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THE HAUSDORFF-DIMENSION OF THE BOUNDARY OF A UNIT-INTERVAL OF A NUMBER SYSTEM

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Dedicated to Professor Karl-Heinz Indlekofer on his fiftieth birthday

Abstract. This paper presents a new method how to calculate the Hausdorff-dimension of the boundary of the unit-interval of a number-system. We show that the nonempty overlaps from a well defined set of feasible translations of the unit-interval with the unit-interval are graph-self-similar.

1. Introduction

Fix a complex number θ (the base) with an absolute value greater than one and a finite set D of complex numbers that contains zero. The elements of D are called "digits". The set

$$H = \left\{ \sum_{i=-\infty}^{-1} d_i b^i : d_i \in D,\ i \in \mathbf{N} \right\}$$

is called the unit-interval of the pair $(\theta; D)$ and

$$W = \left\{ \sum_{i=0}^{M} d_i b^i : d_i \in D, i \in \{0, \dots, M\}, M \in \mathbf{N}_0 \right\}$$

The results presented here are part of the diplomathesis of the author. The diplomathesis was supervised by Prof.Dr.Dr.h.c. K.-H. Indlekofer, Mathematical Faculty at the University of Paderborn.

is called the set of *whole numbers*. They are not integers in general, even they are not closed for the addition.

Several questions arise: Does the equation $W + H = \mathbf{C}$ holds? Is it true that $\lambda((H+\gamma_1)\cap(H+\gamma_2)) = 0$ for all $\gamma_1 \neq \gamma_2 \in W$, where λ is the Lebesgue-measure?

The aim of this paper is to develop a method to calculate the Hausdorff-dimension of ∂H.

In some cases it is not of interest to look at the translates of H by whole numbers, but by elements of a countable ring Δ that contains W. To guarantee that the method works, we suppose that the triple (θ, D, Δ) defines a just-touching-covering, i.e.

$$\bigcup_{\gamma\in\Delta} H + \gamma = \mathbf{C},$$

$$\lambda((H+\gamma_1)\cap(H+\lambda_2)) = 0 \quad (\gamma_1 \neq \gamma_2, \gamma_i \in \Delta)$$

holds. The method introduced, is used to examine the sets $B(\gamma) := \gamma + H \cap H$. The connection between these sets and the boundary of H is given by the following

Lemma 1. *Let (θ, D, Δ) be a just-touching-covering. It follows*

$$x \in \partial H \Leftrightarrow x \in H \cap H + \gamma \quad \textit{for a} \quad \gamma \in \Delta \setminus \{0\}.$$

Proof. The proof is shown only for " $\Leftarrow$ ": Let $x \in H \cap H + \gamma$ and suppose that x is not a boundary point of H. Then there exists an $\varepsilon > 0$ with $B_\varepsilon(x) \subset H$. Using the self-similarity of H with respect to the iterated function-system $f_d(z) := (z+d)/\theta,\ d \in D$, i.e.

$$H = \bigcup_{d\in D} f_d(H),$$

we can deduce that

$$x \in H+\gamma. \Rightarrow \exists j_1 : x \in f_{d_{j_1}}(H)+\gamma. \Rightarrow \exists j_2 : x \in f_{d_{j_1}} \circ f_{d_{j_2}}(H)+\gamma. \Rightarrow \ldots \text{ etc.}$$

By using induction we can find therefore a sequence $j_1, \ldots, j_l$ such that $x \in f_{d_{j_1}} \circ \ldots \circ f_{d_{j_l}}(H) + \gamma =: G$ and diam $(G) < \varepsilon$. It follows that $G \subset$ $\subset B_\varepsilon(x)$, and because of $\lambda(H) > 0$ (Δ is countable), we obtain $\lambda(G) > 0$. But $H \cap H + \gamma \supset G$ implies $\lambda(H \cap H + \gamma) > 0$ which is a contradiction to the just-touching-covering property.

The Lemma shows that the union of all $B(\gamma)$, $\gamma \neq 0$ equals to the boundary of H, if a just-touching-covering is given.

We can prove more than the stated result above: If there is an inner point in H then the set of inner points of H is dense in H. Heuristically spoken: One inner point is distributed through the set H by the iterated functionsystem.

2. The graph $G(S)$

In [5] and [6] it is suggested to use the graph $G(S)$. Let S be the set of those $\gamma \in \Delta$ for which $B(\gamma)$ is nonempty. One can easily prove that $B(\gamma)$ is nonempty, iff γ has an expansion of the form $\gamma = \sum_{i=1}^{\infty} \delta_i \rho^i$, $\delta_i \in B$, where $\rho := 1/\theta$ and $B := D - D$. It follows that $S \subset H - H$ and therefore $|\gamma| \leq 2 \max_{z \in H} |z| =: U$ for all $\gamma \in S$. The graph $G(S)$ is constructed by the following algorithm.

The graph $G(S)$

(1) For all $\gamma \in \Delta$, $\delta \in B$ calculate $\gamma_\delta := \gamma\theta - \delta$. If $|\gamma_\delta| \leq U$, then direct an edge with the name δ from γ to γ_δ.

(2) Delete γ if no edge leaves γ and all edges that end in γ. Continue this process until no appropiate γ remains.

Observe that 0 is a node of the graph $G(S)$ $(0 \in D)$. In most cases it is assumed that (θ, D) satisfies the following properties: (1) θ is an algebraic integer with an absolute value greater than 1, (2) D is a complete residue-system mod θ, and (3) all conjugates of θ have an absolute value greater than 1. We will say, that $(*)$ is satisfied, iff we want to assume that the properties (1)-(3) hold.

In the case that $(*)$ is satisfied only the graph $G(S^*)$ is of interest. $G(S^*)$ is constructed the same way as the graph $G(S)$, one only writes $\gamma \in \Delta \setminus \{0\}$ instead of $\gamma \in \Delta$ and $0 < |\gamma_\delta| \leq U$ instead of $|\gamma_\delta| \leq U$ in the algorithm. Then 0 is no node of the graph $G(S^*)$.

3. Graph-Self-Similarity

For $E \subset \mathbf{C}$ the s-dimensional outer Hausdorff-measure ($s \in [0,\infty]$) is defined by

$$\mathcal{H}^s(E) := \lim_{\delta\to 0}\mathcal{H}^s_\delta(E) := \lim_{\delta\to 0}\left(\inf\sum_{i=1}^{\infty} | U_i |^s\right),$$

where the infimum is over all countable δ-coverings $\{U_i\}$ of E. It can be proved that there exists an $s_0 \in [0,\infty]$ with the property

$$\mathcal{H}^s(E) = \infty \text{ if } s < s_0,$$

$$\mathcal{H}^s(E) = 0 \text{ if } s > s_0.$$

s_0 is called the Hausdorff-dimension of E.

We will use the concept of grahp-self-similarity which has been introduced by Mauldin and Williams [7] and [1].

Let a finite directed multigraph (V, E, i, t) and a real-valued function r defined on E be given. Here V denotes the set of nodes and E the set of edges. The function i gives the initial node of an edge and the function t gives the terminal node of an edge. We will assume that for all $v \in V$ there is an $e \in E$ with $i(e) = v$. We call the graph together with the function r a *Mauldin-Williams-graph*. For all $e \in E$ let a similarity $f_e : \mathbf{C} \to \mathbf{C}$ be given. A list of nonempty compact sets K_v, $v \in V$ is called an ***invariant list*** for the iterative functionsystem $f_e, e \in E$, iff

$$(\sharp) \qquad K_u = \bigcup_{v\in V, e\in E_{uv}} f_u(K_v).$$

In this case the sets $K_v, v \in V$ are called ***graph-self-similar***. It can be proved that the invariant list is uniquely determined, if $r(e) < 1$ for all $e \in E$. In this paper we always assume that $r(e) < 1$ for all $e \in E$. For all $t \geq 0, u, v \in V$ define $A_{uv}(t) := \sum_{e\in E_{uv}} r(e)^t$ and the matrix $A(t)$ by $A(t)[u,v] := A_{u,v}(t)$. The spectral radius of $A(t)$ takes the value 1 for a uniquely determined value of $t_0 = t$. This t_0 is called the ***graph-dimension*** of the Mauldin-Williams-graph (V, E, i, t, r).

If the graph is strictly connected, then the system

$$(**) \qquad q_u^s = \sum_{v\in V, s\in E_{uv}} r(e)^s q_v^s, \quad u \in V$$

has a solution with $q_v > 0$ for all v, iff $s = t_0$.

The graph-dimension can be used to calculate an upper bound of the Hausdorff-dimension of the sets of the invariant list. The graph-dimension equals the Hausdorff-dimension, if the so called open-set-condition is satisfied (see [1]).

We shall use two main results connecting the Hausdorff-dimension with graph-dimension:

Theorem 1. *Let the graph (V, E, i, t) be a strictly connected Mauldin-Williams-graph and let $r(e) < 1$ for all $e \in E$. The graph-dimension is an upper bound of the Hausdorff-dimension of K_v. If the open-set-condition is satisfied, then the Hausdorff-dimension of K_v equals to the graph-dimension.*

Observe that all K_v, $v \in V$ have the same Hausdorff-dimension, if the graph is strictly connected (use the equation ($\sharp$)). What happens if the graph is not strictly connected? To clarify this case we need some preliminary explanations. Two strictly connected components W_1, W_2 of V are called comparable if there exists a path from W_1 to W_2 or a path from W_2 to W_1. Let $SC(V)$ be the set of all strictly connected components of V. The equation $s = \max_{W \in SC(V)} s_W$ holds, where s is the graph-dimension of V and s_W is the graph-dimension of $W \in SC(V)$. Let

$$K := \bigcup_{v \in V} K_v.$$

We can now state the following

Theorem 2. *If the elements of $M := \{W \in SC(V) : s_W = s\}$ are incomparable, then*

$$\dim_H(K) \leq s.$$

If in addition the open-set-condition is satisfied, then equality holds in the formular above.

The proofs of the stated results may be found in [1], [3], [7].

4. The graph $V(S)$

Let the graph $G(S)$ be given. Let $m(\delta)$ count the number possibilities to write δ in the form $\delta = d - d'$. If the graph $G(S)$ contains an edge from A_i to

A_j with the name δ, then direct $m(\delta)$ edges from A_i to A_j in the graph $V(S)$ and define $m(\delta)$ mappings on $B(A_j)$ by

$$f_{d^*} : z \to \frac{z + d^*}{\theta},$$

where d^* runs through all possible elements of D such that $d^* - \delta \in D$.

Proposition. $A_i \xrightarrow{\delta} A_j$, $d^* - \delta \in D$ implies $f_{d^*}(B(A_j)) \subset B(A_i)$.

Observe that for all nodes v of the graph there is an edge starting in v. The main point made is that the sets $B(A_i)$ are the invariant list of the iterated functionsystem f_e.

Proposition.

$$B(A_i) = \bigcup_{A_j \in S, e \in E_{A_i A_j}} f_e(B(A_j)).$$

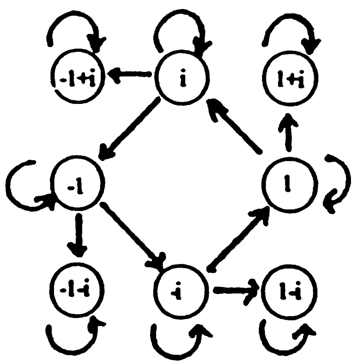
Proof. If $z \in B(A_i)$, then z can be written as $z = \sum_{i=1}^{\infty} d_i \rho^i = A_i + \sum_{i=1}^{\infty} d'_i \rho^i$. Thus $A_i = \sum_{i=1}^{\infty} (d_i - d'_i)\rho^i$. Let $A_j := \theta A_i - (d_i - d'_i) = \sum_{i=1}^{\infty} d_{i+1}\rho^i - d'_{i+1}\rho^i$ and $\delta := d_1 - d'_1$. We conclude that the graph $V(S)$ contains an edge from A_i to A_j with the name δ and that $f_{d_1}(x) = z$, where $x = \sum_{i=1}^{\infty} d_{i+1}\rho^i = A_j + \sum_{i=1}^{\infty} d'_{i+1}\rho^i \in$ $\in B(A_j)$.

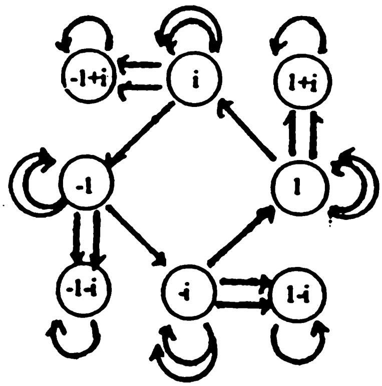
If there is an edge with initial nodes $\gamma \neq 0$ ending in 0, then it is possible to find a smaller copy of H in $B(\gamma)$. If a just-touching-covering is assumed to be satisfied, this is impossible, because the boundary must be a zero-set and H must have a Lebesgue-measure > 0.

As mentioned in Section 2, if $(*)$ (Section 2) is satisfied, then we use the graph $G(S^*)$. From the graph $G(S^*)$ the grahp $V(S^*)$ is constructed in the same fashion as $V(S)$ from $G(S)$.

5. Examples

Example 1. To illustrate the above described method, let $\theta = 2 + i, D =$ $= \{0, 1, -1, -i, i\}$. It is known that $W = Z[i]$. The graphs $G(S)$ resp. $V(S)$ are shown below. $(m(1+i) = m(-1-i) = m(1-i) = m(-1+i) = 2)$. The





nodes $W := \{-1, -i, 1, i\}$ define a strictly connected component of $V(S)$ with the $\alpha_w = 1,3652\ldots$

To calculate the graph-dimension we use the system of equations (**):

$$x_1 = 2\lambda x_1 + \lambda x_2$$

$$x_2 = 2\lambda x_2 + \lambda x_3$$

$$x_3 = 2\lambda x_3 + \lambda x_4$$

$$x_4 = 2\lambda x_4 + \lambda x_1$$

With back-substitution this yields

$$(1 - 2\lambda)x_1 = \lambda x_2$$

$$(1 - 2\lambda)^3 x_2 = \lambda^3 x_1.$$

If we take $x_2 = 1$, then

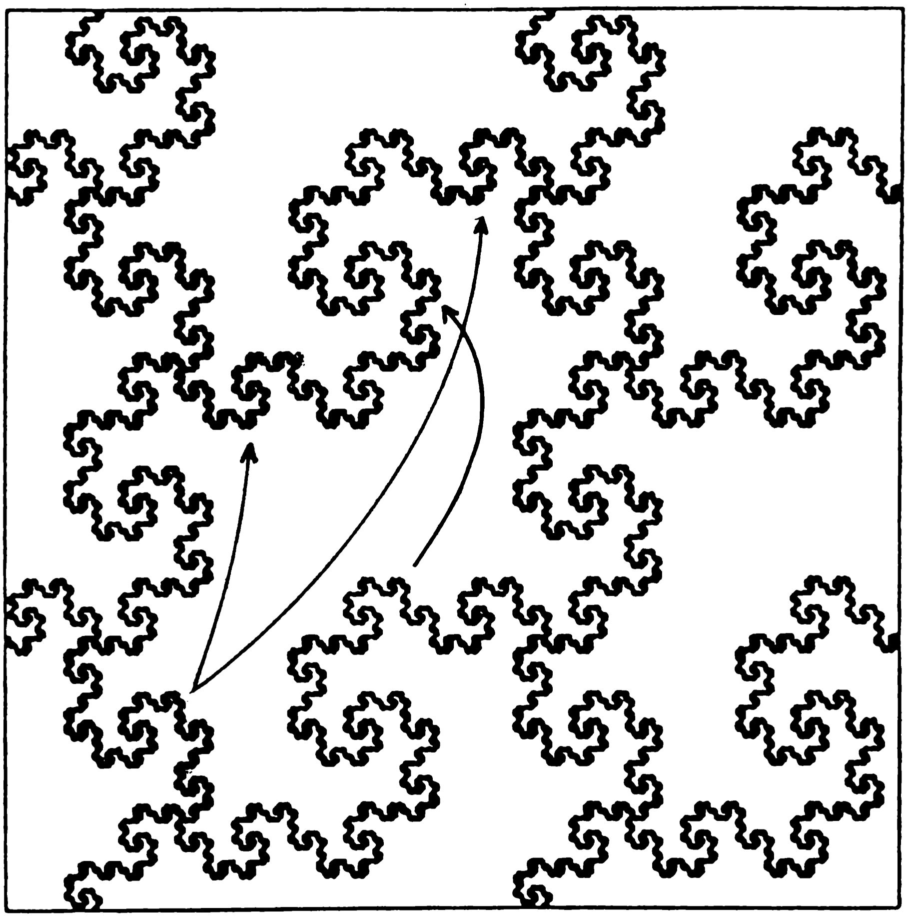
$$(1 - 2\lambda)^4 = \lambda^4.$$

$\lambda = 1/3$ solves this equation, and we obtain $\alpha_K = s = 1.3652\cdots$. The other components of $V(S^*)$ have only one point, and therefore their graph-dimension is zero. The maximum is $\alpha_K = 1.3652\cdots$.

But is this value equal to the Hausdorff-dimension of K? Yes, but it would need some more steps to prove this (see for example [5]). The problem would be solved, if the open-set-condition had been satisfied, but the author was not able to prove or disprove this.

We remark that the graph and the system of equations have symmetry-properties which always arise, if the assumption (*) is satisfied: If the graph $G(S)$ contains an edge from γ to μ with the name δ, then there is an edge from $-\gamma$ to $-\mu$ with the $-\delta$.

Example 2. The following picture shows the graph-self-similarity of the twin-dragon



6. Multiple overlappings

Up to now we have investigated the sets $H \cap H + \gamma$. Now we will look at sets of the form

$$H \cap H + \gamma_1 \cap H + \gamma_2.$$

The analogue to the set S in this context is defined by

$$T := \{(\gamma_1, \gamma_2) : \gamma_i \in \Delta,\ B(\gamma_1) \cap B(\gamma_2) \neq \emptyset\}.$$

Let $(\gamma_1, \gamma_2) \in T$. It follows that there is a $z \in H$ with

$$z = \sum d_i \rho^i = \gamma_1 + \sum d_i' \rho^i = \gamma_2 + \sum d_i'' \rho^i.$$

From this the following expansions result

$$\gamma_1 = \sum (d_i - d_i')\rho^i,$$

$$\gamma_2 = \sum (d_i - d_i'')\rho^i,$$

i.e. γ_1, γ_2 have expansions in $D - D$ with the same first element. This property characterizes completely the element of T: let γ_1, γ_2 have such expansions. Then

$$\gamma_1 + \sum d_i' \rho^i = \sum d_i \rho^i,$$

$$\gamma_2 + \sum d_i'' \rho^i = \sum d_i \rho^i.$$

We conclude that $z := \sum d_i \rho^i \in B(\gamma_1) \cap B(\gamma_2)$. To summarize:

$$B(\gamma_1) \cap B(\gamma_2) \neq \emptyset \Leftrightarrow \gamma_1 = \sum (d_i - d_i')\rho^i,\ \ \gamma_2 = \sum (d_i - d_i'')\rho^i.$$

Let $(\gamma_1^1, \gamma_2^1) \in T$ have expansions as in the equivalence above. Then it holds that

$$\gamma_1^2 := \gamma_1^1 \theta - (d_1 - d_1') = \sum (d_{i+1} - d_{i+1}')\rho^i,$$

$$\gamma_2^2 := \gamma_2^1 \theta - (d_1 - d_1'') = \sum (d_{i+1} - d_{i+1}'')\rho^i.$$

Thus $(\gamma_1^2, \gamma_2^2) \in T$.

Clearly for all $(\gamma_1, \gamma_2) \in T$ we have $| \gamma_i | \leq U$, i.e. $\| (\gamma_1, \gamma_2) \|_\infty \leq U$. Therefore it is obvious how to define the graph $V(T)$:

The graph $V(T)$

(1) For all $(\gamma_1, \gamma_2) \in \Delta^2$ with $\| (\gamma_1, \gamma_2) \|_\infty \leq U$ and for all pairs of the form $(d - d', d - d'')$ calculate $(\gamma_1, \gamma_2)\theta - (d - d', \ d - d'')$. If $\| (\gamma_1, \gamma_2)\theta - (d - d', d - d'') \|_\infty \leq U$, then direct an edge from (γ_1, γ_2) to $(\gamma_1, \gamma_2)\theta - (d - d', d - d'')$ with name (d, d', d'').

(2) Delete (γ_1, γ_2) if no edge leaves (γ_1, γ_2) and all edges that end in (γ_1, γ_2). Continue this process until no appropiate (γ_1, γ_2) remains.

Then next proposition states that the sets $B(\gamma_1) \cap B(\gamma_2)$ are graph-self-similar.

Proposition. (1) Suppose $(\gamma_1^1, \gamma_2^1) \overset{(d,D',d'')}{\rightarrow} (\gamma_1^2, \gamma_2^2)$. It follows

$$f_d(B(\gamma_1^2) \cap B(\gamma_2^2)) \subset B(\gamma_1^1) \cap B(\gamma_2^1).$$

(2) It holds that

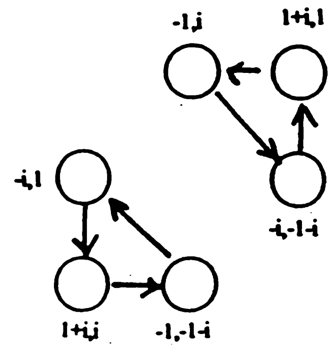
$$B(\gamma_1) \cap B(\gamma_2) = \bigcup_{(\gamma_1,\gamma_2) \overset{(d,d',d'')}{\rightarrow} (\gamma_1^v,\gamma_2^v)} f_d(B(\gamma_1^v) \cap B(\gamma_2^v)).$$

The diagonal elements $(\gamma, \gamma) \in S^2$ are elements of the graph $G(S)$ and the same is true for the elements of the form $(0, \ldots)$ and $(\ldots, 0)$, but these elements do not define real multiple overlappings. To resolve this, assume that $(*)$ is satisfied. Then we can conclude that: (1) If an edge starts from a node not on the diagonal then the terminal node is not on the diagonal; (2) If an edge starts at $(\ldots, 0)$ it also ends in $(\ldots, 0)$ and the same is true for $(0, \ldots)$. If we put this together, it is clear how the graph $V(T^*)$ is defined.

To calculate the graph $V(T)$ it is possible to use the graph $G(S)$:

$$(\gamma_1^1, \gamma_2^1) \overset{(d,d',d'')}{\rightarrow} (\gamma_1^2, \gamma_2^2) \Leftrightarrow \gamma_1^1 \overset{d-d'}{\rightarrow} \gamma_1^2, \ \gamma_2^1 \overset{d-d''}{\rightarrow} \gamma_2^2.$$

Example. The twin-dragon $T^* = \{(1+i, 1), (1+i, i), (-1, i), (-1, -1 - -i), (-i, -1-i), (1, -i)\}$. The graph $V(T^*)$ is



The graph-dimension is 0 for both components.

7. Radixrepresentation in R^n

Let M be an $n \times n$ matrix of integer entries such that M has n distinct eigenvalues the module of which is larger than 1. Let further D be a finite subset of $\mathbf{R}^n$. The sets

$$H = \left\{ \sum_{i=1}^{\infty} M^{-i} d_i : d_i \in D \right\},$$

$$W = \left\{ \sum_{i=0}^{\infty} M^{i} d_i : d_i \in D \right\},$$

are a generalization of the sets H and W of the preceding sections. It is not surprising that the methods useful there can also be applied here. Define S to be the set $S := \{\gamma \in W : H \cap H + \gamma \neq \emptyset\}$. We have

$$B(\gamma) \neq \emptyset \Leftrightarrow \gamma = \sum_{i=1}^{\infty} M^{-i}(d_i - d_i').$$

In an obvious way the graphs $G(S)$ and $V(S)$ can be defined. Again we have that

$$B(A_i) = \bigcup_{A_j \in S, e \in E_{A_i A_j}} f_e(B(A_j)).$$

This means that the overlaps are graph-self-similar.

References

[1] **Edgar G.A.**, *Measure, topology and fractal geometry,* Springer, 1990.

[2] **Falconer K.J.**, *The geometry of fractal sets,* Cambridge University Press, 1985.

[3] **Falconer K.J.**, Dimensions and measures of quasi self-similar sets, *Proc. of AMS,* **106** (1989).

[4] **Falconer K.J.**, *Fractal geometry - Mathematical foundations and applications,* John Wiley and Sons, 1990.

[5] **Indlekofer K.-H., Kátai I. and Racskó P.**, Some remarks on generalized number systems, *Acta Sci. Math. Szeged,* **57** (1993), 543-553.

[6] **Indlekofer K.-H., Kátai I. and Racskó P.**, Number systems and fractal geometry, *Probability theory and applications: essays to the memory of József Mogyoródi,* eds. J. Galambos and I. Kátai, Kluwer, Dordrecht-Boston-London, 1992, 319-334.

[7] **Mauldin R.D. and Williams S.C.**, Hausdorff-dimension in graph directed constructions, *Transaction of American Mathematical Society,* **309** (2) (1988), 811-829.

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