape. Let k, resp. l run over the integers satisfying $-\frac{|d|}{2} < ek \le \frac{|d|}{2}$, $-\frac{|d|}{2} < al \le \frac{|d|}{2}$, respectively. Then F is the collection of all possible vectorials $\begin{bmatrix} k \\ l \end{bmatrix}$.

(3) If $e = a = 0$, then F is of similar shape.

If $D + 1 \not\equiv 0\ (mod\ 4)$, then $\{1, i\sqrt{D}\}$, while for $D \not\equiv -1\ \ (mod\ 4)$ $\{1, \omega\}$ is an integral basis in I, where

$$\omega = \frac{1 + i\sqrt{D}}{2}.$$

Let

$$E = \frac{1 + D}{4}.$$

Let $D \not\equiv -1\ (mod\ 4)$, $\alpha = a + ib\sqrt{D}$, $d := \alpha\overline{\alpha} = a^2 + b^2 D$. We define $\zeta_\alpha := \{e = k + il\sqrt{D}\}$ to be those integers for which $r = \mathrm{Re}\,\overline{\alpha}e$ and $s = \frac{\mathrm{Im}\,\overline{\alpha}e}{\sqrt{D}}$ satisfy the conditions: $r, s \in \left(-\frac{d}{2}, \frac{d}{2}\right]$. Explicitly

$$r = ak + blD,$$

$$s = -bk + al.$$

From Lemma 1 we have that ζ_α is a complete residue system $mod\ d$.

Since $\overline{\alpha}e = r + is\sqrt{D}$,

$$d \mid e \mid^2 = r^2 + s^2 D \le \frac{d^2}{4}(1 + D),$$

whence

$$\mid e \mid \le \frac{\sqrt{d}}{2}\sqrt{1 + D}.$$

Consequently for $\Pi \in P$ we obtain that

$$\mid \Pi \mid \le \frac{1}{2}\,\frac{\sqrt{1 + D}}{1 - 1/\sqrt{d}}. \tag{2.1}$$

Let $D \equiv -1 \ (mod\ 4)$, $\alpha = a + b\omega$, $d = \alpha\overline{\alpha} = \left(a + \frac{b}{2}\right)^2 + \frac{b^2}{4}D$. We define $\zeta_\alpha = \{e = k + l\omega\}$ as the integers for which $\overline{\alpha}e = (a+b)k + b\,lE + (al - bk)\omega =$ $= r + s\omega$ satisfies the conditions $r, s \in \left(-\frac{d}{2}, \frac{d}{2}\right]$.

Since

$$r + s\omega = \left(r + \frac{s}{2}\right) + i\frac{s}{2}\sqrt{D},$$

we obtain that

$$|\, r + s\omega\,|^2 = \left(r + \frac{s}{2}\right)^2 + \frac{s^2}{4}D \leq \left(\frac{3}{4}d\right)^2 + \frac{d^2}{16}D = \frac{d^2}{16}(9 + D),$$

consequently $|\, e\,| \leq \frac{\sqrt{d}}{4}\sqrt{9 + D}$, whence by (1.3) we get that

$$|\, \Pi\,| \leq \frac{\sqrt{9 + D}}{4(1 - \frac{1}{\sqrt{d}})}. \tag{2.2}$$

3. Formulation of the theorem and simple cases

Theorem. *Let α be an arbitrary integer in an imaginary quadratic extension field $Q(i\sqrt{D})$, such that $|\, \alpha\,| > 1$ and $|\, 1 - \alpha\,| \neq 1$ holds. Then $(\mathcal{F}, \alpha)$ is a NS with a suitable coefficient set $\mathcal{F}$.*

Lemma 2. *If $\alpha \in Z$, $\alpha \neq -2, -1, 0, 1$, then (ζ_α, α) is a NS for every extension field $Q(i\sqrt{D})$.*

Proof. If $\alpha \in Z$, then $\alpha = a + 0 \cdot i\sqrt{D}$ or $\alpha = a + 0 \cdot \omega$, $\quad d = a^2$, $\quad \zeta_\alpha =$ $= \left\{\begin{bmatrix} k \\ l \end{bmatrix}\right\}$ for which l, $k \in \left(-\frac{|\,a\,|}{2}, \frac{|\,a\,|}{2}\right]$. Clearly we can expand each $v, u \in Z$ in a NS with base a and coefficient system $\left\{\nu \in \left(-\frac{|\,a\,|}{2}, \frac{|\,a\,|}{2}\right]\right\}$. If $u = \sum k_t a^t$, $\quad v = \sum l_t a^t$, then

$$\beta = u + iv\sqrt{D} = \sum (k_t + l_t D i) a^t,$$

$$\beta = u + v\omega = \sum (k_t + l_t\omega)a^t$$

are the corresponding expansions of the integers β in I.

Lemma 3. *If $\alpha = ib\sqrt{D}$ or $\alpha = b\omega$, then (ζ_α, α) is a NS, except the cases $b = \pm 1$ for $D = 1$ and 3.*

Proof. We can argue similarly as in the proof of Lemma 2. In the exceptional cases $|\alpha|^2 = 1$ $\zeta_\alpha = \{\underline{0}\}$.

Lemma 4. *Let $\Pi \in P$, $\Pi \neq 0$, $\Pi = p + iq\sqrt{D}$ or $\Pi = p + q\omega$, according to whether $D + 1 \not\equiv 0 \pmod 4$ or $D + 1 \equiv 0 \pmod 4$, where $p, q \in Z$. If $q \neq 0$, then*

$$\left(1 - \frac{1}{\sqrt{d}}\right)^2 \leq \frac{D+1}{4D} \quad \text{for} \quad D + 1 \not\equiv 0 \pmod 4, \tag{3.1}$$

and

$$\left(1 - \frac{1}{\sqrt{d}}\right)^2 \leq \frac{1}{4}\frac{D+9}{D+1} \quad \text{for} \quad D + 1 \equiv 0 \pmod 4. \tag{3.2}$$

Proof. From (2.1), (2.2) we have

$$p^2 + q^2 D \leq \frac{D+1}{4(1 - \frac{1}{\sqrt{d}})^2} \qquad (=: R_{D,d}) \tag{3.3}$$

$$\left(p + \frac{q}{2}\right)^2 + \frac{q^2}{4} D \leq \frac{9 + D}{16(1 - \frac{1}{\sqrt{d}})^2} \qquad (=: S_{D,d}) \tag{3.4}$$

whence (3.1), (3.2) immediately follow.

Lemma 5. *All the rational integers $e = k + 0i\sqrt{D}$ satisfying*

$$|k| < \frac{d}{2} \min\left(\frac{1}{|a|}, \frac{1}{|b|}\right)$$

belong to ζ_α if $D + 1 \not\equiv 0 \pmod 4$. All the rational integers $e = k + 0 \cdot \omega$ of the interval

$$|e| \leq \frac{d}{2} \min\left(\frac{1}{|a+b|}, \frac{1}{|b|}\right)$$

belong to ζ_α if $D + 1 \equiv 0 \pmod 4$.

Consequently if $p \in P$, then

$$\frac{d^2}{4} \cdot \frac{1}{\max\{a^2, b^2\}} \le R_{D,d} \tag{3.5}$$

for $D+1 \not\equiv 0$ (mod 4), and

$$\frac{d^2}{4} \cdot \frac{1}{\max\{(a+b)^2, b^2\}} \le S_{D,d} \tag{3.6}$$

for $D+1 \equiv 0$ (mod 4).

Proof. The assertions are obvious consequences of the definition of ζ_α.

4. Proof of the theorem for $D+1 \not\equiv 0(mod\ 4)$

We assume that $D \ge 2$. The case $D = 1$ is completely solved in [2].

Assume first that there is a real $0 \ne \Pi \in P$ for some $\alpha = a + ib\sqrt{D}$. If $1 \le |a| \le |b|$, then from (3.5),

$$\sqrt{d}(\sqrt{d}-1) \le \sqrt{(1+D)b^2} = \sqrt{\frac{(1+D)b^2 \cdot a^2}{|a|^2}} \le \frac{(1+D)b^2 + a^2}{2|a|} =$$

$$= \frac{d}{2|a|} + \frac{b^2}{2|a|} \le \frac{d}{2|a|} + \frac{d}{2|a|D}.$$

$1 \le \frac{1}{\sqrt{d}} + \frac{1}{2|a|} + \frac{1}{2|a| \cdot D}$, which cannot occur if $D \ne 2$.

If $D = 2$, then from (3.5), and from $3b^2 \le 1,5d$ we deduce that $d^2\left(1 - \frac{1}{\sqrt{d}}\right)^2 \le 1,5d$, whence $d \le 4$ follows. Since $d = 4$ implies that either $b = 0$ or $a = 0$, and these cases were treated in Lemmas 2 and 3, we can consider only the case $d = 3$, $|a| = |b| = 1$. We shall treat these cases later.

If $|a| > |b|$, then from (3.5),

$$\sqrt{d}(\sqrt{d-1}) \le \sqrt{(1+D)a^2} = \sqrt{\frac{b^2 + b^2 D}{b^2} \cdot |a|^2} \le$$

$$\leq \frac{1}{2\mid b\mid}\{b^2+d\} = \frac{d}{2\mid b\mid}+\frac{\mid b\mid}{2},$$

whence

$$(2\mid b\mid -1)d - 2\mid b\mid \sqrt{d} - \mid b\mid^2 \leq 0. \tag{4.1}$$

This inequality cannot be true for $d > X_b^2$, where

$$X_b := \frac{\mid b\mid}{2\mid b\mid -1}(1+\sqrt{2\mid b\mid}).$$

Since $x_1 = 1+\sqrt{2}$, (4.1) could be held only for $\mid a\mid=\mid b\mid = 1$, $d=2$. For $\mid b\mid \geq 2$ we have $d \geq b^2(1+D) \geq 3b^2$, and

$$3b^2 > \frac{b^2(1+\sqrt{2\mid b\mid})^2}{(2\mid b\mid -1)^2} = \frac{b^2}{(\sqrt{2\mid b\mid}-1)^2},$$

thus (4.1) cannot be true.

We proved the following

Lemma 6. *Let* $\alpha = a + ib\sqrt{D}$, $\mid \alpha \mid > 1$, $D \geq 2$, $D+1 \not\equiv$ $\not\equiv 0 \pmod 4$, $a \neq 0$, $b \neq 0$. *Then P does not contain real* $\Pi \neq 0$ *except perhaps the cases* $D = 2$, $\mid a\mid=\mid b\mid = 1$.

Furthermore, computing $R_{5,6}$ and $R_{2,7}$ we obtain that (3.1) is not satisfied if $d \geq D+1$ and $D \geq 5$ and for $D=2$, if $d \geq 7$.

To finish the proof we have to consider only the cases $\mid a\mid=\mid b\mid = 1$; $\mid a\mid = 2$, $\mid b\mid = 1$, $D = 2$. If d is odd, then $\zeta_\alpha = -\zeta_\alpha$, furthermore $\zeta_{-\alpha} = \zeta_\alpha$, $\zeta_{\overline{\alpha}} = \overline{\zeta_\alpha}$, thus it is enough to consider one of $\alpha, \overline{\alpha}, -\alpha, -\overline{\alpha}$ in these cases.

Let $\alpha = 1 + i\sqrt{2}$. Then $\zeta_\alpha = \{-1,0,1\}$. Observing that $R_{2,3} < 4$, we get that for $\Pi = p + iq\sqrt{2}$ we have $p^2+2q^2 \leq 3$. Thus $\mid p\mid \leq 1$, $\mid q\mid \leq 1$. $q=0$ cannot occur since then $\Pi = p \in \zeta_\alpha$, and $J(\zeta_\alpha) \to 0$. Thus $\Pi \in \{\pm i\sqrt{2},\ \pm 1 \pm i\sqrt{2}\}$. But $J^2(\alpha) = J(1) = 0$, $J^2(-\alpha) = J(-1) = 0$, furthermore $i\sqrt{2} = -1+$ $+1\cdot\alpha$, $-i\sqrt{2} = 1 + (-1)\alpha$, $\overline{\alpha} = -1 - i\sqrt{2}\alpha$, $-\overline{\alpha} = 1 + i\sqrt{2}\alpha$, and so all the candidates for P have finite expansions.

Let $\alpha = 1 + 2i\sqrt{2}$. Then $d = 6$,

$$\zeta_\alpha = \{0,1,-1,i\sqrt{D},\ -1+i\sqrt{D},\ 1-i\sqrt{D}\}$$

and $R_{2,6} < 3$, whence $p^2 < 3$, $q = 0$ should follow, thus $\Pi = p \in \{-1,0,1,\} \subseteq$ $\subseteq \zeta_\alpha$, so this is a NS as well.

5. Proof of the theorem for $D+1 \equiv 0\ (mod\ 4)$

In the whole section we shall assume that for $\alpha = a + b\omega$ the conditions $b \geq 1$, $a \neq 0$, $a + b \neq 0$ hold. If $(\mathcal{F},\ \alpha)$ is a NS, then $(\overline{\mathcal{F}},\ \overline{\alpha})$ is a NS as well. Since $\overline{\alpha} = (a+b) - b\omega$, and in Lemmas 2, 3 the cases $a = 0$; $b = 0$, consequently $a + b = 0$ were treated it is enough to prove the theorem under the above conditions.

For short let $\mathcal{L}$ be the set

$$\left(-\frac{d}{2},\ \frac{d}{2}\right],\quad d = \left(a + \frac{b}{2}\right)^2 + \frac{b^2 D}{4} = a^2 + ab + b^2 E.$$

Assume that $d > 1$.

Lemma 7. *Every rational integer k, $|k| \leq \frac{|a|}{2}$ belongs to $\zeta\alpha$.*

Proof. We should prove that $k(a+b) \in \mathcal{L}$, $-bk \in \mathcal{L}$ holds for all k, $|k| \leq \frac{|a|}{2}$. If $|a+b| \geq b$, then $(a+b)a > 0$, consequently

$$\frac{|a+b||a|}{2} = \frac{a(a+b)}{2} < \frac{1}{2}\left(a^2 + ab + \frac{b^2 D}{4}\right) < \frac{d}{2}.$$

If $|a+b| < b$, then $a < 0$ and

$$\frac{|a|b}{2} < \frac{d}{2} = \frac{1}{2}\{a^2 + ab + b^2 E\}$$

is equivalent to $|a|b < \frac{1}{2}(a^2 + b^2 E)$, which clearly holds, since $E \geq 1$, $a \neq -b$.

Lemma 8. *Excluding the integers $\alpha = -1 + 2\omega$, $1 + \omega$ in the case $D = 3$, and $\alpha = 1 + \omega$ for $D = 7$, for the others the expansion $(\zeta\alpha,\ \alpha)$ either has a nonreal periodic element, or it is a NS.*

Proof. Assume in contrary that $P \subseteq \mathbb{Z}$ and there is a nonzero $p \in P$.

Let first $b = 1$. If $p_1 = J(p)$, then there is an $e \in \zeta\alpha$ such that $p = e + dp$, consequently $p\overline{\alpha} = e\overline{\alpha} + \alpha p_1$, $e\overline{\alpha} = r + s\omega$, $r, s \in \mathcal{L}$. Thus $(a+1)p = r + dp_1$, $-p = s$, whence $p_1 = \frac{(a+1)p}{d} - \frac{r}{d}$. Hence $|p_1| \leq \frac{|a+1|}{2} + \frac{1}{2}$. Since $p_1 \notin \zeta\alpha$, from Lemma 7 we obtain that

$$\frac{|a|+1}{2} \leq |p_1| \leq \frac{|a+1|+1}{2}. \tag{5.1}$$

(5.1) fails for $a < 0$. Let $a > 0$. Then $|p_1| = \frac{a+1}{2}$ or $\frac{a+2}{2}$ according to the parity of a. So we have that $|p| = \frac{a+1}{2}$ or $\frac{a+2}{2}$ for every $p \in P \setminus \{0\}$. Then either $J(p) = p$ or $J(p) = -p$, $J^2(p) = p$. In the first case $-r = (d-(a+1))p_1 = (a^2+E-1)p_1$, thus $\frac{d}{2} \geq (a^2+E-1)\frac{a+1}{2}$. This cannot hold with the exceptions $E = 1$ and 2, $a = 1$.

In the second case ($J(p) = -p$)) we conclude that $\frac{d}{2} \geq |d+(a+1)||p_1|$ which is impossible for $p_1 \neq 0$.

Let $b \geq 2$. If $p_1 = J(p)$, then $(a+b)p = r + dp_1$, $-bp = s$ hold with some $r, s \in \mathcal{L}$. Thus $p_1 = \frac{r+s}{d} - \frac{a}{d}p$, whence

$$|p_1| \leq 1 + \frac{|a|}{d}|p| = 1 + \frac{|a||s|}{db} \leq 1 + \frac{|a|}{2b}. \tag{5.2}$$

Since $|p_1| \geq \frac{|a|+1}{2}$, and $\frac{|a|}{2b} + 1 < \frac{|a|+1}{2}$, if $b > 2$, or if $b = 2$, and $|a| > 2$, we should consider the cases $b = 2$, $|a| = 1$, 2. If $|a| = 2$, then $|p_1| \geq 2$, and (5.2) cannot hold. If $|a| = 1$, then from (5.2) $P \subseteq \{0, 1, -1\}$. We shall prove finally that $1, -1 \in \zeta\alpha$. This holds if $|a+b| < \frac{d}{2}$ and $b < \frac{d}{2}$ are satisfied. If $a = 1$, then $1 + b < \frac{1}{2}(1 + b + b^2E)$ is valid for $b \geq 2$. If $a = -1$, then $b < \frac{1}{2}(1 - b + b^2E)$ is true with the exception $E = 1$, $b = 2$.

The proof is completed.

Lemma 9. *($\zeta\alpha$, α) is a NS if*

(1) $D > 19$ *and* $d > 1$;
(2) $D = 19$ *and* $d \geq 6$;
(3) $D = 15$ *and* $d \geq 7$;
(4) $D = 11$ *and* $d \geq 8$;
(5) $D = 7$ *and* $d \geq 12$;
(6) $D = 3$ *and* $d \geq 56$.

Proof. If ($\zeta\alpha$, α) is not a NS, then there exists a periodic element $\pi = p + q\omega$ with $q \neq 0$. Then $|\pi|^2 = \left(p + \frac{q}{2}\right)^2 + \frac{q^2D}{4} \geq 4$. From (3.4) we have

$$\left(1 - \frac{1}{\sqrt{d}}\right)^2 \leq \frac{9+D}{16E} = \frac{E+2}{4E},$$

whence

$$\sqrt{d} \le \frac{4E\left(1+\sqrt{\frac{E+2}{4E}}\right)}{3E-2} \quad (=: \lambda_E).$$

Since $d \ge E$, this cannot hold for $E \ge 6$: $\frac{\lambda_E}{\sqrt{E}} \le \frac{\lambda_6}{\sqrt{6}} < 1$ for $E \ge 6$. This proves (1). Furthermore, $\lambda_5^2 = 5,995$; $\lambda_4^2 = 6,65$; $\lambda_3^2 = 7,957$; $\lambda_2^2 = 11,64$; $\lambda_1^2 = 55,35$ whence (2)-(6) follow.

A) Completing the proof. Case D=3

Lemma 10. *Let $D = 3$. If $d > 6$, and $(\zeta\alpha,\ \alpha)$ has a nonzero periodic element π, then it is a unit, $\pi \in \{\pm 1,\ \pm\omega,\ \pm\overline{\omega}\}$.*

Proof. If $|\pi|^2 \ge 2$, then from (3.4) $2 \le \frac{3}{4\left(1-\frac{1}{\sqrt{d}}\right)^2}$, which does not hold for $d > 6$.

Lemma 11. *Let $D = 3$. Then $\{\pm 1,\ \pm\omega,\ \pm\overline{\omega}\} \subseteq \zeta\alpha$ for all α with $d \ge 7$.*

Proof. Since $\overline{\omega} = 1-\omega$, $-\overline{\omega} = -1+\omega$, it is enough to prove that $(a+ +b)k+b\ell$, $-bk+a\ell \in \mathcal{L}$ for all the choices $(k,\ell) = (0,0)$, $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$, $(1,-1)$, $(-1,1)$. This is clear, if

$$m := \max(|a+b|,\ |a|,\ b) < \frac{d}{2}. \tag{5.3}$$

Let first $m = |a+b|$. Then $a > 0$ ($a = 0$ is excluded), (5.3) is equivalent to $a^2+ab+b^2-2a-2b > 0$, which is satisfied with the exception $a = b = 1$.

Let $m = |a|$. Then $a < 0$, $-a > b$. (5.3) is equivalent to $a^2+ab+b^2-2|a| > > 0$. If $b = 1$, then it fails only if $a = -2$. Let $b \ge 2$. Since

$$a^2 - |a|b - 2|a| + b^2 = a^2 - |a|(b+2) + b^2 > 0 \tag{5.4}$$

holds for $b+2 \le |a|$, we have to consider only the cases $a = -(b+1)$. Then (5.4) is equivalent to

$$|a|^2 - |a|(|a|+1) + (|a|-1)^2 = |a|^2 - 3a + 1 > 0,$$

which holds for $a \le -3$, i.e. for all possible choices of a.

Finally we assume that $m = b$. We may assume that $|a+b| < b$, $|a| < b$. Then $1 \leq -a \leq b-1$. (5.4) is equivalent to

$$0 < b^2 + a^2 + ab - 2b = a^2 + b^2 - (2-a)b.$$

If $2 - a \leq b$ then this is true. It remains the case $2 - a = b + 1$, i.e. $a = 1 - b$. The equivalent condition is $0 < b^2 + (1-b)^2 - (b+1)b = b^2 - 3b + 1$. This holds for $b \geq 3$.

We proved (5.4) with the exceptions: $\alpha = -1 + 2\omega$ ($|\alpha|^2 = 4$); $\alpha = -2 + \omega$ ($|\alpha|^2 = 3$); $\alpha = 1 + \omega$ ($|\alpha - 1| = 1$!).

To finish the proof we shall prove

Lemma 12. *Let $\mathcal{F} = \{0,\ 1,\ \omega\}$ and $\alpha = -1 + 2\omega$ or $\alpha = -2 + \omega$. Then $(\mathcal{F},\ \alpha)$ is a NS.*

Proof. Observe that $\beta \in \mathcal{F}$ implies $|\beta| \leq 1$. If $\pi \in \mathcal{F}$, then, by (1.3) $|\pi| \leq \dfrac{1}{\sqrt{3}-1} \approx 1,36$, whence it follows that $\pi = 0$ or π is a unit. $|\alpha| = \sqrt{3}$ holds in both cases.

Let $\alpha = -1 + 2\omega$. Then $-1 = \omega + \alpha(-1+\omega)$, $-\omega = 1 + \alpha(-1+\omega)$, $-1+\omega = 1 + \alpha\omega$, $J(-1+\omega) = \omega$, furthermore $1 - \omega = \omega + 1 \cdot \alpha$. This proves the first case.

Let $\alpha = -2 + \omega$. Then $-1 = \omega + \omega\alpha$, $-\omega = 1 + \omega\alpha$, whence $J^2(-1) = J^2(-\omega) = 0$. Furthermore $-1 + \omega = 1 + \alpha$, $1 - \omega = \omega + \alpha(\omega - 1)$, i.e. $J^2(-1+\omega) = 0$, $J^3(1-\omega) = 0$. The second case is proved.

The theorem is completely proved for $D = 3$.

B) Completion of the proof. Case D=7

The critical values of $\alpha = a + b\omega$ are

$$(a,b) = (-1,1),\ (1,1),\ (-2,1),\ (2,1),\ (-3,1),\ (-1,2),\ (-2,2),\ (1,2),\ (-3,2).$$

$(a,b) = (-1,1)$, $(-2,2)$ are excluded by the condition $a+b \neq 0$. In the notation $d(a,b) = \left(a + \dfrac{b}{2}\right)^2 + \dfrac{b^2 \cdot 7}{4}$ we have

$$d(1,1) = d(-2,1) = 4,\quad d(2,1) = d(-1,2) = 7,\quad d(1,2) = d(-3,2) = 11.$$

The integers in $Q(\sqrt{7}i)$ having norm 2 are $\{-1+\omega,\ \ \omega,\ \ 1-\omega,\ \ -\omega\}$. We show that all they belong to $\zeta\alpha$ if additionally $|\alpha|^2 > 4$ holds.

The inequalities $(a+b)k+2b\ell,\ -bk+a\ell \in \mathcal{L}$ hold for all choices of $(k,\ell) = (-1,1),\ (1,-1),\ (0,1),\ (0,-1)$ and for all $(a,b) = (2,1),\ (-1,2),\ (1,2),\ (-3,2)$. This can be checked immediately.

Furthermore, if there is a $\pi \in P$ with $|\pi|^2 > 2$, then $|\pi|^2 \geq 4$, and from (3.4) we obtain that $d \leq 4$. Since $|a+b| < \frac{d}{2}$, $|b| < \frac{d}{2}$ hold for the listed cases if $d \geq 7$, therefore $1,\ -1 \in \zeta\alpha$ as well.

Thus $(\zeta\alpha,\ \alpha)$ is a NS if $d \geq 6$.

The remaining cases are $\alpha = 1+\omega$, $\alpha = -2+\omega$.

Lemma 13. *Let* $D = 7$, $\mathcal{F}_1 = \{0,1,-1,1-\omega\}$, $\mathcal{F}_2 = \{0,-1,1,\omega\}$, $\alpha_1 = 1+\omega$, $\alpha_2 = -2+\omega$. *Then* $(\mathcal{F}_1,\ \alpha_1)$ *and* $(\mathcal{F}_2,\ \alpha_2)$ *are NS.*

Proof. In both cases $\max\limits_{\beta\in\mathcal{F}_i} |\beta| = \sqrt{2}$, $|\alpha_i| = 2$. If π is periodic, then by (1.3) $|\pi| \leq \sqrt{2}$, consequently it is enough to prove that all integers with norm 2 have finite representations.

Let first $\alpha = \alpha_1 = 1+\omega$. Then $\omega = (-1)+\alpha$, $-\omega = 1+(-1)\alpha$, whence $J^2(\omega) = J^2(-\omega) = 0$. Furthermore $\omega-1 = (1-\omega)+\omega\alpha$ which gives $J^3(\omega-1) = 0$. The assertion is true for $\alpha = \alpha_1$.

Let now $\alpha = \alpha_2 = -2+\omega$. Since $-\omega = \omega+\alpha(\omega-1)$, $\omega-1 = -1+\alpha$, $1-\omega = 1+(-1)\alpha$ we have $J^2(\omega-1) = 0$, $J^2(1-\omega) = 0$, $J^3(-\omega) = 0$. The proof is completed.

C) Completion of the proof for $D = 11,\ 15,\ 19$

1) In the case $D = 11$ only the integers $\alpha = 1+\omega$, $\alpha = -2+\omega$ are remained to consider.

Let $\alpha = 1+\omega$. Then $\zeta\alpha = \{0,1,-1,-1+\omega,1-\omega\}$, $d = |\alpha|^2 = 5$. Since $\max\limits_{\beta\in\zeta\alpha} |\beta| = \sqrt{3}$, for a periodic element π we have $|\pi| \leq \frac{\sqrt{3}}{\sqrt{5}-1} < \sqrt{3}$, whence $\pi \in \{0,1,-1\} \subseteq \zeta\alpha$. Thus $\pi = 0$.

The case $-2+\omega$ can be reduced to the case $1+\omega$ as follows. Since for $\alpha = 1+\omega$, $d = 5$ =odd, $\zeta\alpha = -\zeta\alpha$, therefore $(-\alpha,\ \zeta\alpha)$ is a NS as well, and by complex conjugation $(-\overline{\alpha},\ \overline{\zeta\alpha})$ is a NS. But $-\overline{\alpha} = -2+\omega$, and we are ready.

2) Let $D = 15$. We have to prove the theorem for $\alpha = 1+\omega$, $\alpha = -2+\omega$.

Lemma 14. *Let $D{=}15$, $\alpha_1 = 1+\omega$, $\alpha_2 = -2+\omega$, $\mathcal{F}_1 = \{0,1,-1,1-\omega,-1+\omega,2-\omega\}$, $\mathcal{F}_2 = \{0,1,-1,\omega,-\omega,1+\omega\}$. Then $(\alpha_i,\mathcal{F}_i)$ are NS's for $i = 1,\ 2$.*

Proof. Let $\alpha = 1+\omega$. Then $\max_{\beta\in\mathcal{F}_1} |\beta| = 4$, $|\alpha| = \sqrt{6}$. Thus $\pi \in P$ satisfies $|\pi| \le \dfrac{4}{\sqrt{6}-1}$, whence $|\pi|^2 \le 7$ follows. All the integers with norm ≤ 7 are

$$\{2,\ -2,\ \omega,\ -\omega,\ 1+\omega,\ -1-\omega\} \cup \mathcal{F}_1.$$

All they have finite expansion in $(\mathcal{F}_1,\ \alpha_1)$. This is clear, since $1+\omega = \alpha$, $-1-\omega = -\alpha$, $\omega = -1+\alpha$, $-\omega = 1-\alpha$, $2 = 1-\omega+1\cdot\alpha$, $-2 = (-1+\omega)+(-1)\alpha$.

Let now $\alpha = -2+\omega$. The situation is very similar. We should prove that $\{2,\ -2,\ 1-\omega,\ \omega-1,\ -1-\omega\}$ have finite expansions in $(\mathcal{F}_2,\ \alpha_2)$. Since $2 = \omega+(-1)\alpha$, $-2 = -\omega+\alpha$, $-1+\omega = 1+1\cdot\alpha$, $1-\omega = (-1)+(-1)\alpha$, we are ready.

3) For $D = 19$ the only remained case is $\alpha = -1+\omega$, but this is excluded by $a+b \neq 0$.

The theorem is completely proved.

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