Annales Univ. Sci. Budapest., Sect. Comp. 13 (1992) 195-224

SPLITTING AND MULTISPLITTING OF MATRICES

E.O. Amedjoe (Budapest, Hungary)

1. Introduction

In many physical applications, one must solve an $(n \times n)$ system of linear algebraic equations of the form

$$Ax = b \tag{1.1}$$

where A is the coefficient matrix, which might arise from a finite difference approximation to a partial differential equation, or input-output production and growth in economics, or linear complimentary problems in operations research, or Markov process in probability and statistics. There are many classical ways of solving the linear system (1.1).

In Varga [1] the definition of regular splitting of a matrix A was introduced in order to unify and generalize classical procedures in the numerical solution of systems of linear equations. With the advence of parallel computers O'Leary and White [3] proposed multisplitting as parallel methods for solving linear system of equations iteratively.

In this paper we present an overview of the various splittings and multisplittings with special reference to M-matrices, since many of the tools that are used to establish the convergence and the convergence rates of the basic iterative methods are based up on the theory of these matrices.

2. Definitions

Definition 2.1. A matrix $A > 0$, if for all i, j $a_{i,j} > 0$.

Definition 2.2. Let A, M, N be $(n \times n)$ matrices. Then (M, N) duple is called a *splitting* of A if $A = M - N$ and M is invertible or nonsingular. Usually, we refer to the splitting $A = M - N$ as

(a) *non-trivial* if $N \neq 0$;

(b) *weak-regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;

(c) *regular* if $M^{-1} \geq 0$ and $N \geq 0$;

(d) *M-splitting* if M is an M-matrix and $N \geq 0$;

(e) *proper* if the null spaces of A and M are equal and the ranges of A and M are also equal.

Definition 2.3. A real $(n \times n)$ matrix $A = (a_{i,j})$ with $a_{i,j} \leq 0$ for $i \neq j$ is an M-matrix if A is nonsingular and $A^{-1} \geq 0$.

Definition 2.4. A real $(n \times n)$ matrix $A = (a_{i,j})$ with $a_{i,j} \leq 0$ for $i \neq j$ is a Stieltjes matrix if A is symmetric and positive definite.

Definition 2.5. Let A, M, N be $(n \times n)$ matrices. Let $A = M - N$ be a splitting of A, M nonsingular, $H = M^{-1}N \geq 0$ and $c = M^{-1}b$. Assume that the spectral radius of H $\rho(H) < 1$. Let $x = Hx + c$. Then for

$$\alpha = \sup \left[\lim_{k \to \infty} \|x^k - x\|^{1/k} : x^0 \in C^n \right]$$

the number

$$R_\infty = -\ln \alpha$$

is called the asymptotic rate of convergence of the iteration $x^{k+1} = Hx^k + c$.

Definition 2.6. With $S \subseteq R^n$ we associate two sets S^G, the set generated by S, which consists of all finite nonnegative linear combinations of elements of S and S^*, the dual of S, defined by

$$S^* = [y \subset R^n : x \in S \to (x, y) \geq 0]$$

where $(,)$ denotes the inner product. K is defined to be a *cone* if $K = K^G$.

Definition 2.7. If A is nonsingular, then $X = A^{-1}$ satisfies

$$AXA = A \tag{2.7.1}$$

$$XAX = X \tag{2.7.2}$$

$$AX = (AX)^T \tag{2.7.3}$$

$$XA = (XA)^T \tag{2.7.4}$$

$$XA = AX \tag{2.7.5}$$

$$A^k = XA^{k+1} \tag{2.7.6}$$

(a) If A is nonsingular such that X satisfies (2.7.1), then X is called the *generalized inverse (g-inverse)* of A.

(b) If X satisfies (2.7.1) – (2.7.4), then X is called the *Moore-Penrose generalized inverse* of A.

3. Regular splittings

It is noted that the splittings for many iterative techniques such as Jacobi, Gauss-Seidel and successive overrelaxation are obtained by splitting the coefficient matrix A into the difference of two $(n \times n)$ matrices M and N. From (1.1)

$$Ax = b$$

and from definition 2.2 we have

$$A = M - N \tag{3.1}$$

and substituting (3.1) in (1.1), we have

$$\begin{aligned}(M - N)x &= b\\ Mx &= Nx + b.\end{aligned}$$

The iterative scheme

$$x^{k+1} = M^{-1}Nx + M^{-1}b \tag{3.2}$$

is usually used.
From (3.1)

$$M = A + N,$$

hence the iteration matrix becomes

$$M^{-1}N = (A + N)^{-1}N. \tag{3.3}$$

By the assumption that A is nonsingular, we write (3.3) as

$$M^{-1}N = (I + G)^{-1}G \tag{3.4}$$

where

$$G = A^{-1}N.$$

If x is an eigenvector of G corresponding to the eigenvalue r then

$$\begin{aligned}Gx &= rx\\ (I + G)^{-1}Gx &= \frac{r}{1 + r}x.\end{aligned} \tag{3.5}$$

But from the relation (3.4) it follows that x is also an eigenvector of $M^{-1}N$ corresponding to the eigenvalue μ given by

$$\mu = \frac{r}{1+r}. \tag{3.6}$$

Conversely, if μ is any eigenvalue of $(I+G)^{-1}G$ with corresponding eigenvector z,

$$(I+G)^{-1}Gz = \mu z \tag{3.7}$$

then

$$Gz = \mu(I+G)z.$$

From this expression it follows that μ cannot be unity, so we can write

$$Gz = \frac{\mu}{1-\mu}z \equiv rz \tag{3.8}$$

which is again (3.6).

The convergence of these methods stresses on the condition that the spectral radius of the iteration matrix should be less than 1. The following theorems state the convergence criteria for the regular splittings.

Theorem 3.1. *If $A = M - N$ is a regular splitting of the matrix A and $A^{-1} \geq 0$ then*

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)} < 1.$$

Thus the iteration matrix is convergent and the iterative method (3.2) converges for any initial vector $x^{(0)}$. Conversely, if $\rho(M^{-1}N) < 1$ then $A^{-1} \geq 0$.

Proof. The proof is given in Varga [1, p. 89].

The following corollary of Theorem 3.1 states the condition for symmetric and positive matrices.

Corollary 3.1. *Let A be a Stieltjes matrix, and let $A = M - N$ be a regular splitting of A where N is symmetric and real. Then*

$$\rho(M^{-1}N) \leq \frac{\rho(N)\rho(A^{-1})}{1+\rho(N)\rho(A^{-1})} < 1.$$

Proof. See Varga [1, p. 90].

The following theorem gives the convergence condition for M-matrices.

Theorem 3.2. *Let A be $(n \times n)$ M-matrix. If M is any matrix obtained by setting certain off-diagonal entries to zero, then $A = M - N$ is a regular splitting of A and $\rho(M^{-1}N) < 1$.*

Proof. It follows directly from Theroem 3.1 and the fact the matrix is an M-matrix.

If we have different regular splittings for the same coefficient matrix, we would seek to compare the spectral radii of the iterative matrices. The following theorem summarizes the conditions for the comparison.

Theroem 3.3. *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} > 0$. If $N_2 \geq N_1 \geq 0$, $(N_2 - N_1 \geq 0)$, then*

$$0 < \rho\left(M_1^{-1}N_1\right) < \rho\left(M_2^{-1}N_2\right) < 1.$$

Proof. From Theorem 3.1 it follows that $\rho\left(M_i^{-1}N_i\right)$ $(i = 1, 2)$ is monotone with respect to $\rho(A^{-1}N_i)$.

Corollary 3.3. *Let A an irreducible $(n \times n)$ Stieltjes matrix, and let M_1 and M_2 be $(n \times n)$ matrices, each obtained by setting certain off-diagonal entries of A to zero. If $A = M_1 - N_1 = M_2 - N_2$ and $N_1 \geq N_2 \geq 0$, $(N_1 - N_2 > 0)$, then*

$$0 < \rho\left(M_2^{-1}N_2\right) < \rho\left(M_1^{-1}N_1\right) < 1.$$

Proof. This can be proved using Theorem 3.3 and the fact that a Stieltjes matrix is also an M-matrix and also that when a matrix is irreducible then $A^{-1} > 0$.

An interesting application of the comparison theory is summarized in the following theorem as

Theorem 3.4. *Let $A = I - B$, where $B = L + U$ is a nonnegative irreducible, and convergent $(n \times n)$ matrix, and L and U are strictly lower and upper triangular matrices. Then the successive overrelaxation iteration matrix $\mathcal{L}_\omega$ is convergent for all $0 < \omega \leq 1$. Moreover, if $0 < \omega_1 < \omega_2 \leq 1$ then*

$$0 < \rho(\mathcal{L}_{\omega_2}) < \rho(\mathcal{L}_{\omega_1}) < 1.$$

Proof. Let $A = M_\omega - N_\omega$ where $M_\omega = \dfrac{1}{\omega}(I - \omega L)$ and $N_\omega = \dfrac{1}{\omega}\left(\omega U + (1-\omega)I\right)$, then the splitting is regular for all $0 < \omega \leq 1$. If $0 < \omega_1 < \omega_2 \leq 1$, then $0 \leq N_{\omega_2} \leq N_{\omega_1}$ and as $A^{-1} > 0$, from Theorem 3.3 we have the result.

The results of the earlier theorems have been extended by Ortega and Rheinbolt [4] and Vandergraft [5]. The following theorems 3.6 and 3.7 are new comparison theorems for regular splittings which have been stated in the work of Csordas and

Varga [6] as an extension of the original results of Varga [1]. Before starting these, let us consider the following theorem, a less well-known theorem by Wozincki, but which is nonetheless useful in applications.

Theorem 3.5. *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then $\rho\left(M_2^{-1}N_2\right) \geq \rho\left(M_1^{-1}N_1\right)$. In particular, if $M_1^{-1} > M_2^{-1}$ and if $A^{-1} > 0$, then*

$$\rho\left(M_2^{-1}N_2\right) > \rho\left(M_1^{-1}N_1\right).$$

Proof. The detailed proof can be found in Wozincki [7].

In order to generalize Theorem 3.3 with the assumption that $A^{-1} \geq 0$ and Theorem 3.5, Csordas and Varga [6] proposed the following:

Proposition 3.1. *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. Then*

(i) $N_2 \geq N_1 \Rightarrow M_1^{-1} \geq M_2^{-1}$;

(ii) $M_1^{-1} \geq M_2^{-1} \Rightarrow A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1}$;

(iii) $A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1} \Rightarrow (A^{-1}N_2)^jA^{-1} \geq (A^{-1}N_1)^jA^{-1}$ *for each positive integer* $j > 1$.

Remark. The reverse implication in (i), (ii) or (iii) in the proposition is not in general true.
The generalized comparison theorem is stated as follows:

Theorem 3.6. *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. Assume that there exists a positive integer j for which*

$$(A^{-1}N_2)^jA^{-1} \geq (A^{-1}N_1)^jA^{-1}.$$

Then $\rho\left(M_1^{-1}N_1\right) \leq \rho\left(M_2^{-1}N_2\right) < 1$.

Proof. A detailed proof is given in [6].
Let a set which depends on the matrices A, M_1, N_1, M_2 and N_2 be defined as

$$S := [\textit{positive integers} \;\; j : (A^{-1}N_2)^jA^{-1} \geq (A^{-1}N_1)^jA^{-1}].$$

Proposition 3.2. *The set S is closed under addition.*

Proof. For a proof see [6].
This proposition is incorporated into the following theorem as

Theorem 3.7. *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where it is assumed that $A^{-1} \geq 0$. If $\rho\left(M_2^{-1}N_2\right) > \rho\left(M_1^{-1}N_1\right)$, there exists a positive integer j_0 for which*

$$(A^{-1}N_2)^j A^{-1} > (A^{-1}N_1)^j A^{-1}$$

for all $j \geq j_0$. Consequently, if $\rho\left(M_2^{-1}N_2\right) > \rho\left(M_1^{-1}N_1\right)$, then the set S is not empty and in fact S contains all sufficiently large positive integers. Consequently, if there is a positive integer j for which $(A^{-1}N_2)^j A^{-1} > (A^{-1}N_1)^j A^{-1}$ then

$$\rho\left(M_2^{-1}N_2\right) > \rho\left(M_1^{-1}N_1\right).$$

Proof. See Csordas and Varga [6].

When $j = 1$ in Theorem 3.7, we have the following corollary, which generalizes the second part of the preceding theorem as

Corollary 3.7. *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where it is assumed that $A^{-1} > 0$. If $A^{-1}N_2A^{-1} > A^{-1}N_1A^{-1}$ then $\rho\left(M_2^{-1}N_2\right) > \rho\left(M_1^{-1}N_1\right)$.*

In another paper by Miller and Neumann [9] it was shown that there is a relationship between Csordas and Varga [6] and the paper presented by Beauwens [8].

4. Proper splittings

The concept of proper splitting of a rectangular matrix, that being $A = M - N$, where the ranges and null spaces of A and M are the same was introduced in Plemmons [10]. In the paper M^{-1} of (3.2) was replaced by M^{+}, the Moore-Penrose generalized inverse (g-inverse) of M, and showed that the iteration (3.2) converges if and only if $\rho(M^{+}N) < 1$.

Due to the rapidly increasing applications of g-inverses in network theory, mathematical programming and mathematical statistics Lawson [11] developped and extended the results in [10] using least-square g-inverses, minimum-norm g-inverses and g-inverses.

In this section the matrices are considered to be $(n \times n)$ matrices. The following theorem states the sufficient condition for the convergence of the iteration (3.2).

Theorem 4.1. *Let A, M, N be $(n \times n)$ matrices. Let $A = M - N$. Let A and M be nonsingular and let $A^{-1} \geq 0$, $M^{-1} \geq 0$, $N \geq 0$, then $\rho(M^{-1}N) < 1$.*

Proof. The proof is similar to that for Theorem 3.1.

The following theorem gives a further condition under which the iteration converges for each x^0.

Theorem 4.2. *Let A, M, N be $(n \times n)$ matrices. Let $A = M - N$. Let A and M be nonsingular and let*

$$M^T y \geq 0 \Rightarrow N^T y \geq 0$$
$$A^T y \geq 0 \Rightarrow N^T y \geq 0$$

where T denotes the transpose of the matrix, then

$$\rho(M^{-1}N) < 1.$$

Proof. $A^{-1} > 0 \leftrightarrow N^T A^{-T} \leftrightarrow x \geq 0 \rightarrow N^T A^{-T} \geq 0$. Thus $A^{-T} y \geq 0 \rightarrow N^T y \geq 0$ then for $x = A^T y$, $x \geq 0 \rightarrow N^T A^{-T} \geq 0$ so that $A^{-1}N \geq 0$. Conversely, if $A^{-1}N \geq 0$ then for $x = A^T y \geq 0$ it follows that $N^T y = N^T A^{-T} x \geq 0$ thus the implication $A^T y \rightarrow N^T y \geq 0$ is equivalent to $A^{-1}N \geq 0$. Similarly, $M^T y \geq 0 \rightarrow N^T y \geq 0$ is equivalent to $M^{-1}N \geq 0$. The result follows from Theorem 4.1.

Remark. Theorem 4.1 is included in Theorem 4.2, but not conversely. The theorems for conditions of convergence for the iteration (3.2) using the generalized inverse of $M(M^{-1})$ are stated as follows:

Theorem 4.3. *Let $A = M - N$ be a proper splitting of A $(m \times n)$ matrix and let M^- denote the generalized inverse (g-inverse) of M, then*

(i) $A = M(I - M^- N)$;

(ii) $I - M^- N$ *is nonsingular;*

(iii) $A^{-1} = (I - M^- N)^{-1} M^-$ *is the generalized inverse of A, and*

(iv) $A^- b$ *is the unique solution to the system $x = M^- N x + M^- b$ for any $b \in R^m$.*

Proof. See [11, Theorem 1].

Remarks. For a proper splitting $A = M - N$, $M = A + N$ is a proper splitting of M. Thus if A^- is a g-inverse of A, then $M = A(I + A^- N)$, $I + A^- N$, is nonsingular and $(I + A^- N)^{-1}$ is the g-inverse of M.

Corollary 4.3.1. *Let A be an $(m \times n)$ matrix and $A = M - N$ be a proper splitting of A, then*

(i) $A_m^- = (I - M_m^- N)^{-1} M_m^-$ *is a minimum-norm g-inverse of A;*

(ii) $A_l^- = (I - M_l^- N)^{-1} M_l^-$ *is a least-square g-inverse of A;*

(iii) $A^+ = (I - M^+N)^{-1}M^+$ *is a Moore-Penrose g-inverse of A.*

Proof. See [11, Corollary 1].

Corollary 4.3.2. *Let A be an $(m \times n)$ matrix, $A = M - N$ a proper splitting of A and b m-vector. The iteration $x^{k+1} = M^-Nx^k + M^-b$ converges to A^-b, where A^- is the same type of generalized inverse of A as M^- is of M for every x^0, if and only if $\rho(M^{-1}N) < 1$.*

Proof. See [11, Corollary 2] for detailed proof.

The following theorem states the conditions for which $\rho(M^{-1}N)$ is less than 1.

Theorem 4.4. *Let K be a positive cone in R^n and let $A = M - N$ be a proper splitting of A such that $M^-NK \subseteq K$. If $A^- = (I - M^-N)^{-1}M^-$, then $\rho(M^-N) = \dfrac{\rho(A^-N)}{1 + \rho(A^-N)}$ and hence $\rho(M^-N) < 1$, if and only if $A^-NK \subseteq K$.*

Proof. The proof is similar to that of Theorem 2 in [10].

The next theorem is true only for minimum-norm g-inverses.

Theorem 4.5. *Let $A = M - N$ be a proper splitting of the $(m \times n)$ matrix A. Let M_m^- denote any minimum-norm g-inverse of M and let $A_m^- = (I - M_m^-N)^{-1}M_m$, then the following statements are equivalent:*

(i) $A_m^-N \geq 0$ *and* $M_m^-N \geq 0$;

(ii) $AX = N$ *for some* $X \in R^{n \times n}$, $X \geq 0$, $R(X) \subseteq R(A^T)$, $M^-Y = N$ *for some* $Y \in R^{n \times n}$, $Y \geq 0$, $R(Y) \subseteq R(M^T)$;

(iii) $A^Ty \in R_+^n \oplus \mathcal{N}(A) \Rightarrow N^Ty \geq 0$, $M^Ty \in R_+^n \oplus \mathcal{N}(A) \Rightarrow N^Ty \geq 0$ *where* $\mathcal{N}(A)$ *is the null space of A.*

Proof. See [11] for detailed proof.

Corollary 4.5.1. *Let $A = M - N$ be a proper splitting of an $(m \times n)$ matrix A. Let M_m^- denote any minimum-norm g-inverse of M and let $A_m^- = (I - M_m^-N)^{-1}M_m$. If A, M, N satisfy (i), (ii), or (iii), then $\rho(M_m^-N) = \dfrac{\rho(A_m^-N)}{1 + \rho(A_m^-N)}$.*

Proof. It follows directly from Theorems 4.4 and 4.5 where $K \subseteq R_+^n$.

Theorem 4.6. *Let $L \subseteq R^m$, $K \subseteq R^n$ be positive cones and let $A = M - N$ be a proper splitting of A an $(m \times n)$ matrix, where M^- is the g-inverse of M, and $A^- = (I - M^-N)^{-1}M^-$. Suppose that $M^-L \subseteq K$ and $M^-NK \subseteq K$. Then the following statements are equivalent:*

(i) $A^-L \subseteq K$;

(ii) $A^- NK \subseteq K$; *and*

(iii) $\rho(M^- N) = \dfrac{\rho(A^- N)}{1 + \rho(A^- N)}$.

Proof. The proof is similar to that of Theorem 3 in [10].

Suppose that $A = M_1 - N_1 = M_2 - N_2$ are two different proper splittings, then we have $A_1^- = (I - M_1^- N_1)^{-1} M_1^-$ and $A_2^- = (I - M_2^- N_2)^{-1} M_2^-$ as the two g-inverses. We can therefore compare the spectral radii of $M_1^- N_1$ and $M_2^- N_2$. The following theorem states the comparison conditions.

Theorem 4.7. *Let* $A = M_1 - N_1 = M_2 - N_2$ *be two proper splittings of* A, *where* $N_2 \geq N_1 \geq 0$, $A^- = (I - M_1^- N_1)^{-1} M_1^- = (I - M_2^- N_2)^{-1} M_2^-$. *If* $A^- > 0$, $M_i^- N_i \geq 0$ *and* $M_i \geq 0$ $(i = 1, 2)$, *then* $0 < \rho(M_1^- N_1) < \rho(M_2^- N_2) < 1$.

Proof. From Theorem 4.6 $\rho(M_i^- N_i) = \dfrac{\rho(A^- N_i)}{1 + \rho(A^- N_i)}$ $(i = 1, 2)$. It can be shown that $0 < \rho(A^- N_1) < \rho(A^- N_2)$. The proof is similar to the proof of Theorem 1 in [1].

In the matrix $M_i^- N_i$ converges to zero $(i \to \infty)$, then we can consider the asymptotic rate of convergence $R_\infty(M_i^- N_i)$, where

$$R_\infty(M_i^- N_i) = -\ln \rho(M_i^- N_i).$$

The following corollary states this condition.

Corollary 4.7. *Let* $A = M_1 - N_1 = M_2 - N_2$ *be two proper splittings of* A, *an* $(m \times n)$ *matrix. Let* $A^- = (I - M_1^- N_1)^{-1} M_1^- = (I - M_2^- N_2)^{-1} M_2^-$. *If* $A^- > 0$, $M_i^- N_i \geq 0$ *and* $M_i^- \geq 0$ $(i = 1, 2)$, *then* $R_\infty(M_2^- N_2) < R_\infty(M_1^- N_1)$.
Similar results have been obtained for nonsingular matrices as in Varga [16].

In Berman and Neumann [12] proper splittings of rectangular matrices into $(r \times r)$ nonsingular matrices and their convergence results have been studied. In Berman and Neumann [13] the concepts of consistency and least-square consistency as introduced by Young [14] and Kammerer and Plemmons [15] have been extended to the various types of splittings of A to obtain a general outlook on iterative methods for solving the system (1.1).

5. Parametric splitting

In this section we point out that all the iterative methods can be described from the viewpoint of system (1.1). We assume that all the diagonal entries of the coefficient matrix A are all non-zero. We can then express A as the matrix sum

$$A = D - L - U$$

where $D = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$, L=strictly lower triangular $(n \times n)$ matrix, U=strictly upper triangular $(n \times n)$ matrix. With the choice

(i) $M = D$ and $N = L + U$, we have the point Jacobi iterative method with its operator

$$J = D^{-1}(L + U);$$

(ii) $M = D - L$ and $N = U$ we have the point Gauss-Seidel iterative method with its operator

$$GS = (D - L)^{-1}U;$$

(iii) with the introduction of a relaxation factor or parameter ω and $M = \frac{1}{\omega}(D - \omega L)$ and $N = \frac{1}{\omega}\big(\omega U + (1 - \omega D)\big)$ we have the point successive over-relaxation iterative method with its operator

$$T(\omega) = (D - \omega L)^{-1}\big((1 - \omega)D + \omega U\big).$$

Remark. With the SOR-method, by selecting $\omega = 1$, the method reduces to the point Gauss-Seidel iterative method.

The following theorem of comparison of the rate of convergence concerns the point Jacobi operator and the Gauss-Seidel operator.

Theorem 5.1. *Let the Jacobi matrix $B = L+U$ be nonnegative $(n \times n)$ matrix with zero diagonal entries, and let GS be the Gauss-Seidel operator. Then one and only one of the following mutually exclusive relations is valid:*

(i) $\rho(B) = \rho(GS) = 0$;

(ii) $0 < \rho(GS) < \rho(B) < 1$;

(iii) $1 = \rho(B) = \rho(GS)$;

(iv) $1 < \rho(B) < \rho(GS)$.

Thus, the Jacobi matrix B and the Gauss-Seidel operator are either both convergent, or both divergent.

Proof. See Varga [16, pp. 69-70].

An immediate corollary for this theorem is stated as follows:

Corollary 5.1. *If the nonnegative Jacobi matrix B is such that* $0 < \rho(B) < 1$, *then*

$$R_\infty(GS) > R_\infty(B).$$

With the SOR method for a remarkable decreasing $\rho(M^{-1}N)$ and for increasing the rapidity of convergence is introduced a parameter ω, so that the diagonal entries of A are splitted as in (iii) with the T operator. A further method as examined by Sisler in [17] – [19] resulted in splitting the lower triangular matrix L as

$$M = D - \beta L$$

and

$$N = (1-\beta)L + U$$

with the iterative operator

$$S(\beta) = (D - \beta L)^{-1}\big((1-\beta)L + U\big).$$

Remark. Similar conclusions hold by the splitting of U. Sisler [19] – [21], who studied two-parametric methods by combining the two splittings discussed above for cyclic matrices. Here we have

$$M = \frac{1}{\psi}D - \beta L$$

and

$$N = \left(\frac{1}{\psi} - 1\right) D + (1-\beta)L + U$$

with the iteration operator

$$V(\psi, \beta) = (D - \psi\beta L)^{-1}\big((1-\psi)D + (1-\beta)\psi L + \psi U\big).$$

Here we observed two special cases (i) $V(\psi, 1) = T(\psi)$ the SOR method, and (ii) $V(1, \beta) = S(\beta)$ the Sisler's method.
The following lemma shows that the two-parametric method is closely related to that of the SOR.

Lemma 5.1. *If $T(\omega)$ is the SOR-operator and $V(\psi, \beta)$ is the operator of the two-parametric method, then*

$$V(\psi, \beta) = \alpha T(\omega) + (1-\alpha)D =: Z(\omega, \alpha)$$

holds for $\alpha = \frac{1}{\beta}$, $\omega = \psi\beta$ that is $\psi = \omega\alpha$, $\beta = \frac{1}{\alpha}$, $\alpha, \beta \neq 0$.

Proof.

$$V(\psi, \beta) = (D - \psi\beta L)^{-1}\big((1-\psi)D + (1-\beta)\psi L + \psi U\big).$$

By substituting the parameters α and ω as defined in the lemma, we get

$$V(\psi,\beta) = (D-\omega L)^{-1}\left((1-\omega\alpha)D + \left(1-\frac{1}{\alpha}\right)\omega\alpha L + \omega\alpha U\right) =$$
$$= (D-\omega L)^{-1}(\alpha(1-\omega)D + \omega\alpha U(1-\alpha)D - \omega(1-\alpha)L) =$$
$$= \alpha(D-\omega L)^{-1}((1-\omega)D + \omega U) + (1-\alpha)(D-\omega L) - 1(D-\omega L) =$$
$$= \alpha T(\omega) + (1-\alpha)D =: Z(\omega,\alpha).$$

Remarks. $T(\omega) = Z(\omega,1)$ which is the SOR operator and for $S(\omega) = = Z\left(\omega,\frac{1}{\omega}\right) = \frac{1}{\omega}T(\omega) + \left(1-\frac{1}{\omega}\right)D$ we have the Sisler's operator. Assuming that $r(\omega)$ is the eigenvalue of $T(\omega)$ then $\zeta(\omega,\alpha) = \alpha r(\omega) + (1-\alpha)$ is the eigenvalue of $Z(\omega,\alpha)$. It follows that $|\zeta(\omega,\alpha)| < 1$, if and only if $\left|r(\omega) - \left(1-\frac{1}{\alpha}\right)\right| < 1$ for $\alpha > 0$.

The following lemma determines the region in (ω,α) plane where $Z(\omega,\alpha)$ is convergent for the noncyclic case.

Lemma 5.2. *Let $A = D-L-U$ and let r be an eigenvalue of $T(\omega)$ with the eigenvector y and $y*y=1$. Then the following holds:*

$$r = \frac{\gamma - a + \sigma}{\gamma + a + \sigma}$$

with $\gamma = \frac{(2-\omega)}{\omega}$, $\alpha = y*Ay$ *and* $\sigma = y*(U-L)y$.

Proof. A proof is given in [23].

Theorem 5.1. *If A is positive definite with eigenvalues $0 < \lambda_1 \le \ldots \le \lambda_n$ and $\lambda_1 < 1$, $Z(\omega,\alpha)$ is convergent for all (ω,α) with $0<\omega\le 2$, $0<\alpha<\frac{\gamma+\lambda_n}{\lambda_n}$ with $\gamma = \frac{2-\omega}{\omega}$, $2<\omega<\frac{2}{1-\lambda_1}$, $0<\alpha<\frac{\gamma+\lambda_1}{\lambda_1}$ where $\gamma=\frac{2-\omega}{\omega}$.*

Proof. For the proof of this theorem see Niethammer [24].
An immediate corollary to this theorem is stated as follows:

Corollary 5.1. *Under the assumption of Theorem 5.1 and if $\lambda_n > 1$ the iteration*

$$(D-\beta L)x^{m+1} = (1-\beta)Lx^m + Ux^m + b$$

with operator $S(\omega)$ *is convergent for*

$$\omega_l =: \max\left(0, \frac{2-\lambda_n}{1-\lambda_n}\right) < \omega < \omega_r =: \frac{2-\lambda_1}{1-\lambda_1}.$$

Remark. $\rho\big(S(0)\big) = \rho(B)$, where $B = L - L^T$ is a symmetric matrix, is valid, so if $\rho(B) < 1$, then $S(\omega)$ is convergent for some $\tilde{\omega}_l < \omega \leq 0$.
The following lemma determines the region of convergence for cyclic matrices.

Lemma 5.3. *Let* $A = D - B$ *be consistently ordered p-cyclic matrix, where* $B = L - L^T$ *is symmetric. If* $\omega \neq 0$ *and if* $\xi \neq (1-\alpha)$ *is an eigenvalue of* $Z(\omega, \alpha)$ *and if* μ *satisfies*

$$(\xi - 1 + \omega\alpha)^p = \alpha(\xi + \alpha - 1)^{p-1}\omega^p\mu^p$$

then ξ *is an eigenvalue of* B. *Conversely, if* μ *is an eigenvalue of* B *and satisfies*

$$(\xi - 1 + \omega\alpha)^p = \alpha(\xi + \alpha - 1)^{p-1}\omega^p\mu^p$$

then ξ *is an eigenvalue of* $Z(\omega, \alpha)$.

Proof. See Sisler [20, theorem 1].
The following theorem is a special case of Lemma 5.2, that is when $p = 2$.

Theorem 5.2. *Let* $A = D - B$ *be symmetric, positive definite and let* B *be a weakly 2-cyclic consistently ordered matrix with eigenvalues* μ *and* $0 \leq \delta^2 \leq \mu^2 \leq$ $\leq \beta^2 = \rho^2(B) < 1$. *Let*

$$r_{1,2}(\beta) = 1 - \omega - \frac{\left(\beta^2\omega^2 \pm |\omega\mu|\sqrt{\omega^2\beta^2 + 4(1-\omega)}\right)}{2}$$

be the eigenvalues of the SOR operator $T(\omega)$ *corresponding to the eigenvalues* $\pm\beta$ *of* B. *Then the region* $\aleph$ *such that* $Z(\omega, \alpha)$ *is convergent for* $(\omega, \alpha) \in \aleph$ *is described by*

(a) $$\omega < 0, \qquad \frac{2}{1 - r_1(\beta)} < \alpha < 0$$

(b) $$0 < \omega < \omega'(\delta) := \frac{2}{1 + \sqrt{1 - \delta^2}}, \qquad 0 < \alpha < \frac{2}{1 - r_2(\beta)}$$

(c) $$\omega'(\delta) \leq \omega \leq 2, \qquad 0 < \alpha < \frac{2 - \omega\delta^2}{\omega(1 - \delta^2)}$$

(d) $$2 \le \omega < \frac{2}{\beta^2}, \qquad 0 < \alpha < \frac{2-\omega\beta^2}{\omega(1-\beta^2)}.$$

Remark. For an $(n \times n)$ matrix A, where n is odd, we have δ^2 to be necessarily zero and where n is even, a positive lower bound δ^2 for μ^2 cannot be determined. In (c) when $\delta^2 = 0$, the bound is $0 < \alpha < \frac{2}{\omega}$.

By letting $S(\omega)$ be convergent for $\omega_l < \omega < \omega_r$ from theorem 5.1, relations (a) and (d) we get the following conditions

$$\omega_l = \frac{2}{1 - r_1(\beta)}, \qquad \frac{1}{\omega_r} = \frac{2-\omega_r\beta^2}{\omega_r(1-\beta^2)}.$$

Thus, we have the following corollary.

Corollary 5.2. *Under the conditions of Theorem 5.2, Sisler's method with operator $S(\omega)$ is convergent for*

$$\frac{1}{2}\left(1 - \frac{1}{\beta^2}\right) =: \omega_l < \omega < \omega_r := 1 + \frac{1}{\beta^2}.$$

Proof. The proof can found in Sisler [17, theorem 4].

In Niethammer [24], the optimal choice of the parameters to maximize the two-parametric method with operator $Z(\omega, \alpha)$ is studied.

6. Quadrant interlocking splitting

In the previous sections the iterative methods discussed for solving the linear system (1.1) used the splitting of the coefficient matrix primarily into the form $A = D - L - U$, where D, L, U are the diagonal, strictly lower and strictly upper triangular entries of A respectively. In this section we shall introduce a new splitting of A by Evans [25] known as the Quadrant interlocking splitting.

Let $A = X - W - Z$ be the splitting of A, where the entries for X, W, Z are defined as

$$X = \begin{cases} a_{i,j} \\ a_{i,n-i+1}, & i = 1(1)n \end{cases}$$

$$-W = (a_{i,j}) = \begin{cases} 1 \le j < \lfloor \frac{n-1}{2} \rfloor, & j < i < n-j+1 \\ \lfloor \frac{n+2}{2} \rfloor < j \le n, & n-j+1 < i < j \\ 0, & \text{elsewhere} \end{cases}$$

and

$$-Z = (a_{i,j}) = \begin{cases} 1 \le i < \lfloor \frac{n+1}{2} \rfloor, & i < j < n-i+1 \\ \lfloor \frac{n+2}{2} \rfloor < i \le n, & n-j+1 < j < i \\ 0, & \text{elsewhere} \end{cases}$$

where the symbol $\lfloor \alpha \rfloor$ denotes the largest integer $\le \alpha$. The X, W, Z matrices are indicated by the following zero/nonzero patterns

$$X = \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\ 0 & X & 0 & 0 & 0 & 0 & 0 & X & 0 \\ 0 & 0 & X & 0 & 0 & 0 & X & 0 & 0 \\ 0 & 0 & 0 & X & 0 & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X & 0 & X & 0 & 0 & 0 \\ 0 & 0 & X & 0 & 0 & 0 & X & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & X & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\ X & X & 0 & 0 & 0 & 0 & 0 & X & X \\ X & X & X & 0 & 0 & 0 & X & X & X \\ X & X & X & X & 0 & X & X & X & X \\ X & X & X & 0 & 0 & 0 & X & X & X \\ X & X & 0 & 0 & 0 & 0 & 0 & X & X \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & X & X & X & X & X & X & X & 0 \\ 0 & 0 & X & X & X & X & X & 0 & 0 \\ 0 & 0 & 0 & X & X & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X & X & X & 0 & 0 & 0 \\ 0 & 0 & X & X & X & X & X & 0 & 0 \\ 0 & X & X & X & X & X & X & X & 0 \end{bmatrix}$$

Fig. 1.

The most important property of this splitting is the matrix X, in that it is an easily soluble matrix, consisting of (2×2) subsystems which can be obtained by standard procedures.

In Evans and Haghighi [26] this new splitting was extended to the corresponding iterative methods for solving linear systems of equations. In that paper the system (1.1) was replaced by an equivalent system

$$x = Bx + c$$

where

$$B = X^{-1}(W + Z) \quad \text{and} \quad c = X^{-1}b,$$

so that if

$$\Delta_i \equiv \det \begin{bmatrix} a_{i,i} & a_{i,n-i+1} \\ a_{n-i+1,i} & a_{n-i+1,n-i+1} \end{bmatrix} \neq 0, \quad i = 1(1)\left\lfloor \frac{n+1}{2} \right\rfloor$$

then X^{-1} exists.

Thus, it is analogous to the already familiar $A = D - L - U$ in Varga [16]. We can then formulate the following iterative methods:

(i) The simultaneous (Jacobi) Quadrant Interlocking iterative method (JQI) in the matrix form as

$$x^{k+1} = X^{-1}(W + Z)x^k + X^{-1}b = Bx^k + c.$$

(ii) Related to (i) with the introduction of a parameter ω we have the simultaneous overrelaxation Quadrant Interlocking iterative method (JOQI)

$$x^{k+1} = B_\omega x^k + \omega c$$

where

$$B_\omega = \omega B + (1 - \omega)I = \omega\big(X^{-1}(W + Z)\big) + (1 - \omega)I$$

so that when $\omega = 1$, we have the JQI method.

(iii) The successive Quadrant Interlocking iterative method (SQI) is formulated as

$$x^{k+1} = (X - W)^{-1}Zx^k + (X - W)^{-1}b$$

or

$$x^{k+1} = \mathcal{L}x^k + c$$

where

$$\mathcal{L} = (X - W)^{-1}Z \quad \text{and} \quad c = (X - W)^{-1}b.$$

(iv) Related to (iii) and the introduction of a parameter ω we have the successive overrelaxation Quadrant Interlocking iterative method (SOQI) as

$$x^{k+1} = \mathcal{L}_\omega x^k + \omega c$$

where

$$\mathcal{L}_\omega = (X - \omega W)^{-1}(\omega Z - (1-\omega)X).$$

Clearly, if $\omega = 1$, then the (SOQI) method reduces to the (SQI) method. The following theorem establishes the fact that the Quadrant interlocking splitting is a regular splitting.

Theorem 6.1. *The quadrant interlocking matrix splitting is a regular splitting.* We shall quote some essential theorems for the convergence of these iterative schemes. A more detailed proof of these is given in [26].

Theorem 6.2. *For any coefficient matrix, if the (JQI) method converges then the (JOQI) method will also converge for* $0 < \omega \leq 1$.

Theorem 6.3. *For the (SOQI) iteration matrix* $\mathcal{L}_\omega$ *and for all real* ω *we have that* $\rho(\mathcal{L}_\omega) \geq |\omega - 1|$. *Moreover, if the (SOQI) method converges, then we have* $0 < \omega < 2$.

Theorem 6.4. *Let* A *be* $(n \times n)$ *diagonally dominant matrix. Then the (JQI) method converges.*

Theorem 6.5. *Let* A *be an irreducible and diagonally dominant matrix, then the (SQI) method converges, and the (SOQI) method converges for* $0 < \omega \leq 1$.

Theorem 6.6. *Let* A *be a real, symmetric, nonsingular matrix, and let the crossed diagonal matrix of* A *(i.e.* X*) be positive definite. Then the (JOQI) method converges, if and only if* A *and* $2\omega^{-1}X - A$ *are positive definite. The condition that* $2\omega^{-1}X - A$ *is positive definite may be replaced by the condition*

$$0 < \omega < \frac{2}{1 - \mu_{min}} \leq 2,$$

where $\mu_{min} \leq 0$ *is the smallest eigenvalue of (JQI) iteration matrix* $B = X^{-1}(W + Z)$.

Corollary 6.6. *Under the hypotheses of Theorem 6.6, the (JQI) method converges if and only if* A *and* $2X - A$ *are positive definite.*

Theorem 6.7. *If* A *is a real symmetric matrix with positive definite crossed diagonal matrix* X*, then the (SOQI) method converges if and only if* A *is positive definite and* $0 < \omega < 2$.

Corollary 6.7. *Under the hypotheses of Theorem 6.7, the (SQI) method converges if and only if A is positive definite.*

Theorem 6.8. *The (SOQI) method converges for $0 < \omega < 2$ and corresponds to a (2×2) block SOR method with $\omega = \dfrac{2}{1 + \sqrt{1 - \rho^2(J)}}$, where $\rho(J)$ is the spectral radius of the corresponding block Jacobi method.*

Remark. This theorem establishes a relation between the eigenvalues of the matrix $\mathcal{L}_\omega$ associated with (SOQI) method and the eigenvalues of the matrix B associated with the (JQI) method.

7. Multisplitting

In this section, we present the concepts of multisplitting of matrices which are known to be parallel iterative methods for solving a linear system (1.1). This was proposed by O'Leary and White [3]. These methods are based on a collection of different splittings of the coefficient matrix of the linear system. In a parallel computing environment each processor performs iterations corresponding to one of the splittings and the multisplitting iterates are obtained by combining the results of each processor in an appropriate manner to yield the next iteration step.

Definition 7.1. Let A, M_i, N_i and D_i be $(n \times n)$ matrices. Then (M_i, N_i, D_i) triplets are called *multisplitting* of A if

(i) $A = M_i - N_i$, $i = 1(1)K$, where each M_i is invertible;

(ii) $\sum D_i = I$ where D_i are diagonal matrices and $D_i \geq 0$.

The system (1.1) may be written as

$$M_i x = N_i x + b, \qquad i = 1(1)K.$$

Considering the iteration

$$M_i x_i^{k+1} = N_i x^k + b$$

we have

$$x_i^{k+1} = M_i^{-1} N_i x^k + M_i^{-1} b$$

and now using the weighting matrices D_i to combine the K splitted iterates we have

$$x^{k+1} = \sum_{i=1}^{K} D_i x_i^{k+1} = \sum_{i=1}^{K} D_i M_i^{-1} N_i x^k + \sum_{i=1}^{K} D_i M_i^{-1} b$$

and by letting

$$H := \sum_{i=1}^{K} D_i M_i^{-1} N_i$$

and

$$G := \sum_{i=1}^{K} D_i M_i^{-1}$$

we have an algorithm

$$x^{k+1} = Hx^k + Gb. \tag{7.1}$$

This iteration is very similar to the iterations we saw in Sections 3 and 4. The only problem worth noting here is that the choice of D_i affects the spectral radius of H and hence the rate of convergence of this iteration.

The following theorem gives conditions on the multisplitting (M_i, N_i, D_i), $i = 1(1)K$ to ensure convergence from standard results, see Berman and Plemmons [27] and Ortega [28].

Theorem 7.1.

(a) *If for* $i = 1(1)K$ (M_i, N_i) *is a weak regular splitting of a matrix* A *satisfying* $A^{-1} \geq 0$ *and* $H := \sum_{i=1}^{K} D_i M_i^{-1} N_i$ *then the iteration (7.1) is convergent.*

(b) *If for* $i = 1(1)K$ (M_i, N_i) *is P-regular splitting of a symmetric positive definite matrix* A *and* $D_i = \alpha_i I$ *then the iteration (7.1) is convergent.*

(c) *If for* $i = 1(1)K$ $\|M_i^{-1} N_i\|_\infty < 1$ *then the iteration (7.1) is convergent.*

Proof. The proof is similar to the standard proof found in [28].

Remark. This theorem actually implies that, if we have a collection of convergent splittings of a matrix, then under certain conditions we could construct a convergent multisplitting.

Another way of constructing a convergent multisplitting is by *dissolution*. Here the matrix is broken into simple pieces A_i and adding diagonal matrices E_i to ensure that $M_i = A_i + E_i$ is invertible.

Definition 7.2. Let A_i, M_i, N_i, D_i be $(n \times n)$ matrices. Then the (A_i, E_i, D_i) triplet is called a dissolution of A if

(i) $A = \sum A_i$, $i = 1(1)K$;

(ii) E_i and D_i are diagonal matrices;

(iii) (M_i, N_i, D_i) is a multisplitting of A, where $M_i = A_i + E_i$ and $N_i = E_i - \sum_{j \neq i} A_j$.

Remark. The (A_i, E_i, D_i) triplet is a convergent dissolution of A, if it is a dissolution for which the multisplitting (M_i, N_i, D_i) leads to a convergent iteration.

The following theorems give explicit conditions for a convergent dissolution of the matrix A.

Theorem 7.2. $A = \sum_{i=1}^{K} A_i$ *be an M-matrix and let the matrices* E_i *be nonnegative diagonal matrices with diagonal components equal to* e_{li}. *Then if the matrices satisfy*

(a) $0 \leq -a^i_{lm} \leq -a_{lm}, \quad l \neq m$;

(b) $e_{li} + a^i_{ll} > -\sum_{m \neq l} a^i_{lm}$;

(c) $e_{li} + a^i_{ll} \geq a_{ll}$;

then for all nonnegative diagonal matrices D_i *with* $\sum_i D_i = I$, (A_i, E_i, D_i) *is a convergent dissolution.*

Proof. Using (a) and the fact that $A = \sum A_i$ for $l \neq m$ $m^i_{lm} = -\sum_{j \neq i} a^j_{lm} = a^i_{lm} - a_{lm} \geq 0$. By (c) $m^i_{ll} = e_{li} + a^i_{ll} - a_{ll} \geq 0$, $M_i = A_i + E_i$ satisfies $m^i_{lm} = a^i_{lm} \leq 0$, $l \neq m$ by (a) and $m^i_{ll} = a^i_{ll} + e_{li} > 0$ by (c). Further, by (b) M_i is strictly row diagonally dominant matrix. This implies M_i is an M-matrix and $M_i^{-1} \geq 0$. Thus, $A = M_i - N_i$ is a weak regular splitting for each i and by Theorem 7.1(a), the multisplitting is convergent.

Theorem 7.3 *Let* $A = \sum_i A_i$ *be a symmetric positive definite matrix, and let* $A_i + E_i$ *be nonsingular and* $2(A_i + E_i) - A$ *be positive definite* $i = 1(1)K$. *Then for nonnegative diagonal matrices* $D_i = \alpha_i I$, *the* (A_i, E_i, D_i) *triplet is a convergent dissolution of* A.

Proof. From the conditions $M_i = A_i + E_i$ is invertible. $M_i + N_i = A_i + 2E_i - \sum_{j \neq i} A_j = 2(A_i + E_i) - A$ is positive definite. From Theorem 7.1(b), (A_i, E_i, D_i) is a convergent dissolution.

Neumann and Plemmons [29] studied in detail the work of O'Leary and White [3] to develop a convergence theory for multisplitting iterative methods where A is an M-matrix. Here the multisplitting process in $R^{n \times n}$ is presented as an ordinary iterative process for a certain block matrix $\mathcal{A} \in R^{kn \times kn}$ where k is the number of processors. The i-th processor receives the approximation y_j and computes the vector

$$y_{j+1} = E_i M_i^{-1} N_i y_j + E_i M_i^{-1} b.$$

They introduced an iteration scheme in R^{kn} as

$$z_{i+1} = \begin{bmatrix} E_1 M_1^{-1} N_1 & & E_k M_k^{-1} N_k \\ \vdots & & \vdots \\ E_1 M_1^{-1} N_1 & & E_k M_k^{-1} N_k \end{bmatrix} z_i \oplus_{j=1}^{k} \left(\sum_{i=1}^{k} E_i M_i^{-1} b \right) := \tag{7.2}$$

$$:= B z_i + \oplus_{i=1}^{k} \left(\sum_{i=1}^{k} E_i M_i^{-1} b \right)$$

where for an n-vector y, $\oplus_{j=1}^{k} y_j$ denotes the kn-vector by taking the direct sum of k copies of the vector y. For

$$B := \begin{bmatrix} E_1 & & E_k \\ \vdots & & \vdots \\ E_1 & & E_k \end{bmatrix} \begin{bmatrix} M_1^{-1} N_1 & & 0 \\ \vdots & & \vdots \\ 0 & & M_k^{-1} N_k \end{bmatrix} \tag{7.3}$$

we get the same set of eigenvalues for the matrix

$$\tilde{B} := \begin{bmatrix} M_1^{-1} N_1 & & 0 \\ \vdots & & \vdots \\ 0 & & M_k^{-1} N_k \end{bmatrix} \begin{bmatrix} E_1 & & E_k \\ \vdots & & \vdots \\ E_1 & & E_k \end{bmatrix} \tag{7.4}$$

as B by disregarding multiplicities. The matrix $\tilde{B}$ is induced by splitting a $kn \times kn$ matrix $\mathcal{A}$:

$$\mathcal{A} = \begin{bmatrix} M_1 & 0 & & 0 \\ 0 & M_2 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & M_k \end{bmatrix} - \begin{bmatrix} N_1 & 0 & & 0 \\ 0 & N_2 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & N_k \end{bmatrix} \begin{bmatrix} E_1 & E_2 & & E_k \\ \vdots & \vdots & & \vdots \\ E_1 & E_2 & & E_k \end{bmatrix}, \tag{7.5}$$

$$\mathcal{A} := \begin{bmatrix} M_1 - N_1 E_1 & -N_1 E_2 & & -N_1 E_k \\ -N_2 E_1 & M_2 - N_2 E_2 & & -N_2 E_k \\ \vdots & \vdots & & \vdots \\ -N_k E_1 & -N_k E_2 & & M_k - N_k E_k \end{bmatrix}. \tag{7.6}$$

We shall quote the essential lemmas and theorems which ensure convergence for the iteration in $R^{n\times n}$. Detailed proofs are given in [29].

Lemma 7.1. *For the block matrix $\mathcal{A}$ and the splitting (7.2) the following hold:*

(i) *If the splitting $A = M_i - N_i$, $i = 1(1)k$ are (weak) regular splittings, then (7.6) is a (weak) splitting.*

(ii) *If x is an n-vector for which $Ax \geq 0$, then $\mathcal{A}\left(\oplus_{i=1}^{k} x\right) \geq 0$.*

(iii) *If A is a monotone matrix for each $i = 1(1)k$, $A = M_i - N_i$ is a weak regular splitting, then $\mathcal{A}$ is a monotone matrix.*

(iv) *If A is a nonsingular M-matrix and for each $i = 1(1)k$, the splitting $A = M_i - N_i$ is an M-splitting, then $\mathcal{A}$ is a nonsingular M-matrix.*

By denoting $\rho(B)$ as the spectal radius of a matrix $B \in R^{n\times n}$, if $B \geq 0$, then $\rho(B)$ is the eigenvalue of B and has a nonnegative eigenvector corresponding to $\rho(B)$ known as *Perron vector* of B.

Lemma 7.2. *Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splittings of the monotone matrices A_1 and A_2 respectively such that $M_2^{-1} \geq M_1^{-1}$. If there exists a positive vector x such that $0 \leq A_1 x \leq A_2 x$, then for the monotonic norm associated with x,*

$$\|M_2^{-1}N_2\|_x \leq \|M_1^{-1}N_1\|_x. \tag{7.7}$$

In particular, if $M_1^{-1}N^{-1}$ has a positive Perron vector, then

$$\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1). \tag{7.8}$$

Moreover, if x is a Perron vector of $M_1^{-1}N_1$ and $\|M_2^{-1}N_2\|_x < \|M_1^{-1}N_1\|_x$ then

$$\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1). \tag{7.9}$$

Proof. Since $M_2^{-1} \geq M_1^{-1}$ we have, by the condition $0 \leq A_1 x \leq A_2 x$, that

$$(I - M_2^{-1}N_2)x = M_2^{-1}A_2 x \geq M_1^{-1}A_1 x = (I - M_1^{-1}N_1)x$$

from which

$$M_2^{-1}N_2 \leq M_1^{-1}N_1.$$

The results follow from Rheinboldt and Vandergraft [31, pp. 140-141].

Let

$$H := \sum_{i=1}^{k} E_i M_i^{-1} N_i, \tag{7.10}$$

$$\mathcal{M} := \operatorname{diag}(M_1, \ldots, M_k) \tag{7.11}$$

and

$$\mathcal{N} := \begin{bmatrix} N_1 \\ \vdots \\ N_k \end{bmatrix} \quad (E_1, \ldots, E_k) \tag{7.12}$$

then from (7.2) we have

$$\mathcal{A} = \mathcal{M} - \mathcal{N}. \tag{7.13}$$

Theorem 7.4. *Let A be a monotone matrix. Suppose there exist weak regular splittings $A = \hat{M}_i - \hat{N}_i$ such that $\hat{M}_i^{-1} \geq M_i^{-1}$ for all $i = 1(1)k$. Then for any choice of nonnegative diagonal matrices $E_i, \hat{E}_i$ with $\sum_{i=1}^{k} E_i = \sum_{i=1}^{k} \hat{E}_i = I$ and for any choice of a vector $x \gg 0$ such that $Ax \geq 0$,*

$$\left\| \begin{pmatrix} \hat{M}_1^{-1} \hat{N}_1 x \\ \vdots \\ \hat{M}_k^{-1} \hat{N}_k x \end{pmatrix} \right\| \oplus_{i=1}^{k} x = \|\hat{\mathcal{M}}^{-1} \hat{\mathcal{N}}\| \oplus_{i=1}^{k} x \leq \|\mathcal{M}^{-1} \mathcal{N}\| \oplus_{i=1}^{k} x = \tag{7.14}$$

$$= \left\| \begin{pmatrix} M_1^{-1} N_1 x \\ \vdots \\ M_k^{-1} N_k x \end{pmatrix} \right\| \oplus_{i=1}^{k} x.$$

In particular, if in addition $\oplus_{i=1}^{k} x$ is a Perron vector of $\mathcal{M}_{-1}\mathcal{N}$, then

$$\rho(\hat{\mathcal{M}}^{-1} \hat{\mathcal{N}}) \leq \rho(\mathcal{M}^{-1} \mathcal{N}) \tag{7.15}$$

and consequently,

$$\rho(\hat{H}) \leq \rho(H). \tag{7.16}$$

Proof. By Lemma 7.1 the splittings $\mathcal{A} = \mathcal{M} - \mathcal{N}$ and $\hat{\mathcal{A}} = \hat{\mathcal{M}} - \hat{\mathcal{N}}$ are both weak regular splittings. Since A is monotone, then $\mathcal{A}$ and $\hat{\mathcal{A}}$ are monotone matrices by Lemma 7.1 and from (7.6) and the fact that

$$\sum_{i=1}^{k} E_i = \sum_{i=1}^{k} \hat{E}_i = I,$$

$$\hat{\mathcal{A}}(\oplus x) = \mathcal{A}(\oplus x).$$

The splittings therefore satisfy the requirements of Lemma 7.2, hence

$$\|\hat{\mathcal{M}}^{-1}\hat{\mathcal{N}}\|(\oplus_{i=1}^{k}) \leq \|\mathcal{M}^{-1}\mathcal{N}\|(\oplus_{i=1}^{k} x).$$

Moreover, if $\oplus_x$ is a Perron vector of $\mathcal{M}^{-1}\mathcal{N}$, then $\rho(\hat{\mathcal{M}}^{-1}\hat{\mathcal{N}}) \leq \rho(\mathcal{M}^{-1}\mathcal{N})$ and as an immediate consequence of $\rho(\hat{H}) \leq \rho(H)$.

We now consider the multisplittings for the classical methods. If we assume that the diagonal entries of the coefficient matrix are all unity, then let $A = I - L - U$ be a nonsingular M-matrix, where L and U are strictly lower and upper triangular matrices respectively. Then each of the splittings for the classical methods is a regular splitting of the coefficient matrix A.

$$A = I - (L + U)$$

where

$$J = L + U \tag{7.17}$$

is the Jacobi iteration matrix;

$$A = (I - L) - U$$

where

$$\mathcal{L} = (I - L)^{-1} U \tag{7.18}$$

is the Gauss-Seidel iteration matrix;

$$A = (I - L)(I - U) - LU$$

where

$$S = (I - U)^{-1} L (I - L)^{-1} U \tag{7.19}$$

is the symmetric Gauss-Seidel iteration matrix.

The next theorem states the comparison theorem for the multisplittings for the above classical methods.

Theorem 7.5. *Let $A = I - L - U$ be an $(n \times n)$ nonsingular M-matrix and consider the multisplittings*

$$A = (I - L_i - U_i) - (U + L - L_i - U_i) \tag{7.20}$$

$$A = (I - L_i) - (U + L - L_i) \tag{7.21}$$

$$A = (I - L_i)(I - U_i) - (U + L + L_iU_i - L_i - U_i) \tag{7.22}$$

where

$$0 \leq L_i \leq L \quad \text{and} \quad 0 \leq U_i \leq U \quad \text{for} \quad i = 1(1)k. \tag{7.23}$$

Then for any choice of nonnegative diagonal matrices E_i such that $\sum_{i=1}^{k} E_i = I$ and any vector $x \gg 0$ such that $Ax \leq 0$,

$$\left\| \begin{pmatrix} (I - L_1 - U_1)^{-1}(U + L - L_1 - U_1)x \\ \vdots \\ (I - L_k - U_k)^{-1}(U + L - L_k - U_k)x \end{pmatrix} \right\|_{\oplus_{i=1}^{k} x} \leq \|J\|_x, \tag{7.24}$$

$$\|\mathcal{L}\|_x \leq \left\| \begin{pmatrix} (I - L_1)^{-1}(U + L - L_1)x \\ \vdots \\ (I - L_k)^{-1}(U + L - L_k)x \end{pmatrix} \right\|_{\oplus_{i=1}^{k} x} \leq \|J\|_x, \tag{7.25}$$

and

$$\begin{aligned} &\|S\|_x \leq \\ &\leq \left\| \begin{pmatrix} (I - U_1)^{-1}(I - L_1)^{-1}(U + L - L_1U_1 - L_1 - U_1)x \\ \vdots \\ (I - U_k)^{-1}(I - L_k)^{-1}(U + L - L_kU_k - L_k - U_k)x \end{pmatrix} \right\|_{\oplus_{i=1}^{k} x} \leq \\ &\leq \left\| \begin{pmatrix} (I - L_1)^{-1}(U + L - L_1)x \\ \vdots \\ (I - L_k)^{-1}(U + L - L_k)x \end{pmatrix} \right\|_{\oplus_{i=1}^{k} x}. \end{aligned} \tag{7.26}$$

In particular, if A is (also) irreducible, then

$$\rho\left(\sum_{i=1}^{k} E_i(I - L_i - U_i)^{-1}(U + L - L_i - U_i)\right) \le \rho(J), \tag{7.27}$$

$$\rho\left(\sum_{i=1}^{k} E_i(I - L_i)^{-1}(U + L - L_i)\right) \le \rho(J), \tag{7.28}$$

$$\rho\left(\sum_{i=1}^{k} E_i(I - U_i)^{-1}(I - L_i)^{-1}(U + L + L_iU_i - L_i - U_i)\right) \le \rho(J). \tag{7.29}$$

Proof. Since A is a nonsingular M-matrix, each of the splittings (7.20) – (7.22) is a regular splitting.

Since $I - L_i$, $I - U_i$ and $I - L_i - U_i$, $i = 1(1)k$ are nonsingular M-matrices such that by (7.23)

$$(I - L_i - U_i) \le I \tag{7.30}$$

and

$$(I - L) \le (I - L_i) \le I \tag{7.31}$$

we have that

$$I \le (I - L_i - U_i)^{-1}, \tag{7.32}$$

$$I \le (I - L_i)^{-1} \le (I - L)^{-1} \tag{7.33}$$

and

$$I \le (I - L_i)^{-1} \le (I - U_i)^{-1}(I - L_i)^{-1}. \tag{7.34}$$

Then by (7.17), (7.20), (7.32) and Theorem 7.4 it follows that (7.24) holds. By (7.17), (7.18), (7.21), (7.33) and Theorem 7.4 (7.25) is valid. And (7.26) follows from (7.20) – (7.22), (7.34) and Theorem 7.4.

Suppose now that A is (also) irreducible. Then by (7.17) J is a nonnegative and irreducible matrix and hence has a Perron vector x. Moreover, $Ax \gg 0$. Now let $M_i = I$ and $N_i = L + U$, $i = 1(1)k$, then from (7.11) – (7.13) and Lemma 7.2 it follows that $\oplus_{i=1}^{k} x$ is a Perron vector of $\mathcal{M}^{-1}\mathcal{N}$, hence (7.27) – (7.29) follows.

Neumann and Plemmons [29] in their paper discussed the upper bounds for the convergence rate of iterative procures based on multisplittings, but quite recently, in a paper by Elsner [30], the comparison results for weak regular splittings of monotone matrices and the establishment of lower bounds for both weak regular splittings and multisplittings were studied.

Finally, we remark that most of the splittings discussed in this paper are eventually parallel iterative algorithms and hence in a parallel computing environment they are widely employed.

References

[1] **Varga R.S.**, Factorization and normalized iterative methods, *Boundary Problems in Differential Equations*, ed. R. E. Langer, University of Wisconsin Press, Madison, 1960.

[2] **Ostrowski A.M.**, Determinanten mit überwiegender Hauptdiagonale und die absolute Konvergenz von linearen Prozeßen, *Comment Helv.*, **30** (1956), 175-210.

[3] **O'Leary D.P. and White R.E.**, Multisplittings of matrices and parallel solutions of linear systems, *SIAM J.Alg. Discrete Methods*, **6** (1985), 630-640.

[4] **Ortega J.M. and Rheinboldt W.C.**, Monotone iterations for nonlinear equations with applications to Gauss-Seidel methods, *SIAM J.Numer.Anal.*, **4** (1979), 171-190.

[5] **Vandergraft J.S.**, Applications of partial orderings to the study of positive definiteness, monotonicity and convergence of iterative methods of linear systems, *SIAM J.Numer.Anal.*, **4** (1972), 97-104.

[6] **Csordas G. and Varga R.S.**, Comparisons of regular splittings of matrices, *Numer.Math.*, **44** (1984), 23-35.

[7] **Wozincki Z.**, *Two-sweep iterative methods for solving large systems and the applications to the numerical solution of multigroup multi-dimensional neutron diffusion equation*, Ph.D. thesis, Institute of Nuclear Reseach, Swierk/Otwocka, Poland, 1973.

[8] **Beauwens R.**, Factorization iterative methods: M-operators and H-operators, *Numer.Math.*, **31** (1979), 335-357.

[9] **Miller V.A. and Neumann M.**, A note on comparison theorems for nonnegative matrices, *Numer.Math.*, **47** (1985), 427-434.

[10] **Berman A. and Plemmons R.J.**, Cones and iterative methods for least-squares solution of linear systems, *SIAM J.Numer.Anal.*, **11** (1974), 145-154.

[11] **Lawson L.M.**, Computational methods for generalized inverse matrices arising from proper splittings, *SIAM J.Lin.Alg.Appl.*, **12** (1975), 111-126.

[12] **Berman A. and Neumann M.**, Proper splittings of rectangular matrices, *SIAM J.Appl.Math.*, **31** (1976), 307-315.

[13] **Berman A. and Neumann M.**, Consistency and splittings, *SIAM J.Numer. Anal.*, **13** (1976), 877-888.

[14] **Young D.M.**, On the consistency of linear stationary iterative methods, *SIAM J.Numer.Anal.*, **9** (1972), 89-96.

[15] **Kammer W.J. and Plemmons R.J.**, Direct iterative methods for least-squares solutions to singular equations, *J.Math.Anal.Appl.*, **49** (1975), 512-526.

[16] **Varga R.S.**, *Matrix iterative analysis*, Prentice-Hall, Engelwood Cliffs, New Jersey, 1962.

[17] **Sisler M.**, Über ein Iterationsverfahren für zyklische Matrizen, *Appl.Math.*, **17** (1972), 225-233.

[18] **Sisler M.**, Über die Konvergenz eines gewissen Iterationsverfahrens für zyklische Matrizen, *SIAM J.Numer.Anal.*, **18** (1973), 89-98.

[19] **Sisler M.**, Bemerkungen zur Optimierung eines gewissen Iterationsverfahrens, *SIAM J.Numer.Anal.*, **18** (1973), 314-324.

[20] **Sisler M.**, Über ein zweiparametriges Iterationsverfahren, *SIAM J. Numer. Anal.*, **18** (1973), 325-332.

[21] **Sisler M.**, Über die Optimierung eines zweiparametrigen Iterationsverfahrens, *SIAM J.Numer.Anal.*, **20** (1973), 126-142.

[22] **Sisler M.**, Bemerkungen zur Optimierung eines zweiparametrigen Iterationsverfahrens, **SIAM J.Numer.Anal.**, **21** (1976), 213-220.

[23] **Niethammer W.**, Über- und Unterrelaxation bei linearen Gleichungssystemen, *Computing*, **55** (1970), 303-311.

[24] **Niethammer W.**, On different splittings and the associated methods, *SIAM J.Numer.Anal.*, (1979), 186-200.

[25] **Evans D.J.**, Parallel algorithms for linear equations, *Parallel Numerical Algorithms*, ed. E.Cavalli, Academic Press, N.Y., 357-383.

[26] **Evans D.J. and Haghighi R.S.**, Parallel iterative methods for solving linear equations, *Intern.J.Computer Math.*, **11** (1982), 247-284.

[27] **Berman A. and Plemmons R.J.**, *Nonnegative Matrices in Mathematical Sciences*, Academic Press, N.Y., 1979.

[28] **Ortega J.M.**, *Numerical Analysis: A Second Course*, Academic Press, N.Y., 1972.

[29] **Neumann M. and Plemmons R.J.**, Convergence of parallel multisplitting methods, *Linear Alg.Appl.*, **88/89** (1987), 559-573.

[30] **Elsner L.**, Comparisons of weak regular splittings and multisplitting methods, *Numer.Math.*, **56** (1989), 283-289.

[31] **Rheinboldt W.C. and Vandergraft J.S.**, A simple approach to the Perron-Frobenius theory for positive operators on general partially ordered finite dimensional linear spaces, *Math.Comp.*, **27** (1973), 139-145.

(Received November 14, 1991)

E.O. Amedjoe
Deparment of Numerical Analysis
Eötvös Loránd University
H-1117 Budapest, XI. Bogdánfy u. 10/b.
Hungary