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WALSH WAVELETS

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1. Introduction

In theoretical and practical investigations two types of wavelets are considered. Weyl-Heisenberg coherent states, which arise from translations and modulations of a single function, called *mother wavelet*, and affine coherent states, which arise as translations and dilations of a single function (see [1], [2], [3]).

In this paper we shall investigate a new type of wavelets obtained from the mother wavelet using dilation and modulation. Instead of usual modulation we consider the dyadic modulation, i.e. multiplication by the characters of the dyadic group. Thus the functions in question are of the form

$$(1)\qquad \varphi_{mn}(x) = w_m(x)\rho(2^n x) \qquad (x \in [0,\infty), \quad m, n \in \mathbf{N}),$$

where $(w_m, m \in \mathbf{N})$ is the Walsh system in Paley's ordering and ρ is a 1-periodic, locally integrable function on $[0,\infty)$ (see [4]).

In this paper we give sufficient conditions for periodic functions such that the system $\{\varphi_{mn} : 2^{n-1} \leq m < 2^n, n \geq 1\}$ defined by (1) is an exact frame in

$$L_0^2[0,1] := \left\{ f \in L^2[0,1] : \int_0^1 f = 0 \right\}.$$

The condition in question can be expressed with the help of a new semi-norm of the mother wavelet ρ, namely by

$$\|\rho\|_* = \sum_{j=1}^{\infty} \left(\sum_{l(k)=j} |\hat{\rho}(k)|^2 \right)^{\frac{1}{2}},$$

where

$$\hat{\rho}(k) = \int_0^1 \rho(t) w_k(t) dt \qquad (k \in \mathbf{N})$$

are the Walsh-Fourier coefficients of ρ and $l(k)$ is the sum of non-zero digits in the binary expansion of k. This number is called the *diversity* of k.

Obviously the Walsh system can be obtained as a special case, corresponding to the function $\rho(x) = 1$ $(x \in \mathbf{R})$. A subsequence of the integrated Walsh functions is also a system of this type. In this special case we give an explicit form for the inverse frame. Moreover, starting from the Walsh system and using integration we can get infinitely many systems of the above type.

2. Frames in Hilbert spaces

In this paper we shall investigate frame expansions connected with the integrated Walsh system. The notion of the frame is a generalization of the base. We consider here frames in Hilbert spaces.

Let $(H, \langle , \rangle)$ be a Hilbert space and $x_n \in H$ $(n \in \mathbf{N})$ be a sequence in H.

The collection $X = (x_n, n \in \mathbf{N})$ is called *frame* if there exist constants $0 < m \leq M < \infty$ such that

$$m\|x\|^2 \leq \sum_{n \in \mathbf{N}} |\langle x, x_n \rangle|^2 \leq M\|x\|^2 \tag{2}$$

holds for all $x \in H$ (see [1], [3]).

The constants m and M are called *frame constants*. The frame is called *tight* if the frame constants coincide. A frame is called *exact* if it ceases to be a frame if even one element is deleted from the sequence.

It is known that $X = (x_n, n \in \mathbf{N})$ is a frame if and only if the series

$$F(x) := \sum_{n \in \mathbf{N}} \langle x, x_n \rangle x_n \tag{3}$$

converges for all $x \in H$ and its sum F is $1-1$ map from H onto H.

The map F is called the *frame operator* of X. It is obvious that the frame operator F is positive definite and

$$\langle Fx, x \rangle = \sum_{n \in \mathbf{N}} |\langle x, x_n \rangle|^2 \quad (x \in H).$$

If F^{-1} is the inverse of the frame operator then the sequence $\tilde{x}_n = F^{-1}(x_n) \in H$ $(n \in \mathbf{N})$ is also a frame called the inverse frame of X and its frame operator is $\tilde{F} = F^{-1}$. The operators F and F^{-1} are symmetric.

Frame can be used to define two expansions in H. Namely from (3) we get for all $x, y \in H$

$$x := \sum_{n \in \mathbf{N}} \langle x, x_n \rangle \tilde{x}_n,$$

$$y := \sum_{n \in \mathbf{N}} \langle y, \tilde{x}_n \rangle x_n.$$

The sequences $(\langle x, x_n \rangle, n \in \mathbf{N})$, $(\langle y, \tilde{x}_n \rangle, n \in \mathbf{N})$ are called the frame coefficients of x and y, respectively. These series converge in the norm of H, and these expansions are analogous to the biorthogonal expansion. It is known that the frame X is exact if and only if $\langle x_m, \tilde{x}_m \rangle = 1$ for all $m \in \mathbf{N}$ and exact frames are minimal systems. In this case $(\tilde{x}_n, n \in \mathbf{N})$ and $(x_n, n \in \mathbf{N})$ are biorthogonal.

It is convenient to represent frames and frame operators in a suitable complete orthonormal system. Let $(e_n, n \in \mathbf{N})$ be a complete orthonormal system in H and denote

$$x_k = \sum_{n=0}^{\infty} a_{nk} e_n \qquad (k \in \mathbf{N}) \tag{4}$$

the Fourier expansion of $x_k \in H$ with respect to this basis. First we show that the matrix of the frame operator can be expressed by the matrix

$$A = [a_{mn}]_{m,n=0}^{\infty}.$$

Namely if $B = [\langle F e_n, e_m \rangle]_{m,n=0}^{\infty}$ is the matrix of F in the basis in question then

$$B = AA^*. \tag{5}$$

Indeed, since

$$F e_n = \sum_{k=0}^{\infty} \langle e_n, x_k \rangle x_k$$

and this series converges in norm, therefore

$$\langle F e_n, e_m \rangle = \sum_{k=0}^{\infty} \langle e_n, x_k \rangle \langle x_k, e_m \rangle = \sum_{k=0}^{\infty} \bar{a}_{nk} a_{mk} = (AA^*)_{mn}$$

and (5) is proved.

The inverse matrix B^{-1} of B is the matrix of the inverse frame operator $\tilde{F}$. We shall show that if A is invertible then the Fourier expansion of the inverse frame can be expressed by $\tilde{A} = A^{-1} = [\tilde{a}_{ij}]_{i,j=0}^{\infty}$.

Proposition 1. *Let (4) be the expansion of the frame X with respect to the orthonormal basis $e_n \in H$ $(n \in \mathbf{N})$. If A has an inverse matrix, then the inverse frame can be expressed by the matrix $\tilde{A} = (A^{-1})^* = [\tilde{a}_{nk}]_{n,k=0}^{\infty}$ in the form*

$$\tilde{x}_k = \sum_{n=0}^{\infty} \tilde{a}_{nk} e_n \qquad (k \in \mathbf{N}). \tag{6}$$

Proof. Denote F the frame operator. Then by definition $\tilde{x}_k = F^{-1}(x_k)$, or equivalently

$$x_k = F(\tilde{x}_k). \tag{7}$$

Using the fact that B is a matrix representation of F, equality (7) can be written in the form

$$\langle x_k, e_n \rangle = \sum_{l=0}^{\infty} b_{nl} \langle \tilde{x}_k, e_l \rangle.$$

Hence, using (4) and (6) we get

$$a_{nk} = \sum_{l=0}^{\infty} b_{nl} \tilde{a}_{lk} \qquad (n, k \in \mathbf{N})$$

or in matrix form

$$A = B\tilde{A} = AA^*\tilde{A}.$$

Since A^{-1} exists, therefore A^* has also an inverse $(A^*)^{-1} = (A^{-1})^*$, and consequently

$$\tilde{A} = (A^*)^{-1}$$

and our claim is proved.

We shall give a useful sufficient condition for the matrix A to guarantee the existence of the inverse A^{-1}. Write A in the form

$$A = I - C,$$

where $I = [\delta_{ij}]_{i,j=0}^{\infty}$ is the identity matrix, and denote

$$J_k = \{n : c_{nk} \neq 0\} \qquad (k \in \mathbf{N})$$

the support of k-th column of C.

Proposition 2. *Let $(a_{nk}, n \in \mathbf{N})$ be the Fourier-coefficients of $x_k \in H$ $(k \in \mathbf{N})$ with respect to the complete orthonormal system $(e_n, n \in \mathbf{N})$ and set $C = I - A$. Suppose that there exists a number $0 \leq \kappa < 1$ such that for every $k \in \mathbf{N}$*

$$\sum_{n=0}^{\infty} |c_{nk}|^2 \leq \kappa^2 \tag{8}$$

and the supports of the columns are pairwise disjoint, i.e.

$$J_n \cap J_m = \emptyset \quad if \quad m \neq n. \tag{9}$$

Then the induced norm by the l^2-norm of C^* satisfies

i) $\|C^*\| \leq \kappa$ and consequently A has an inverse, and

ii) $(x_k, k \in \mathbf{N})$ is a frame.

Proof. i) To prove that the the norm of C^* is not greater than κ, let $(x_j, j \in \mathbf{N})$ be the coordinates of $x \in H$ and $(y_j, j \in \mathbf{N})$ the coordinates of $y = C^*x$. Then

$$y_j = \sum_{i=0}^{\infty} \overline{c}_{ij} x_i$$

and by definition of J_j, using Cauchy's inequality we get

$$|y_j|^2 = \left| \sum_{i=0}^{\infty} \overline{c}_{ij} x_i \right|^2 \leq \left(\sum_{i=0}^{\infty} |c_{ij}|^2 \right) \left(\sum_{i \in J_j} |x_i|^2 \right) \leq \kappa^2 \sum_{i \in J_j} |x_i|^2.$$

Thus by condition (9)

$$\|C^*x\|^2 = \sum_{j=0}^{\infty} |y_j|^2 \leq \kappa^2 \sum_{j=0}^{\infty} \sum_{i \in J_j} |x_i|^2 \leq \kappa^2 \sum_{n \in \mathbf{N}} |x_n|^2 = \kappa^2 \|x\|^2.$$

Consequently $\|C^*x\| \leq \kappa\|x\|$, i.e. $\|C^*\| \leq \kappa$. Since $\|C^*\| = \|C\| < \kappa$, therefore the inverse of $A = I - C$ exists.

ii) We shall prove that if $x \in H$ then

$$\|x\|(1 - \kappa) \leq \left(\sum_{k \in \mathbf{N}} |\langle x, x_k \rangle|^2 \right)^{\frac{1}{2}} \leq (1 + \kappa)\|x\|. \tag{10}$$

Using the notations $\check{x}(n) = \langle x, x_n \rangle$, $\hat{x} = \langle x, e_n \rangle$ $(n \in \mathbf{N})$ for the Fourier coefficients with respect the complet orthonormal system $(e_n, n \in \mathbf{N})$, the equality

$$x_k = e_k - \sum_{n \in \mathbf{N}} c_{nk} e_n$$

can be expressed in the form

$$\check{x} = \hat{x} - C^* \hat{x}.$$

Hence we have that

$$\|\hat{x}\|_{l^2} - \|C^*\| \, \|\hat{x}\|_{l^2} \leq \|\check{x}\|_{l^2} = \|\hat{x} - C^* \hat{x}\|_{l^2} \leq \|\hat{x}\|_{l^2} + \|C^*\| \, \|\hat{x}\|_{l^2}$$

and (10) is proved.

Proposition 2 will be used in the case when the matrix C is generated by one sequence $(b_n, n \in \mathbf{P})$, where $\mathbf{P} = \mathbf{N} \backslash \{0\}$.

Starting from the sequence $(b_n, n \in \mathbf{P})$ we introduce the matrix C with rows satisfying certain lacunarity conditions given by the diversity. Fix $j \in \mathbf{P}, 2^{n-1} \leq m < 2^n$ for some $n \in \mathbf{P}$ and set

$$c_{lm} := \begin{cases} b_k, & \text{if } \ l = k2^n + m \ \text{ and } \ l(k) = j; \\ 0, & \text{elsewhere.} \end{cases}$$

The norm of $C = (c_{lm})_{l,m \in \mathbf{P}}$ induced by the l^2-norm is estimated in

Proposition 3. *Let $(b_n, n \in \mathbf{P})$ be in l^2. Then*

$$\|C\| \leq \|b\|_{l^2}.$$

Proof. We show that proposition 2 can be applied. To this end it is enough to prove that condition (9) is satisfied. We have to prove that the sets

$$J_m := \{k2^n + m : k = 1, 2, \ldots, l(k) = j\}$$

$2^{n-1} \leq m < 2^n, n \in \mathbf{P}$ are pairwise disjoint. In fact, let $m, \tilde{m} \in \mathbf{P}$ such that

$$2^{n-1} \leq m < 2^n, \ 2^{\tilde{n}-1} \leq \tilde{m} < 2^{\tilde{n}}, \qquad n, \tilde{n} \in \mathbf{P}.$$

Suppose that $n \leq \tilde{n}$ and $k2^n + m = \tilde{k}2^{\tilde{n}} + \tilde{m} \in J_m \cap J_{\tilde{m}}$. Since $l(k) = l(\tilde{k}) = j$, therefore k and $\tilde{k}$ are of the form

$$k = 2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_j}, \qquad \tilde{k} = 2^{\tilde{\nu}_1} + 2^{\tilde{\nu}_2} + \ldots + 2^{\tilde{\nu}_j}$$

with some $\nu_1 > \nu_2 > \ldots > \nu_j \geq 0$ and $\tilde{\nu}_1 > \tilde{\nu}_2 > \ldots > \tilde{\nu}_j \geq 0$.
Thus

$$2^{\nu_1+n} + 2^{\nu_2+n} + \ldots + 2^{\nu_j+n} + m = 2^{\tilde{\nu}_1+\tilde{n}} + 2^{\tilde{\nu}_2+\tilde{n}} + \ldots + 2^{\tilde{\nu}_j+\tilde{n}} + \tilde{m},$$

where $2^n > m$ and $2^{\tilde{n}} > \tilde{m}$ and consequently

$$\nu_i + n = \tilde{\nu}_i + \tilde{n} \qquad \text{for } \; i = 1, 2, \ldots, j.$$

Hence we get $m = \tilde{m}$.

We shall see that conditions (8) and (9) are satisfied in many cases. To show this, let $f : [0, 1] \to \mathbf{R}$ defined by

$$f = 1 - \sum_{k=1}^{\infty} b_k w_k \tag{11}$$

extended to $\mathbf{R}$ by periodicity with period 1 and satisfying the following condition

$$\|f\|_* = \sum_{j=1}^{\infty} \left(\sum_{l(k)=j} |b_k|^2 \right)^{\frac{1}{2}} < 1. \tag{12}$$

Using Proposition 2 we shall prove that this function generates an exact frame in $L_0^2[0, 1]$.

Theorem 1. *Let f be a function given by (11) and satisfying condition (12). Then the system $\phi = (\varphi_m, m \in \mathbf{P})$ defined by*

$$\varphi_m(x) := w_m(x) f(2^n x) \qquad (2^{n-1} \leq m < 2^n, n \in \mathbf{P}) \tag{13}$$

is an exact frame in $L_0^2[0, 1]$.

Proof. Since for any $m < 2^n$ and $k, n \in \mathbf{N}$

$$w_k(2^n x) = w_{2^n k}(x), \qquad w_m(x) w_{k2^n}(x) = w_{m+k2^n}(x)$$

the Fourier expansion of φ_m is of the form

$$\varphi_m = w_m - \sum_{k=1}^{\infty} b_k w_{k2^n+m} \qquad (2^{n-1} \leq m < 2^n, n \in \mathbf{P}).$$

This can be written in the form

$$\varphi_m = w_m - \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} c_{lm}^{(j)} w_l \qquad (2^{n-1} \le m < 2^n, n \in \mathbf{P}),$$

where for any $j, n \in \mathbf{P}$, $2^{n-1} \le m < 2^n$

$$c_{lm}^{(j)} = \begin{cases} b_k, & \text{if } l = k2^n + m \text{ and } l(k) = j; \\ 0, & \text{elsewhere.} \end{cases}$$

Using the notation introduced before, the matrix of the frame operator is

$$A = I - C,$$

where $C = \sum\limits_{j=1}^{\infty} C^{(j)}$ and $C^{(j)}$ is defined by

$$C^{(j)} = \left[c_{lm}^{(j)}\right]_{l,m\in\mathbf{P}}.$$

By Proposition 3 we have that $\|C^{(j)}\| \le \left(\sum\limits_{l(k)=j} |b_k|^2\right)^{\frac{1}{2}}$, consequently

$$\|C\| = \sum_{j=1}^{\infty} \|C^{(j)}\| \le \sum_{j=1}^{\infty} \left(\sum_{l(k)=j} |b_k|^2\right)^{\frac{1}{2}}$$

Then by Proposition 2 we have that ϕ is a frame in $L_0^2[0,1]$.

It is clear that

$$\|b\|_{l^2} \le \sum_{j=1}^{\infty} \left(\sum_{l(k)=j} |b_k|^2\right)^{\frac{1}{2}} \le \|b\|_{l^1}.$$

This implies

Corollary 1. *If $\|b\|_{l^1} < 1$ then the system (13) is an exact frame.*

3. Examples

In this section we give some examples of functions satisfying the conditions of Theorem 1. In these examples we use the Walsh-Fourier expansion of the integrated

Walsh functions. It is known (see [4], p. 27) that for $m = 2^n + m'$ with $0 \leq m' < 2^n, n \in \mathbf{N}$

$$(14) \qquad J_m(x) := \int_0^x w_m(t)dt = w_{m'}(x)J_{2^n}(x)$$

and

$$(15) \qquad J_{2^n}(x) = 2^{-n-2}\left(1 - \sum_{j=1}^{\infty} 2^{-j} w_{2^{n+j}+2^n}(x)\right) \qquad (x \in \mathbf{R}).$$

In the case $m = 0$, we have

$$J_0(x) = x = 2^{-1} - 2^{-2}\sum_{j=0}^{\infty} 2^{-j} w_{2^j}(x) \qquad (0 \leq x < 1).$$

3.1. First we investigate the frame obtained from

$$(16) \qquad \rho_1(x) := 2^2 \int_0^x w_1(t)dt = 2^2 J_1(x) \qquad (x \in \mathbf{R})$$

via the formula (1), using the sequence

$$(17) \qquad \varphi_m(x) = w_m(x)\rho_1(2^n x) \qquad (2^{n-1} \leq m < 2^n, n \in \mathbf{P}).$$

(Compare with (13) in Theorem (1).) From (14) and (15) we get

$$\rho_1(x) = 1 - \sum_{i=1}^{\infty} 2^i w_{2^i+1},$$

and it is easy to check that

$$\|\rho_1\|_* = \frac{1}{\sqrt{3}}.$$

Consequently by Theorem 1 the system φ_m in (17) is an exact frame. Later we give an explicit form for its inverse frame.

3.2 From the function (16) by integration we obtain

$$\rho_2(x) := 2^2 \int_0^x \rho_1(2t)w_1(t)dt =$$

$$= 1 - \frac{2}{3} w_3 - \sum_{j=2}^{\infty} 2^{-j} w_{2^j+1} + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2^{-2j-k-1} w_{2^{j+k+1}+2^{j+1}+2+1}.$$

It is easy to verify that

$$\|\rho_2\|_* = \frac{1}{6} \left(\sqrt{19} + \frac{1}{\sqrt{5}} \right),$$

consequently by Theorem 1 the system

$$\varphi_m(x) := w_m(x) \rho_2(2^n x) \qquad (2^{n-1} \le m < 2^n, n \in \mathbf{P})$$

is an exact frame in $L_0^2[0,1]$.

3.3 Generalizing the procedure introduced in (16) we define a sequence of mother wavelets by the recursion

$$\rho_n(x) := 2^2 \int_0^x \rho_{n-1}(2t) w_1(t) dt \qquad (n \in \mathbf{P}, x \in \mathbf{R}),$$

where $\rho_0(t) \equiv 1$ for $t \in \mathbf{R}$. The functions ρ_n are periodic with period 1 and $\|\rho_n\|_*$ can be estimated, can be used to construct exact frames.

4. The inverse frame

In this section we give an explicit form for the inverse frame in the case when the mother wavelet is given by (16). In this case the frame φ_m given by (17) can be written in the form

$$\varphi_m = w_m - \sum_{j=1}^{\infty} 2^{-j} w_{m+2^n+2^{n+j}} = w_m - \sum_{n=0}^{\infty} c_{lm} w_l,$$

where for m with $2^{n-1} \le m < 2^n$, $n \in \mathbf{P}$ we have

$$(18) \qquad c_{lm} = \begin{cases} 2^{-j}, & \text{if } l = m + 2^n + 2^{n+j},\ j \in \mathbf{P}; \\ 0, & \text{elsewhere.} \end{cases}$$

In this case already we have seen in 3.1 that

$$\|\rho_1\|_* = \|C\| = \frac{1}{\sqrt{3}},$$

consequently the proof of Proposition 2 (ii) shows that the frame constants are $\left(1-\frac{1}{\sqrt{3}}\right)$ and $\left(1+\frac{1}{\sqrt{3}}\right)$.

To get the inverse frame $(\psi_m, m \in \mathbf{N})$ of $(\varphi_m, m \in \mathbf{N})$ we use Proposition 1. Namely the Fourier expansion of the inverse frame $(\psi_m, m \in \mathbf{N})$ can be expressed by the inverse of A, which can be obtained in the next explicit form.

Proposition 4. *Let C be defined by (18). Then $A = I - C$ has an inverse matrix and $D := A^{-1}$ has the following form: if $2^{n-1} \le k < 2^n$ with some $n \in \mathbf{P}$ and $m \in \mathbf{P}$ is of the form*

$$m = k + 2^n + 2^{n_1} + 2^{n_1+1} + \ldots + 2^{n_{l-1}} + 2^{n_{l-1}+1} + 2^{n_l}, \tag{19}$$

where $n < n_1 < n_1 + 1 < n_2 < \ldots < n_{l-1} < n_{n-l} + 1 < n_l$, then

$$d_{mk} = 2^{-(n_l - n - (l-1))}$$

and $d_{mk} = 0$ elsewhere.

Proof. Since $||C|| < 1$, C has an inverse which can be written in the form

$$D = I + C + C^2 + \ldots + C^n + \ldots.$$

We denote the entries of the matrices C^l by $[c_{ij}^{(l)}]_{i,j\in\mathbf{P}}$ and define them by recursion. If $n = 1$ then $c_{ij}^{(1)} = c_{ij}$ is defined by (18) and for $i, j \in \mathbf{P}$ we have

$$c_{ij}^{(l+1)} = \sum_{k=1}^{\infty} c_{ik}^{(l)} c_{kj}.$$

We prove that for all $l \in \mathbf{P}$ and $2^{n-1} \le k < 2^n, n \in \mathbf{P}$ and for m of the form (19) we have

$$c_{mk}^{(l)} = 2^{-(n_l - n - (l-1))} \tag{20}$$

and $c_{mk}^{(l)} = 0$ elsewhere. This claim can be proved by induction.

Let $n \in \mathbf{P}$ and $2^{n-1} \le k < 2^n$, then

$$c_{mk}^{(2)} = \sum_{j=1}^{\infty} c_{mj} c_{jk}.$$

Since $c_{jk} \neq 0$ if $j = k + 2^n + 2^{n_1}, n_1 > n$ and $c_{mj} \neq 0$ if $m = j + 2^{n_1+1} + 2^{n_2}, n_2 > n_1 + 1$ it follows that

$$c^{(2)}_{mk} = c_{k+2^n+2^{n_1+1}+2^{n_2},k+2^n+2^{n_1}} c_{k+2^n+2^{n_1},k} = 2^{-(n_2-n-1)}.$$

Applying induction suppose that (20) holds. Then from

$$c^{(l+1)}_{mk} = \sum_{j=1}^{\infty} c^{(l)}_{mj} c_{jk}$$

using hypothesis (20) we get that $c_{jk} \neq 0$ if $j = k + 2^n + 2^{n_1}, n_1 > n$ and $c^{(l)}_{mj} \neq 0$ if m is of the form

$$m = j + 2^{n_1+1} + 2^{n_2} + 2^{n_2+1} + \ldots + 2^{n_l} + 2^{n_l-1} + 2^{n_l+1},$$

where $n_1 + 1 < n_2 < n_2 + 1 < \ldots < n_l < n_l + 1 < n_{l+1}$.
Consequently

$$c^{(l+1)}_{mk} = 2^{-(n_{l+1}-(n_1+1)-(l-1))} 2^{-(n_1-n)} = 2^{-(n_{l+1}-n-l)}$$

and $c^{(l+1)}_{mk} = 0$ elsewhere, which completes the proof.

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